Compito di
Matematica per le Applicazioni Economiche 2
22 dicembre 2008

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (10)
Si dica per quali valori di $a \in \mathbb{R}$ il seguente sistema ammette una, nessuna o infinite soluzioni:

$$\begin{cases} x_1 + x_2 - x_3 & = & 1 \\ 2x_1 + 3x_2 + ax_3 & = & 0 \\ x_1 + ax_2 + 3x_3 & = & 0 \end{cases}$$

Esercizio 2. (10)
Data la funzione

$$l : \mathbb{R}^3 \to \mathbb{R}^3, \quad l\,(x_1, x_2, x_3) = \begin{pmatrix} x_1 - x_2, & x_2 - x_3, & x_3 - x_1 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si calcoli $\dim \ker l$ e $\dim \operatorname{Im} l$.

Esercizio 3. (10)
Dato l'insieme

$$S \equiv \cap_{n \in \mathbb{N}} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right)$$

a. si dimostri che $S = [0, 1]$;
b. si dimostri che $Int\ S = (0, 1)$.

Esercizio 4. (30)
a. Si discuta il seguente problema. Per dati $\pi \in (0, 1)$, $a \in (0, +\infty)$,

$$\max_{(x,y)} \pi \cdot u\,(x) + (1 - \pi)\,u\,(y) \quad \text{s.t.} \quad \begin{array}{l} y \leq a - \frac{1}{2}x \\ y \leq 2a - 2x \\ x \geq 0 \\ y \geq 0 \end{array}$$

dove $u : \mathbb{R} \to \mathbb{R}$ e' una funzione C^2 tale che $\forall z \in \mathbb{R}$, $u'\,(z) > 0$ e $u''\,(z) < 0$.

b. Si dica se esistono dei valori di (π, a) per i quali $(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{2}{3}a, \frac{2}{3}a, 0, \lambda_2, 0, 0\right)$, con $\lambda_2 > 0$, e' una soluzione delle condizioni di Kuhn-Tucker - dove λ_j e' il moltiplicatore associato al vincolo j.

c. "Assumendo" che

il primo, il terzo e il quarto vincolo valgano in forma di disuguaglianza stretta,

il secondo vincolo valga in forma di uguaglianza e l'associato moltiplicatore sia strettamente positivo,

si descriva in dettaglio la procedura per il calcolo dell'effetto di una variazione di a e π sulla soluzione (x, y) del problema.

Traccia delle Soluzioni del compito del 22 dicembre 2008

Esercizio 1.

$$\begin{cases} x_1 + x_2 - x_3 & = & 1 \\ 2x_1 + 3x_2 + ax_3 & = & 0 \\ x_1 + ax_2 + 3x_3 & = & 0 \end{cases}$$

La matrice associata al sistema e'

$$\left[\begin{array}{cccc} 1 & 1 & -1 & | & 1 \\ 2 & 3 & a & | & 0 \\ 1 & a & 3 & | & 0 \end{array}\right]$$

$$\det \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & a \\ 1 & a & 3 \end{bmatrix} = \det \begin{bmatrix} 3 & a \\ a & 3 \end{bmatrix} - \det \begin{bmatrix} 2 & a \\ 1 & 3 \end{bmatrix} - \det \begin{bmatrix} 2 & 3 \\ 1 & a \end{bmatrix} =$$

$$= 9 - a^2 - 6 + a - 2a + 3 = -a^2 - a + 6$$

$a^2 + a - 6 = 0$, $a = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = -3, 2$

Se $a = -3$,

$$\left[\begin{array}{cccc} 1 & 1 & -1 & | & 1 \\ 2 & 3 & -3 & | & 0 \\ 1 & -3 & 3 & | & 0 \end{array}\right]$$

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 0 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix} = -6 - 3 = -9 \neq 0$$

Se $a = 2$

$$\left[\begin{array}{cccc} 1 & 1 & -1 & | & 1 \\ 2 & 3 & 2 & | & 0 \\ 1 & 2 & 3 & | & 0 \end{array}\right]$$

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix} = 4 - 3 \neq 0$$

Esercizio 2.

Data la funzione

$$l : \mathbb{R}^3 \to \mathbb{R}^3, \quad l(x_1, x_2, x_3) = \begin{pmatrix} x_1 - x_2, & x_2 - x_3, & x_3 - x_1 \end{pmatrix}$$

a. si verifichi che l e' lineare,

$l(\alpha x + \beta y) = l(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) =$

$= \begin{pmatrix} \alpha x_1 + \beta y_1 - (\alpha x_2 + \beta y_2), & (\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3), & (\alpha x_3 + \beta y_3) - (\alpha x_1 + \beta y_1) \end{pmatrix} =$

$= \begin{pmatrix} \alpha x_1 + \beta y_1 - (\alpha x_2 + \beta y_2), & (\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3), & (\alpha x_3 + \beta y_3) - (\alpha x_1 + \beta y_1) \end{pmatrix}.$

$\alpha l(x) + \beta l(y) = \alpha \begin{pmatrix} x_1 - x_2, & x_2 - x_3, & x_3 - x_1 \end{pmatrix} + \beta \begin{pmatrix} y_1 - y_2, & y_2 - y_3, & y_3 - y_1 \end{pmatrix} =$

$= \begin{pmatrix} \alpha(x_1 - x_2) + \beta(y_1 - y_2) & \alpha(x_2 - x_3) + \beta(y_2 - y_3) & \alpha(x_3 - x_1) + \beta(y_3 - y_1) \end{pmatrix}$

b. si calcoli la matrice A associata a l

$l(1,0,0) = \begin{pmatrix} 1 - 0, & 0 & 0 - 1 \end{pmatrix} = (1, 0, -1)$

$l(0,1,0) = \begin{pmatrix} 0 - 1 & 1 - 0, & 0 \end{pmatrix} = (-1, 1, 0)$

$l(0,0,1) = \begin{pmatrix} 0 & 0 - 1, & 1 - 0 \end{pmatrix} = (0, -1, 1)$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

c. e si calcoli $\dim \ker l$ e $\dim \operatorname{Im} l$

$$\det \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \neq 0$$

e dunque $\dim \ker l = 1$ e $\dim \operatorname{Im} l = 2$.

Esercizio 3.
Si osservi che

$$S \equiv \cap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$$

a. $\cap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) \subseteq [0,1]$

Vogliamo dimostrare che $x \in \cap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) \Rightarrow x \in [0,1]$, ovvero che $x \notin [0,1] \Rightarrow x \notin \cap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$, ovvero $x \notin [0,1] \Rightarrow \exists n \in \mathbb{N}$tale che $x \notin \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$: se $x < 0$ e' sufficiente prendere n tale che $x < -\frac{1}{n}$; se $x > 1$, e' sufficiente prendere n tale che $1+\frac{1}{n} < x$.

b. $[0,1] \subseteq \cap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$.

$\forall x \in [0,1]$, $-\frac{1}{n} < x < 1+\frac{1}{n}$.

$Int\ [0,1] = (0,1)$.

a. $Int\ [0,1] \subseteq (0,1)$.

Vogliamo dimostrare che $x \notin (0,1) \Rightarrow x \notin Int\ [0,1]$.

Se $x \notin [0,1]$, allora $x \notin Int\ [0,1]$, per definizione di Int. Dobbiamo dunque dimostrare che se $x \in \{0,1\}$, allora $x \notin Int\ [0,1]$.

Se $x = 0$, $\forall r > 0$, $-\frac{r}{2} \in I(0,r)$ e $-\frac{r}{2} \notin [0,1]$.

Se $x = 1$, $\forall r > 0$, $1+\frac{r}{2} \in I(1,r)$ e $1+\frac{r}{2} \notin [0,1]$.

b. $(0,1) \subseteq Int\ [0,1]$

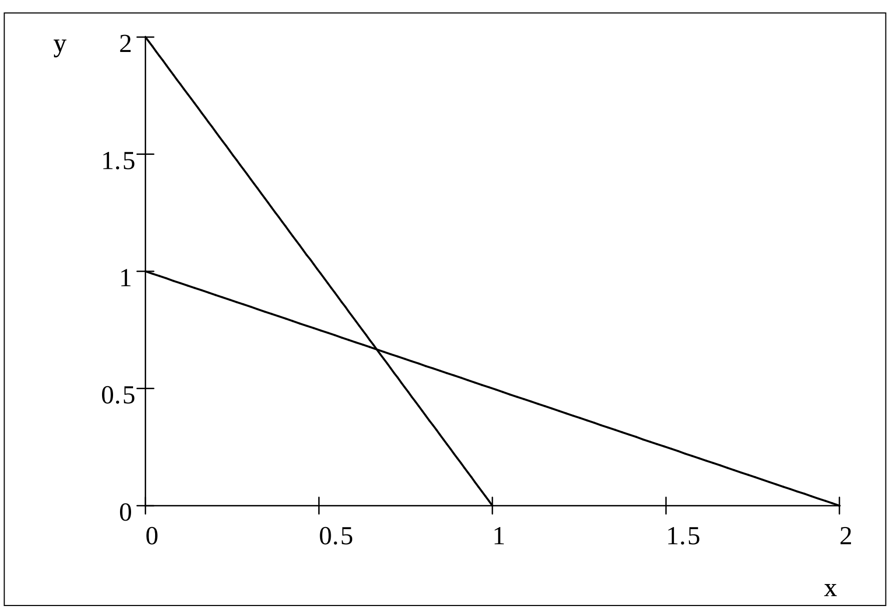
Vogliamo dimostrare che $\forall x \in (0,1)$ e' un punto interno a $[0,1]$.

Si prenda $r = \min\{x, 1-x\}$. $I(x,r) \subseteq [0,1]$.

1. Canonical form.
Per dati $\pi \in (0,1)$, $a \in (0,+\infty)$,

$$\begin{array}{llllr} \max_{(x,y)\in\mathbb{R}^2} & \pi \cdot u(x) + (1-\pi)\, u(y) & \text{s.t.} & a - \frac{1}{2}x - y \geq 0 & \lambda_1 \\ & & & 2a - 2x - y \geq 0 & \lambda_2 \\ & & & x \geq 0 & \lambda_3 \\ & & & y \geq 0 & \lambda_4 \end{array}$$

$$\begin{cases} y = a - \frac{1}{2}x \\ y = 2a - 2x \end{cases}, \text{soluzione: } \left[x = \frac{2}{3}a, y = \frac{2}{3}a\right]$$



2. The set X and the functions f and g.
a. The domain of all function is $\mathbb{R}^2$.
b. $X = \mathbb{R}^2$.
c. $\mathbb{R}^2$ is open and convex.
d. $Df(x,y) = (\pi \cdot u'(x), (1-\pi) u'(y))$. La matrice Hessiana e'

$$\begin{bmatrix} \pi \cdot u''(x) & 0 \\ 0 & (1-\pi) u''(y) \end{bmatrix}$$

Dunque f, g sono C^2 e f e' strettamente concava e g sono affini.
3. Existence.
C e' chiuso. Limitato sotto da $(0,0)$ e sopra da (a,a) :
$y \leq a - \frac{1}{2}x \leq a$
$2x \leq 2a - y \leq 2a$.
4. Number of solutions.
La soluzione e' unica perche' f e' strettamente concava e g_j sono affini e dunque concave.
5. Necessity of K-T conditions.
g_j sono affini e dunque concave.
$x^{++} = \left(\frac{1}{2}a, \frac{1}{2}a\right)$

$$\begin{array}{l} a - \frac{1}{2}\frac{1}{2}a - \frac{1}{2}a = \frac{1}{4}a > 0 \\ 2a - 2\frac{1}{2}a - \frac{1}{2}a = \frac{1}{2}a > 0 \\ \frac{1}{2}a > 0 \\ \frac{1}{2}a > 0 \end{array}$$

6. Sufficiency of K-T conditions.
f e' strettamente concava e g sono affini.
7. K-T conditions.
$\mathcal{L}(x, y, \lambda_1, ..., \lambda_4; \pi, a) = \pi \cdot u(x) + (1-\pi) u(y) + \lambda_1 \left(a - \frac{1}{2}x - y\right) + \lambda_2 (2a - 2x - y) + \lambda_3 x + \lambda_4 y$.

$$\begin{cases} \pi \cdot u'(x) - \frac{1}{2}\lambda_1 - 2\lambda_2 + \lambda_3 & = 0 \\ (1-\pi) u'(y) - \lambda_1 - \lambda_2 + \lambda_4 & = 0 \\ \min\left\{\lambda_1, a - \frac{1}{2}x - y\right\} & = 0 \\ \min\left\{\lambda_2, 2a - 2x - y\right\} & = 0 \\ \min\left\{\lambda_3, x\right\} & = 0 \\ \min\left\{\lambda_4, y\right\} & = 0 \end{cases}$$

$$\begin{cases} \pi \cdot u'(x) - \frac{1}{2}\lambda_1 - 2\lambda_2 + \lambda_3 & = 0 \\ (1-\pi) u'(x) - \lambda_1 - \lambda_2 + \lambda_4 & = 0 \\ a - \frac{1}{2}x - y & \geq 0 \\ \lambda_1 & \geq 0 \\ \text{prodotto} & = 0 \\ 2a - 2x - y & \geq 0 \\ \lambda_2 & \geq 0 \\ \text{prodotto} & = 0 \\ x & \geq 0 \\ \lambda_3 & \geq 0 \\ \text{prodotto} & = 0 \\ y & \geq 0 \\ \lambda_4 & \geq 0 \\ \text{prodotto} & = 0 \end{cases}$$

8. "Solve the K-T conditions"

$$\begin{cases} \pi \cdot u'\left(\frac{2}{3}a\right) - 2\lambda_2 & = 0 \\ (1-\pi)\, u'\left(\frac{2}{3}a\right) - \lambda_2 & = 0 \\ a - \frac{1}{2}\frac{2}{3}a - \frac{2}{3}a & = 0 \\ \lambda_1 & > 0 \\ \text{prodotto} & = 0 \\ 2a - 2\frac{2}{3}a - \frac{2}{3}a & = 0 \\ \lambda_2 & = 0 \\ \text{prodotto} & = 0 \\ \frac{2}{3}a & > 0 \\ \lambda_3 & = 0 \\ \text{prodotto} & = 0 \\ \frac{2}{3}a & > 0 \\ \lambda_4 & = 0 \\ \text{prodotto} & = 0 \end{cases}$$

$$\lambda_2 = \frac{1}{2}\pi \cdot u'\left(\frac{2}{3}a\right) = (1-\pi) \cdot u'\left(\frac{2}{3}a\right) > 0$$

$$\frac{1}{2}\pi = 1 - \pi$$

$$\pi = \frac{2}{3}, a \in \mathbb{R}_{++}$$

c.

$$\begin{cases} \pi \cdot u'\left(\frac{2}{3}a\right) - 2\lambda_2 & = 0 \\ (1-\pi)\, u'\left(\frac{2}{3}a\right) - \lambda_2 & = 0 \\ a - \frac{1}{2}x - y & > 0 \\ \lambda_1 & = 0 \\ \text{prodotto} & = 0 \\ 2a - 2x - y & = 0 \\ \lambda_2 & > 0 \\ \text{prodotto} & = 0 \\ x & > 0 \\ \lambda_3 & = 0 \\ \text{prodotto} & = 0 \\ y & > 0 \\ \lambda_4 & = 0 \\ \text{prodotto} & = 0 \end{cases}$$

$$\begin{cases} \pi \cdot u'(x) - 2\lambda_2 & = 0 \\ (1-\pi)\, u'(y) - \lambda_2 & = 0 \\ 2a - 2x - y & = 0 \end{cases}$$

$$\begin{array}{llllll} & x & y & \lambda_2 & \pi & a \\ \\ \pi \cdot u'(x) - 2\lambda_2 & \pi \cdot u''(x) & 0 & -2 & u'(x) & 0 \\ (1-\pi)\, u'(y) - \lambda_2 & 0 & (1-\pi)\, u''(y) & -1 & -u(y) & 0 \\ 2a - 2x - y & -2 & -1 & 0 & 0 & 2 \end{array}$$

$$\det \begin{bmatrix} \pi \cdot u''(x) & 0 & -2 \\ 0 & (1-\pi)\, u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix} =$$
$$= \pi u''(x) \det \begin{bmatrix} (1-\pi)\, u''(y) & -1 \\ -1 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & -2 \\ (1-\pi)\, u''(y) & -1 \end{bmatrix} =$$
$$= -\pi u''(x) - 4(1-\pi)\, u''(y) > 0$$

$$D_{(\pi,a)}(x, y, \lambda_2) = - \begin{bmatrix} \pi \cdot u''(x) & 0 & -2 \\ 0 & (1-\pi)\, u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} u''(x) & 0 \\ -u''(y) & 0 \\ 0 & 2 \end{bmatrix}$$

Per curiosita':

$$\begin{bmatrix} \pi\cdot u''(x) & 0 & -2 \\ 0 & (1-\pi)\,u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix}^{-1} =$$
$$= \frac{1}{-\pi u''(x)-4(1-\pi)u''(y)} \begin{bmatrix} -1 & 2 & 2u''(y)-2\pi u''(y) \\ 2 & -4 & \pi u''(x) \\ 2u''(y)-2\pi u''(y) & \pi u''(x) & \pi u''(x)\,u''(y)-\pi^2 u''(x)\,u''(y) \end{bmatrix}$$

$$D_{(\pi,a)}(x,y,\lambda_2) =$$
$$= -\frac{1}{-\pi u''(x)-4(1-\pi)u''(y)} \begin{bmatrix} -1 & 2 & 2u''(y)(1-\pi) \\ 2 & -4 & \pi u''(x) \\ 2u''(y)(1-\pi) & \pi u''(x) & \pi u''(x)\,u''(y)(1-\pi) \end{bmatrix} \begin{bmatrix} u'(x) & 0 \\ -u'(y) & 0 \\ 0 & 2 \end{bmatrix} =$$
$$= \frac{1}{\pi u''(x)+4(1-\pi)u''(y)} \begin{bmatrix} -u'(x)-2u'(y) & 4u''(y)(1-\pi) \\ 2u'(x)-4u'(y) & 2\pi u''(x) \\ 2u''(y)(1-\pi)\cdot u'(x)+\pi u''(x)(-u'(y)) & 2\pi u''(x)\,u''(y)(1-\pi) \end{bmatrix}$$

Compito di
Matematica per le Applicazioni Economiche 2
14 genio 2009

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Esercizio 1. (10)
Discutere al variare di $b \in \mathbb{R}$, il sistema

$$\begin{cases} bx + y & = & b - z \\ x + by - z & = & -y \\ 3x + 4y & = & 2 \end{cases}$$

Esistono valori di b per i quali il sistema ammette infinite soluzioni? In tal caso esplicitare le soluzioni.

Esercizio 2. (8)
Data la funzione

$$l : \mathbb{R}^3 \to \mathbb{R}^3, \quad l(x_1, x_2, x_3) = \begin{pmatrix} x_1, & -x_2, & x_1 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si descriva $\ker l$ e $\operatorname{Im} l$.

Esercizio 3. (12)
Si considerino sottoinsiemi di $\mathbb{R}$. Si dimostri la verita' o falsita' delle seguenti affermazioni:
a. Unione finita di compatti e' compatta;
b. Intersezione finita di compatti e' compatta;
c. Unione infinita di compatti e' compatta (suggerimento: $[0, 1] \cup [1, 2] \cup ... \cup [n, n + 1] \cup ...$).
b. Intersezione infinita di compatti e' compatta.
Esercizio 4. (30)
a. Si discuta il seguente problema

$$\max_{(x,y)} \quad h(x) + \ln(y + 1) \quad s.t. \quad \begin{array}{l} x + y \leq a, \\ x \geq 0, \end{array}$$

con $a \in (0, +\infty)$, e $h : \mathbb{R} \to \mathbb{R}$ di classe C^2 e tale che $\forall x \in \mathbb{R}$, $h'(x) > \frac{1}{a+1}$ e $h''(x) < 0$.
b. Si dica se $(x = 0, y = a)$ e' soluzione del problema per qualche valore dei parametri;
c. Restringendosi all'insieme dei parametri per i quali esiste una soluzione del problema di massimo con $x \in (0, a)$, si studi l'effetto di una variazione di a sul valore della coppia (x^*, y^*) soluzione del problema.

Traccia delle Soluzioni.
Esercizio 1.

$$\left[\begin{array}{ccc|c} b & 1 & 1 & b \\ 1 & b+1 & -1 & 0 \\ 3 & 4 & 0 & 2 \end{array}\right]$$

che sommando la prima riga alla seconda diventa:

$$\left[\begin{array}{ccc|c} b & 1 & 1 & b \\ b+1 & b+2 & 0 & b \\ 3 & 4 & 0 & 2 \end{array}\right]$$

$$\det A = \det \begin{bmatrix} b & 1 & 1 \\ b+1 & b+2 & 0 \\ 3 & 4 & 0 \end{bmatrix} = \det \begin{bmatrix} b+1 & b+2 \\ 3 & 4 \end{bmatrix} = 4b+4-3b-6 = b-2$$

Dunque se $b \neq 2$, il sistema ammette un'unica soluzione.
Se $b = 2$, si ha

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 3 & 4 & 0 & 2 \\ 3 & 4 & 0 & 2 \end{array}\right]$$

Dunque il sistema ha come matrice associata

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 3 & 4 & 0 & 2 \end{array}\right]$$

Poiche'

$$\det \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} = -4$$

il sistema ha ∞ soluzioni;

$$\begin{cases} y+z & = & 2-2x \\ 4y & = & 2-3x \end{cases}$$

Esercizio 2. (10)
Data la funzione

$$l : \mathbb{R}^3 \to \mathbb{R}^3, \quad l\,(x_1, x_2, x_3) = \begin{pmatrix} x_1, & -x_2, & x_1 \end{pmatrix}$$

a. si verifichi che l e' lineare;
...
b. si calcoli la matrice A associata a l;

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

c. si descriva $\ker l$ e $\operatorname{Im} l$.
rank $A = 2$ e percio' $\dim \ker l = 1$. In effetti,

$$\ker l = \left\{x \in \mathbb{R}^3 : x_1 = x_2 = 0\right\}$$

e

$$\operatorname{Im} l = \left\{x \in \mathbb{R}^3 : x_1 = x_3\right\}$$

Esercizio 3. (12)
Sia $\{K_i\}_{i=1}^n$ una famiglia di compatti.
a. Unione finita di compatti e' compatta: VERO.

Unione di chiusi e' chiusa; $\forall i$, $\exists a_i', a_i'' \in \mathbb{R}$ tale che $\forall x_i \in K_i$

$$a_i' < x_i < a_i''$$

Si prenda $a^* = \min_i \{a_i'\}$ e $a^{**} = \max_i \{a_i'\}$. ...
b. Intersezione finita di compatti e' compatta. VERO.
Intersezione finita di chiusi e' chiusa e $\forall i$, $\cap_{j=1}^n K_j \subseteq K_i$. ...
Sia $\{K_\alpha\}_{\alpha \in A}$ una famiglia infinita di compatti. Falso
c. Unione infinita di compatti e' compatta (suggerimento: $[0,1] \cup [1,2] \cup ... \cup [n, n+1] \cup ...$).
L'insieme suggerito e' l'unione infinita di compatti, ma tale unione e' $\mathbb{R}_+$ che non e' limitato.
b. Intersezione infinita di compatti e' compatta.
Intersezione infinita di chiusi e' chiusa e $\forall i$, $\cap_{j=1}^n K_j \subseteq K_i$. ...

Esercizio 4.

$\boxed{\text{a.}}$

$$\max_{(x,y)\in\mathbb{R}\times(-1,+\infty)} \quad h(x) + \ln(y+1) \quad s.t. \quad \begin{array}{l} x + y \leq a, \\ x \geq 0, \end{array}$$

0. $\mathbb{R} \times (-1, +\infty)$ e' aperto e convesso.
1. Esistenza. L'insieme vincolato non e' compatto.
2. Forma Canonica.

$$\max_{(x,y)\in\mathbb{R}\times(-1,+\infty)} \quad h(x) + \ln(y+1) \quad s.t. \quad \begin{array}{ll} a - x - y \geq 0, & \mu \\ x \geq 0, & \lambda_x \end{array}$$

3. Unicità'.

L'insieme vincolato e' chiaramente convesso perché' definito tramite diseguaglianza relative a funzioni lineari. La matrice Hessiana della funzione obiettivo e' $\begin{bmatrix} h''(x) & 0 \\ 0 & -\frac{1}{(y+1)^2} \end{bmatrix}$,

Dunque la funzione obiettivo e' strettamente concava e la soluzione e' unica.

4. Necessita' delle condizioni di K-T.

Le funzioni vincolo sono lineari e dunque pseudo-concave; $(x,y)^{++} = \left(\frac{a}{4}, \frac{a}{4}\right)$ soddisfa i vincoli in forma di disuguaglianza stretta.

5. Sufficienza delle condizioni di K-T.

La funzione obiettivo e'strettamente concava e dunque pseudo-concava e le funzioni vincolo sono lineari e dunque quasi-concave.

6. Condizioni di K-T.

$\mathcal{L}(x,y) = x + \ln(y+1) + \mu(a - x - y) + \lambda_x x$.

$$\begin{array}{ll} & \min\{a - x - y, \mu\} = 0 \\ & \min\{x, \lambda_x\} = 0 \\ \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow & h'(x) - \mu + \lambda_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow & \frac{1}{y+1} - \mu = 0 \end{array}$$

$\boxed{\text{b.}}$

Se $(x = 0, y = a)$ si ha

$$\begin{array}{l} \min\{0, \mu\} = 0 \Rightarrow \mu \geq 0 \\ \min\{0, \lambda_x\} = 0 \Rightarrow \lambda_x \geq 0 \\ h'(0) - \mu + \lambda_x = 0 \\ \frac{1}{a+1} - \mu = 0 \end{array}$$

Dunque $\mu = \frac{1}{a+1}$ e $\lambda_x = -h'(0) + \frac{1}{a+1} < 0$, una contraddizione.

$\boxed{\text{c.}}$

$$\begin{array}{ll} & \min\{a - x - y, \mu\} = 0 \\ & \min\{x, \lambda_x\} = 0 \Rightarrow \lambda_x = 0 \\ \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow & h'(x) - \mu = 0 \Rightarrow a - x - y = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow & \frac{1}{y+1} - \mu = 0 \end{array}$$

Dunque si ha

$$\left\{\begin{array}{lccc} & x & y & \mu \\ h'(x)-\mu=0 & h''(x) & 0 & -1 \\ \frac{1}{y+1}-\mu=0 & 0 & -\left(\frac{1}{y+1}\right)^2 & -1 \\ a-x-y=0 & -1 & -1 & 0 \end{array}\right.$$

$$\det\begin{bmatrix} h''(x) & 0 & -1 \\ 0 & -\left(\frac{1}{y+1}\right)^2 & -1 \\ -1 & -1 & 0 \end{bmatrix} = h''(x)(-1)-(-1)\left(\frac{1}{y+1}\right)^2 = -h''(x)+\left(\frac{1}{y+1}\right)^2 > 0.$$

E' forse piu' semplice riscrivere il sistema come segue

$$\frac{1}{a-x+1}-h'(x)=0$$

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27 gennaio 2009

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Esercizio 1. (10)
Dato il sistema lineare

$$\begin{cases} a\,x - 3\,x + y + z = a - 3 \\ 3\,x - y = 2 \\ x - a\,y - 3\, - y = a - 5 + z \end{cases}$$

dopo averlo riportato in forma normale, discuterlo al variare del parametro $a \in \mathbb{R}$. Dire se esistono valori del parametro per cui il sistema ha infinite soluzioni e, in questo caso, esplicitarle.

Esercizio 2. (8)
Siano dati $X \subseteq \mathbb{R}^n$ aperto e convesso, le funzioni $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ e la funzunione $h : \mathbb{R} \to \mathbb{R}$ strettamente crescente.
Sia P il problema

$$\max_{x \in X} f(x) \quad s.t. \quad g(x) \geq 0$$

e Q il problema

$$\max_{x \in X} h(f(x)) \quad s.t. \quad g(x) \geq 0$$

Si dimostri che

$$\arg\max(P) = \arg\max(Q)$$

(Suggerimento: si scriva la definizione di $\arg\max$; $\Rightarrow$ e' ovvio; per dimostrare $\Leftarrow$ si proceda per assurdo).

Esercizio 3. (12)
Si dimostri la verita' o falsita' delle seguenti affermazioni:
a. Esistono insiemi chiusi e convessi e illimitati;
b. Unione di insiemi convessi e' convessa;
c. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -|x|$ e' una funzione concava.

Esercizio 4. (30)
1. Si discuta il seguente problema di massimo:
Per dati $\pi \in (0,1)\,, w_1, w_2 \in \mathbb{R}_{++}$,

$$\max_{(x,y,m) \in \mathbb{R}^2_{++} \times \mathbb{R}} \; \pi \log x + (1-\pi) \log y \qquad s.t \qquad w_1 - m - x \geq 0, \quad w_2 + m - y \geq 0$$

Si tralasci il problema di esistenza di soluzione; e' comunque possibile dimostrare che esiste una soluzione.

2. Si diano condizioni sui parametri che implicano che tutti i vincoli sono attivi e che i due moltiplicatori siano uguali.

3. Usando quando detto al punto 2, si calcoli l' effetto di una variazione di w_1 sulla soluzione x^* del problema.

Traccia delle Soluzioni.
Esercizio 1.

$$A=\begin{bmatrix} k & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & k \end{bmatrix}, b=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\det A=-\det\begin{bmatrix} -1 & 1 \\ -1 & k \end{bmatrix}+\det\begin{bmatrix} k & 1 \\ 1 & k \end{bmatrix}=-(-k+1)+k^2-1=k^2+k-2=0$$

Le soluzioni sono: $k=\frac{-1\pm\sqrt{1+8}}{2}$, cioe', -2 e 1.
Dunque per $k\neq-2,1$, il sistema ammette una unica soluzione.
Per $k=-2$,

$$[A|b]=\begin{bmatrix} -2 & -1 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 1 & -1 & -2 & | & 1 \end{bmatrix}$$

Dunque, poiche' $\det\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\neq 0$, $rango A=2$. Inoltre

$$rango\,[A|b]=rango\begin{bmatrix} -2 & -1 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}=3$$

Dunque il sistema non ammette soluzioni.
Per $k=1$,come sopra $rango A=2$ e

$$rango\,[A|b]=rango\begin{bmatrix} -2 & -1 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}=3$$

Dunque il sistema non ammette soluzioni.
Esercizio 2. (10)
Data la funzione

$$l:\mathbb{R}^3\to\mathbb{R}^3,\quad l(x_1,x_2,x_3)=\begin{pmatrix} x_1, & -x_2, & x_1 \end{pmatrix}$$

a. si verifichi che l e' lineare;
...
b. si calcoli la matrice A associata a l;

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

c. si descriva $\ker l$ e $\operatorname{Im} l$.
rank $A=2$ e percio' $\dim\ker l=1$. In effetti,

$$\ker l=\left\{x\in\mathbb{R}^3: x_1=x_2=0\right\}$$

e

$$\operatorname{Im} l=\left\{x\in\mathbb{R}^3: x_1=x_3\right\}$$

Esercizio 3. (12)
Sia $\{K_i\}_{i=1}^n$ una famiglia di compatti.
a. Unione finita di compatti e' compatta: VERO.
Unione di chiusi e' chiusa; $\forall i$, $\exists a_i', a_i''\in\mathbb{R}$ tale che $\forall x_i\in K_i$

$$a_i'<x_i<a_i''$$

Si prenda $a^*=\min_i\{a_i'\}$ e $a^{**}=\max_i\{a_i'\}$. ...
b. Intersezione finita di compatti e' compatta. VERO.
Intersezione finita di chiusi e' chiusa e $\forall i$, $\cap_{j=1}^n K_j\subseteq K_i$. ...

Sia $\{K_\alpha\}_{\alpha\in A}$ una famiglia infinita di compatti. Falso
c. Unione infinita di compatti e' compatta (suggerimento: $[0,1]\cup[1,2]\cup...\cup[n,n+1]\cup...$).
L'insieme suggerito e' l'unione infinita di compatti, ma tale unione e' $\mathbb{R}_+$ che non e' limitato.
b. Intersezione infinita di compatti e' compatta.
Intersezione infinita di chiusi e' chiusa e $\forall i,\ \cap_{j=1}^{n} K_j \subseteq K_i$. ...
Esercizio 4.
1. OK
2. Esistenza ...
3. Forma canonica.

$$\begin{array}{lll} \max_{(x,y,m)\in\mathbb{R}^2_{++}\times\mathbb{R}} & \pi\log x+(1-\pi)\log y & s.t \\ & & w_1-m-x\geq 0 \quad \lambda_x \\ & & w_2+m-y\geq 0 \quad \lambda_y \end{array}$$

2. Unicita'.
La funzione obiettivo e' strettamente concava ... e i vincoli sono lineari.
3. Necessity of Kuhn-Tucker conditions.
Le funzioni vincolo sono lineari o concave. Si scelga $(x,y,m)^{++}=\left(\frac{w_1}{2},\frac{w_2}{2},0\right)$.
4. Sufficiency of Kuhn-Tucker conditions.
OK
5. Condizioni di Kuhn-Tucker.

$$\begin{array}{ll} D_xL=0\Rightarrow & \frac{\pi}{x}-\lambda_x=0 \\ D_yL=0\Rightarrow & \frac{1-\pi}{y}-\lambda_y=0 \\ D_mL=0\Rightarrow & -\lambda_x+\lambda_y=0 \\ & \min... \\ & \min... \end{array}$$

6. Soluzioni.
I vincoli sono chiaramente attivi. $\lambda_x=\lambda_y\equiv\lambda>0$.
$\left\{\begin{array}{l} w_1-m-x=0 \\ w_2+m-y=0 \end{array}\right.$
$\left\{\begin{array}{c} w_1-m-\frac{\pi}{\lambda}=0 \\ w_2+m-\frac{1-\pi}{\lambda}=0 \end{array}\right.$
$\left\{\begin{array}{c} w_1-\frac{\pi}{\lambda}=-w_2+\frac{1-\pi}{\lambda} \\ ... \end{array}\right.$
$w_1-\frac{\pi}{\lambda}=-w_2+\frac{1-\pi}{\lambda}$.
$\lambda=\frac{1}{w_1+w_2}$
$m=w_1-\pi(w_1+w_2)$
$x=\pi(w_1+w_2)$
$y=(1-\pi)(w_1+w_2)$
Il calcolo delle derivate e' immediato.
Applicazione del Teorema della funzione implicita (e' sufficiente impostare i calcoli).
Sia F la funzione definita dal lato di sinistra delle condizioni di K-T del problema:
$\left\{\begin{array}{l} \frac{\pi}{x}-\lambda_x=0 \\ \frac{1-\pi}{y}-\lambda_y=0 \\ -\lambda_x+\lambda_y=0 \\ w_1-m-x=0 \\ w_2+m-y=0 \end{array}\right.$
Il risultato voluto si ottiene dalla seguente espressione:

$$D_{(\pi,w_1,w_2)}(x^*,y^*,m^*)=\left[D_{(x,y,m)}F(x,y,m,\pi,w_1,w_2)\right]^{-1}D_{(\pi,w_1,w_2)}F(x,y,m,\pi,w_1,w_2)$$

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Esercizio 1. (10)
Si discuta il seguente sistema al variare del parametro $a \in \mathbb{R}$:

$$\begin{bmatrix} 1 & 0 & -a \\ 1-a & 0 & a \\ 1 & a & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 1 \\ 0 \end{bmatrix}$$

Esercizio 2. (8)
Si consideri la funzione $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & se \quad x \neq 0 \\ \\ 0 & se \quad x = 0 \end{cases}$$

Si verifichi per tale funzione che e' falsa la seguente affermazione

per ogni insieme A aperto, si ha che $f^{-1}(A)$ e' aperto

(il che dimostra che f non e' continua).
Esercizio 3. (12)
Dato $n \in \mathbb{N} \backslash \{0\}$, e la funzione

$$l : \mathbb{R}^n \to \mathbb{R}^{n-2}, \quad l\left((x_i)_{i=1}^n\right) = \begin{pmatrix} x_1, & \ldots & x_{n-3}, & \sum_{i=1}^n x_i \end{pmatrix}$$

a. si verifichi che l e' lineare;
b. si calcoli la matrice A associata a l;
c. si dia una base di $\ker l$ e $\operatorname{Im} l$.
Suggerimento. Si usi il seguente fatto: per trovare una base di $\operatorname{Im} l$, e' sufficiente considerare un sottoinsieme di vettori linearmente indipendenti dei vettori colonna di A, sottoinsieme di cardinalita' uguale alla $\dim \operatorname{Im} l$.

Esercizio 4. (30)
1. Si discuta il seguente problema di massimo:
Per dato $a \in \mathbb{R}_{++}$,

$$\max_{(x,y)\in\mathbb{R}^2_{++}} \quad \ln x + \ln y \qquad s.t. \qquad \begin{array}{ll} -\frac{x}{10} - \frac{y}{60} + a \geq 0 & \lambda_1 \\ -\frac{x}{30} - \frac{y}{30} + a \geq 0 & \lambda_2 \\ x - a \geq 0 & \mu_x \\ y - a \geq 0 & \mu_y \end{array}$$

2. Si calcoli l' effetto di una variazione di a sulla soluzione x^* del problema.

<u>Traccia</u> delle Soluzioni di alcuni esercizi.
Esercizio 1.
Si discuta il seguente sistema al variare del parametro $a \in \mathbb{R}$:

$$\begin{bmatrix} 1 & 0 & -a \\ 1-a & 0 & a \\ 1 & a & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 1 \\ 0 \end{bmatrix}$$

Soluzione.

$$[A(a)\,|b(a)] = \begin{bmatrix} 1 & 0 & -a & 0 \\ 1-a & 0 & a & a \\ 1 & a & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\det[A(a)\,|b(a)] = \det \begin{bmatrix} 1 & 0 & -a & 0 \\ 1-a & 0 & a & a \\ 0 & a & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} =$$

$$\det \begin{bmatrix} 1 & 0 & -a & 0 \\ 1-a & 0 & a & a \\ 0 & a & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = -a\det \begin{bmatrix} 1 & -a & 0 \\ 1-a & a & a \\ 1 & -1 & 0 \end{bmatrix} = a^2 \det \begin{bmatrix} 1 & -a \\ 1 & -1 \end{bmatrix} =$$

$$= a^2(-1+a).$$

Dunque, il sistema non ha soluzioni per $a \in \mathbb{R}\backslash\{0,1\}$.
Caso $a = 0$.

$$rango \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = rango \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} =$$

$$= rango \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = 3 \tag{1}$$

$$rango \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} = rango \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} = 2$$

e il sistema non ha soluzioni.

Ovviamente si poteva concludere che il sistema non ha soluzioni dal fatto che dalla seconda riga della matrice in (2) si ha che il sistema e' equivalente a un sistema contenente la equazione $0 = 1$.

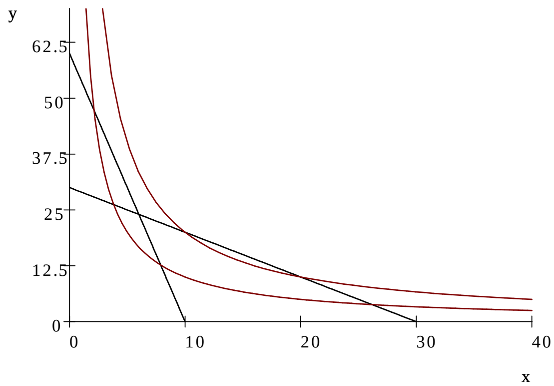
Caso $a = 1$.

$$rango \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = rango \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = 3$$

$$rango \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = 3$$

e dunque il sistema ammette un'unica soluzione.
...
Esercizio 4.
Analizziamo solo il caso $a = 1$.

$$\max_{(x,y)\in\mathbb{R}^2_{++}} \quad \ln x+\ln y \quad s.t. \quad \begin{array}{ll} -\frac{x}{10}-\frac{y}{60}+1\geq 0 & \lambda_1 \\ -\frac{x}{30}-\frac{y}{30}+1\geq 0 & \lambda_2 \\ x-1\geq 0 & \mu_x \\ y-1\geq 0 & \mu_y \end{array}$$



1. Esistenza.

...

2. Unicita'.

L'insieme vincolato e' convesso. La matrice Hessiana della funzione obiettivo. e'

$$\begin{array}{cc} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{array}$$

e dunque, poiche' l'Hessiano e' negativo definito, la funzione obiettivo e' strettamente concava. Dunque la soluzione e' unica.

3. Forma canonica.

$$\max_{(x,y)\in\mathbb{R}^2_{++}} \quad \ln x+\ln y \quad s.t. \quad \begin{array}{ll} -\frac{x}{10}-\frac{y}{60}+1\geq 0 & \lambda_1 \\ -\frac{x}{30}-\frac{y}{30}+1\geq 0 & \lambda_2 \\ x-1\geq 0 & \mu_x \\ y-1\geq 0 & \mu_y \end{array}$$

4. Necessita' delle condizioni di Kuhn-Tucker.

I vincoli sono lineari e dunque pseudo-concavi. Si prenda $(x^{++},y^{++})=(2,2)$:

$-\frac{2}{10}-\frac{2}{60}+1=\frac{23}{30}$

$-\frac{2}{30}-\frac{2}{30}+1=\frac{13}{15}$

5. Sufficienza delle condizioni di Kuhn-Tucker.

La funzione obbiettivo e' strettamente concava e dunque pseudo-concava. I vincoli sono lineari e dunque quasi-concavi.

6. Funzione Lagrangiana e Condizioni di Kuhn-Tucker.

$L\left(x,y,\lambda_1,\lambda_2,\mu_x,\mu_y\right)=\ln x+\ln y+\lambda_1\cdot\left(-\frac{x}{10}-\frac{y}{60}+1\right)+$

$+\lambda_2\cdot\left(-\frac{x}{30}-\frac{y}{30}+1\right)+\mu_x\left(x-1\right)+\mu_y\left(y-1\right)$

$D_x\mathcal{L}=0\Rightarrow \frac{1}{x}-\frac{1}{10}\lambda_1-\frac{1}{30}\lambda_2+\mu_x=0$

$D_y\mathcal{L}=0\Rightarrow \frac{1}{y}-\frac{1}{60}\lambda_1-\frac{1}{30}\lambda_2+\mu_y=0$

$\min\left\{-\frac{x}{10}-\frac{y}{60}+1,\lambda_1\right\}=0$

$\min\left\{-\frac{x}{30}-\frac{y}{30}+1,\lambda_2\right\}=0$

$\min\left\{x-1,\mu_x\right\}=0$

$\min\left\{y-1,\mu_y\right\}=0.$

Sulla positivita' dei moltiplicatori. Si congetturi che $\mu_x=\mu_y=0$, $\lambda_1,\lambda_2>0$.

$D_x\mathcal{L}=0\Rightarrow \frac{1}{x}-\frac{1}{10}\lambda_1-\frac{1}{30}\lambda_2=0$

$D_y\mathcal{L}=0\Rightarrow \frac{1}{y}-\frac{1}{60}\lambda_1-\frac{1}{30}\lambda_2=0$

$-\frac{x}{10}-\frac{y}{60}+1=0$

$-\frac{x}{30}-\frac{y}{30}+1=0$

$$\begin{cases} -\frac{x}{10} - \frac{y}{60} + 1 = 0 \\ -\frac{x}{30} - \frac{y}{30} + 1 = 0 \end{cases}, \text{ Solution is : } \{y = 24, x = 6\}$$

$$\begin{cases} \frac{1}{6} - \frac{1}{10}\lambda_1 - \frac{1}{30}\lambda_2 = 0 \\ \frac{1}{24} - \frac{1}{60}\lambda_1 - \frac{1}{30}\lambda_2 = 0 \end{cases}, \text{ Solution is : } \left\{\lambda_2 = \frac{1}{2}, \lambda_1 = \frac{3}{2}\right\}.$$

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Esercizio 1. (10)
Si discuta il sistema
$$A(k) \cdot x = b(k)$$
al variare del parametro $k \in \mathbb{R}$, dove
$$A(k) \equiv \begin{bmatrix} 1 & 0 & -k \\ 1-k & 0 & k \\ 1 & k & -1 \\ 1 & 0 & -1 \end{bmatrix}, \qquad b(k) \equiv \begin{bmatrix} 0 \\ k \\ 1 \\ 0 \end{bmatrix}$$

Esercizio 2. (10)
Si discuta la verita' o falsita' delle seguenti affermazioni:
1. Un insieme chiuso e convesso e' compatto.
2. $\{x \in \mathbb{R} : 0 \le x < 1\}$ e' un insieme convesso.
3. La seguente funzione e' concava: $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) = \begin{cases} 0 & \text{se} \quad x \le 0 \\ \\ x & \text{se} \quad x > 0 \end{cases}$$

Esercizio 3. (8)
Dato $n \in \mathbb{N} \backslash \{0\}$, $x = (x_i)_{i=1}^n$ e la funzione
$$f : \mathbb{R}^n \to \mathbb{R}^{n+2}, \quad f(x) = \begin{pmatrix} x \\ \sum_{i=1}^n x_i \\ x_1 + x_2 - \sum_{i=3}^n x_i \end{pmatrix}$$
a. si verifichi che l e' lineare;
b. si calcoli la matrice A associata a l;
c. si dia una base di $\ker l$ e $\operatorname{Im} l$.

Esercizio 4. (30)
1. Si discuta il seguente problema di massimo:
Per dato $k \in (1, +\infty)$,
$$\max_{(x,y,z) \in \left(-\frac{1}{4}, +\infty\right)^3} \ln(1+x) + \ln(2+x+y) + \ln(1+z) \qquad s.t. \qquad \begin{array}{l} x^2 + y^2 + z^2 \le 1 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{array}$$
Non si calcolino le eventuali soluzioni (x^*, y^*, z^*). Si dica solo se e' possibile averne una in cui $x^* = 0$.
2. Si calcoli l' effetto di una variazione di k sulla soluzione x^* del problema.

———————————————————————

Esercizio 3.
$$[A(k) | b(k)] = \begin{bmatrix} 1 & 0 & -k & 0 \\ 1-k & 0 & k & k \\ 1 & k & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\det\left[A\left(k\right)|b\left(k\right)\right]=\det\begin{bmatrix}1 & 0 & -k & 0\\ 1-k & 0 & k & k\\ 0 & k & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}=$$

$$\det\begin{bmatrix}1 & 0 & -k & 0\\ 1-k & 0 & k & k\\ 0 & k & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}=-k\det\begin{bmatrix}1 & -k & 0\\ 1-k & k & k\\ 1 & -1 & 0\end{bmatrix}=k^2\det\begin{bmatrix}1 & -k\\ 1 & -1\end{bmatrix}=$$

$$=k^2\left(-1+k\right).$$

Dunque, il sistema non ha soluzioni per $k\in\mathbb{R}\setminus\{0,1\}$.
Caso $k=0$.

$$rango\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}=rango\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}=$$

$$=rango\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}=3 \tag{2}$$

$$rango\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & -1\end{bmatrix}=rango\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & -1\end{bmatrix}=2$$

e il sistema non ha soluzioni.

Ovviamente si poteva concludere che il sistema non ha soluzioni dal fatto che dalla seconda riga della matrice in (2) si ha che il sistema e' equivalente a un sistema contenente la equazione $0=1$.

Caso $k=1$.

$$rango\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}=rango\begin{bmatrix}0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}=3$$

$$rango\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\end{bmatrix}=3$$

e dunque il sistema ammette un'unica soluzione.

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Esercizio 1. (10)
Si discuta il seguente sistema al variare del parametro $a \in \mathbb{R}$:

$$\begin{cases} x - ay + z = a \\ 2x + y + az = 0 \\ 3x - y + 3z = a \end{cases}$$

Esercizio 2. (10)
Si discuta la verita' o falsita' delle seguenti affermazioni:
1. 1 e' un punto di accumulazione per l'insieme

$$\left\{ x \in \mathbb{R} : \exists n \in \mathbb{N} \text{ tale che } x = 1 + (-1)^n \frac{1}{3^n} \right\}$$

2. $\{x \in \mathbb{R} : x \geq 0\}$ e' un insieme convesso, chiuso e limitato.
3. La seguente funzione e' concava: $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{se} \quad x \leq 0 \\ \\ 1 + x & \text{se} \quad x > 0 \end{cases}$$

Esercizio 3. (8)
Data la funzione

$$l : \mathbb{R}^4 \to \mathbb{R}^4, \quad l(x_1, x_2, x_3, x_4) = \begin{pmatrix} -x_1, & +x_2, & -x_3 & +x_4 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si descriva $\ker l$ e $\operatorname{Im} l$.

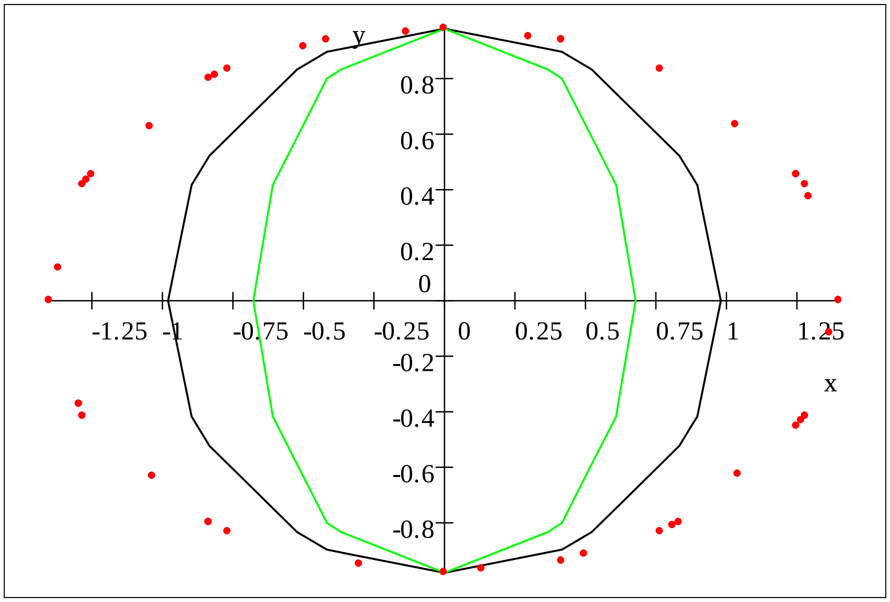
Esercizio 4. (30)
1. Si discuta il seguente problema di massimo:
Per dati $a, b, c \in (0, +\infty)$,

$$\min_{(x,y)\in\mathbb{R}^2} \quad ax^2 + y^2 \qquad s.t. \qquad \begin{array}{l} y \geq \frac{2}{b}x + 2 \\ y \geq -\frac{2}{c}x + 2 \end{array}$$

Non si calcolino le eventuali soluzioni (x^*, y^*).

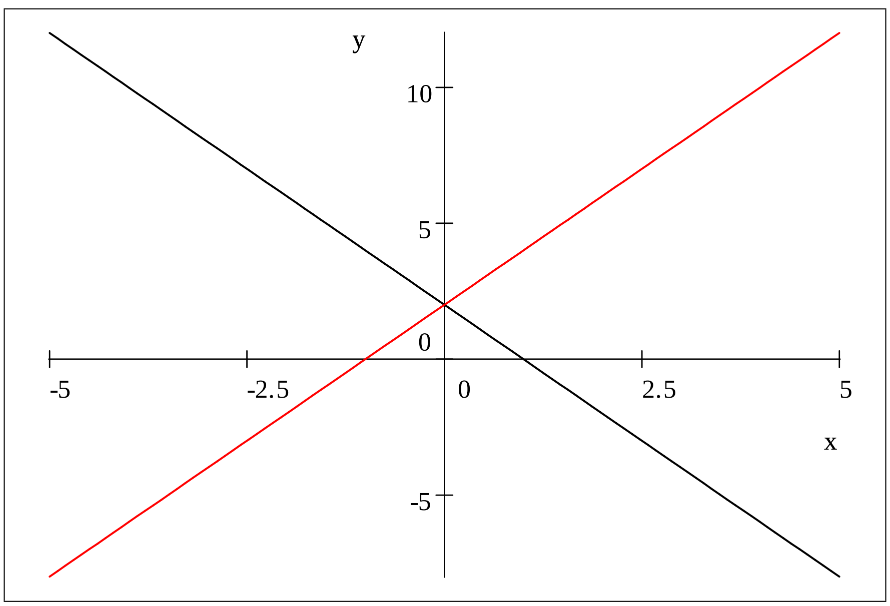
2. Usando il teorema della funzione implicita, si calcoli l' effetto di una variazione di b sulla soluzione x^* del problema, assumendo che il moltiplicatore associato al primo vincolo sia strettamente positivo e che il seconodo vincolo valga in forma di disuguaglianza stretta.

$x^2 + y^2 = 1$



y
0.8
0.6
0.4
0.2
0
-1.25
-1
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
1
1.25
-0.2
-0.4
-0.6
-0.8
x

$-2x + 2$



y
10
5
0
-5
-2.5
0
2.5
5
x
-5

Compito di
Matematica per le Applicazioni Economiche 2
9 giugno 2009

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (10)
Dato il sistema lineare

$$\begin{cases} a\,x - 3\,x + y + z + 3 = a \\ 3\,x - y = 2 \\ x - a\,y + 3\,y - z + 5 = a \end{cases}$$

dopo averlo riportato in forma normale, discuterlo al variare del parametro $a \in R$. Dire se esistono valori del parametro per cui il sistema ha infinite soluzioni e, in questo caso, esplicitarle.

Esercizio 2. (10)
Si **dimostri** la verita' o falsita' delle seguenti affermazioni:
1. 1 e' un punto di accumulazione, interno e di frontiera per $\mathbb{N}$.
2. Se un insieme e' chiuso, allora e' limitato.
3. Il seguente insieme e' chiuso

$$\left\{ x \equiv (x_i)_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n (x_i)^2 \geq 1 \right\}$$

Esercizio 3. (10)
Data la funzione

$$l : \mathbb{R}^4 \to \mathbb{R}^3, \quad l\,(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + x_2 & x_1 + x_2 + x_3 & x_1 + x_2 + x_3 + x_4 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si descriva $\ker l$ e $\operatorname{Im} l$.

Esercizio 4. (30)
1. Si discuta il seguente problema:

$$\min_{(x,y)\in\mathbb{R}^2} \quad px + y \qquad s.t. \qquad u\,(x) + v\,(y) \geq a$$

dove $p, a \in (0, +\infty)$, $u, v : \mathbb{R} \to \mathbb{R}$, $\forall, x, y \in \mathbb{R}$, $u'(x) > 0, u''(x) < 0, v'(y) > 0, v''(y) < 0$. Per semplicita', si assuma l'esistenza di soluzione.

2. Usando il teorema della funzione implicita, si calcoli l' effetto di una variazione di p sulla soluzione x^* del problema, assumendo che il moltiplicatore associato al primo vincolo sia strettamente positivo.

Compito di
Matematica per le Applicazioni Economiche 2
25 giugno 2009

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (10)
Dato il sistema lineare

$$\begin{cases} x + y + 2\,z\beta - 2\,z = 6 - 4\,\beta \\ 2\,x\beta - 2\,x + 3\,y + z\beta - z = 3 \\ -x + 2\,z = -5 \end{cases}$$

dopo averlo riportato in forma normale, discuterlo al variare del parametro $\beta \in \mathbb{R}$. Dire se esistono valori del parametro per cui il sistema ha infinite soluzioni e, in questo caso, esplicitarle.

Esercizio 2. (10)
Si **dimostri** la verita' o falsita' delle seguenti affermazioni:
1. Se x e' un punto interno di $A \subseteq \mathbb{R}$, allora x e' un punto di accumulazione per A .
2. Se un intervallo e' aperto e limitato, allora ha frontiera non vuota.
3. Data la funzione $f : \mathbb{R} \to \mathbb{R}$, il seguente insieme e' aperto

$$\{x \equiv \mathbb{R} : f(x) > 1\}$$

Esercizio 3. (10)
Data la funzione

$$l : \mathbb{R}^3 \to \mathbb{R}^4, \quad l(x_1, x_2, x_3) = \begin{pmatrix} x_1 - x_2 - x_3 & -x_1 + x_2 + x_3 & x_1 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si descriva $\ker l$ e $\operatorname{Im} l$.

Esercizio 4. (30)
1. Si discuta il seguente problema:

$$\max_{(x,y)\in\mathbb{R}^2} \quad u(x) + v(y) \qquad s.t. \qquad \begin{array}{l} x + y \geq a \\ x \geq 0 \\ y \geq 0 \end{array}$$

dove $a \in (0, +\infty)$, $u, v : \mathbb{R} \to \mathbb{R}$, $\forall, x, y \in \mathbb{R}$, $u'(x) > 0, u''(x) < 0, v'(y) > 0, v''(y) < 0$.

2. Usando il teorema della funzione implicita, si calcoli l' effetto di una variazione di a sulla soluzione (x^*, y^*) del problema, assumendo che i moltiplicatore associati al secondo e terzo vincolo siano strettamente positivi.

Compito di
Matematica per le Applicazioni Economiche 2
10 luglio 2009

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (10)
Dire quale condizione devono verificare i parametri $a, b \in \mathbb{R}$ affinchè il sistema

$$\begin{cases} x + y & = b + z \\ -x + 4y + z & = a - y \\ -x + 17y + z & = a - b \end{cases}$$

sia compatibile (ossia ammetta almeno una soluzione). Specificare un caso in cui tale condizione è soddisfatta e determinare le soluzioni esplicitamente.

Esercizio 2. (10)
Si **dimostri** la verita' o falsita' delle seguenti affermazioni:
1. Se x e' un punto isolato di $A \subseteq \mathbb{R}$, allora x e' un punto di frontiera per A .
2. Se un insieme contiene infiniti punti, allora contiene almeno un punto di accumulazione.
3. Data la funzione continua $f : \mathbb{R} \to \mathbb{R}$, il seguente insieme e' chiuo

$$\left\{ x \equiv \mathbb{R} : f(x) + x^2 \geq 1 \right\}$$

Esercizio 3. (10)
Sia $n \in \mathbb{N}$, $n \geq 2$. Data la funzione

$$l : \mathbb{R}^n \to \mathbb{R}^3, \quad l\left((x_i)_{i=1}^n\right) = \begin{pmatrix} \sum_{i=1}^n x_i & x_1 & x_2 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si descriva $\ker l$ e $\operatorname{Im} l$.

Esercizio 4. (30)
1. Si discuta il seguente problema:

$$\max_{(x,y) \in \mathbb{R}^2} \quad px + qy \qquad s.t. \qquad G(x, y) \geq 0$$

dove $p, q \in (0, +\infty)$ e $G : \mathbb{R}^2 \to \mathbb{R}$, G e' $C^2, \forall (x, y) \in \mathbb{R}^2$, $DG(x, y) < 0$, $D^2G(x, y)$ e' negativa definita. Si tralasci il problema di esistenza

2. Usando il teorema della funzione implicita, si calcoli l' effetto di una variazione di q sulla soluzione (x^*, y^*) del problema.

Compito di
Matematica per le Applicazioni Economiche 2
10 settembre 2009

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (10)
Discutere il il sistema lineare per qualsiasi valore reale del parametro β:

$$\begin{cases} x + y - z = \beta - 1 \\ x - 2\,y + z\beta - z = 0 \\ 3\,x + y = 2 \end{cases}$$

Esercizio 2. (10)
Si **dimostri** la verita' o falsita' delle seguenti affermazioni:
1. Se x e' un punto interno di $A \subseteq \mathbb{R}$, allora x e' un punto di frontiera per A .
2. Se un sottoinsieme di $\mathbb{R}$ e' inferiormente e superiormente limitato, allora a. e' chiuso e b. e' convesso
3. Data la funzione $f : \mathbb{R} \to \mathbb{R}$, il seguente insieme e' chiuso

$$\{x \equiv \mathbb{R} : f(x) \geq 1\}$$

Esercizio 3. (10)
Data la funzione

$$l : \mathbb{R}^3 \to \mathbb{R}^2, \quad l(x_1, x_2, x_3) = \begin{pmatrix} x_1 + x_2 + x_3 & -x_1 - x_2 - x_3 \end{pmatrix}$$

a. si verifichi che l e' lineare,;
b. si calcoli la matrice A associata a l;
c. si descriva $\ker l$ e $\operatorname{Im} l$.
d. Si dica se

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad l(x, y) = \begin{pmatrix} x + y & x^2 \end{pmatrix}$$

e' lineare.

Esercizio 4. (30)
1. Si discuta il seguente problema:

$$\begin{array}{ll} \max_{(x,y,z) \in \mathbb{R}^3_{++}} & \ln x + a \ln y \\ & s.t. \\ & x + z \leq b \\ & y \leq f(z) \end{array}$$

dove $a \in (0, 1)$, $b \in \mathbb{R}_{++}$. Si tralasci il problema di esistenza di soluzione.
2. Usando il teorema della funzione implicita, si calcoli l' effetto di una variazione di b sulla soluzione (x^*, y^*, z^*) del problema, assumendo che i moltiplicatori associati al primo e secondo vincolo siano strettamente positivi.

Compito di
Matematica per le Applicazioni Economiche 2
14 dicembre 2009
(programma anno 2009-2010)
FAC-SIMILE

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (6) (Esercizio 18, capitolo2)
Show that
a.

$$V = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\right\}$$

is a vector subspace of $\mathbb{R}^3$;
b.

$$S = \{(1, -1, 0), (0, 1, -1)\}$$

is a basis for V.

Esercizio 2. (6) (Ultimo esempio del Capitolo 6)
Si consideri il sistema $Ax = b$, dove

$$A = \begin{bmatrix} k - \sqrt{3} & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -\sqrt{3} \\ -k \\ 0 \end{bmatrix},$$

e x e' il vettore di variabili incognite. Si dica per quali valori di $k \in \mathbb{R}$, il sistema ammette soluzioni.

Esercizio 3. (6) (Esercizio 7, capitolo 7)
Dato

$$S \equiv \left\{x \in \mathbb{R} : \exists n \in \mathbb{N} \backslash \{0\} \text{ tale che } x = \frac{1}{n}\right\} \cup [1, 2] \cup \{3\}$$

si dimostri che
a. 0 non e' un punto interno per S;
b. 0 e' un punto di frontiera per S.

Esercizio 4. (6) (Esercizio 6, capitolo 10)
Compute total derivative and directional derivate at x_0 in the direction u.

$$f : \mathbb{R}^3_{++} \to \mathbb{R}, \qquad f(x_1, x_2, x_3) = \frac{1}{3} \log x_1 + \frac{1}{6} \log x_2 + \frac{1}{2} \log x_3$$

$x_0 = (1, 1, 2)$, $u = \frac{1}{\sqrt{3}}(1, 1, 1)$.

Esercizio 5. (6) (Esercizio 354)
Given the utility function $f : \mathbb{R}^2_{++} \to \mathbb{R}$,

$$f(x, y) = xy$$

compute the Marginal Rate of Substitution in (x_0, y_0).

Compito di
Matematica per le Applicazioni Economiche 2
14 dicembre 2009
(programma anno 2009-2010)

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (6)
Data la funzione lineare

$$l : \mathbb{R}^2 \to \mathbb{R}^4, \quad l_4(x_1, x_2) = \begin{pmatrix} x_1, & x_1 + x_2, & x_1 - x_2, & -x_1 \end{pmatrix}$$

si calcoli $\ker l$ e $\operatorname{Im} l$.

Esercizio 2. (6)
Dato il sistema lineare

$$\begin{cases} 4\,x + z = 1 \\ x + \alpha\, y = 1 \\ \alpha\, x - 3\, y + z = 0 \end{cases}$$

dopo averlo riportato in forma normale, discuterlo al variare del parametro $\alpha \in \mathbb{R}$. Dire se esistono valori del parametro per cui il sistema ha infinite soluzioni e, in questo caso, esplicitarle.

Esercizio 3. (6)
Usando la caratterizzazione di funzione continua in termini di insiemi chiusi, si verifichi che la seguente funzione non e' continua:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{se} \quad x \leq 0 \\ \\ -1 & \text{se} \quad x > 0. \end{cases}$$

Esercizio 4. (6) Siano date le funzioni differenziabili

$$g : \mathbb{R} \to \mathbb{R}, \quad x_1 \mapsto g(x_1)$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = (g(x_1), x_1 + x_2)$$

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \quad h(y_1, y_2) = \left(e^{2y_1}, y_1 \cdot y_2\right)$$

Se possibile, si calcoli il differenziale di $h \circ f$.

Esercizio 5. (6)
Siano date la funzione di utilitita' $u : \mathbb{R} \to \mathbb{R}$, $x \to u(x)$ di classe C^1 e tale che $\forall x \in \mathbb{R}$, $u'(x) > 0$ e la probabilita' $\pi \in (0, 1)$. Si definisca

$$U : \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto \pi \cdot u(x_1) + (1 - \pi) \cdot u(x_2).$$

Si calcoli il saggio marginale di sostituzione in un punto arbitrario.

Compito di
Matematica per le Applicazioni Economiche 2
13 gennaio 2010
(programma anno 2009-2010)

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (6)
Si dica se i seguenti insiemi sono sottospazi di $\mathbb{R}^n$, con $n \in \mathbb{N} \setminus \{0\}$
a.

$$\left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \right\},$$

b.

$$\left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1 \right\},$$

c.

$$\left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{1}{x_i} = 0 \right\},$$

dove $x = (x_i)_{i=1}^n$.

Esercizio 2. (6)
Data la funzione lineare

$$l : \mathbb{R}^n \to \mathbb{R}^{n+2}, \quad l(x) = \begin{pmatrix} x, & 0, & 0 \end{pmatrix}$$

con $n \in \mathbb{N} \setminus \{0\}$ e $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, si calcoli $\ker l$ e $\operatorname{Im} l$.

Esercizio 3. (6)
Discutere al variare di $b \in \mathbb{R}$ il sistema

$$\begin{cases} x - by + z + t & = & 0 \\ x + y - z - t & = & b \\ 3x + y & = & 2 \end{cases}$$

Esercizio 4. (6)
Sia

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 + e^x - \cos y$$

e

$$S = \left\{ (x, y) \in \mathbb{R}^2 : f(x, y) \geq 3 \right\}.$$

Si dimostri che S e' un insieme chiuso.

Esercizio 5. (6)
Siano date le funzioni

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, x_2) = e^x + \sin y$$

$$g : \mathbb{R} \to \mathbb{R}^2, \quad g(z) = \begin{pmatrix} z^2 + z \\ 2\cos z \end{pmatrix}$$

Se possibile si calcoli la derivata direzionale di $h = g \circ f$ nel punto $(0, 0)$ nella direzione $(1, 1)$.

Esercizio 6. (6)
Si consideri la funzione

$$\pi : \mathbb{R}_{++} \to \mathbb{R}, \qquad \pi(x) = pf(x) - wx$$

con $f : \mathbb{R}_{++} \to \mathbb{R}$ di classe C^2 e tale che $\forall x \in \mathbb{R}_{++}$, $f'(x) > 0$ and $f''(x) < 0$,
$p, w \in \mathbb{R}_{++}$ (π e' una funzione di profitto).

Assumendo che il massimo globale di π esista e sia unico, si scriva l'equazione che lo caratterizza. Come varia la soluzione al variare di p? (ovvero, il profitto aumenta se aumenta il prezzo dell'output?).

Compito di
Matematica per le Applicazioni Economiche 2
10 febbraio 2010
(programma anno 2009-2010)

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (6)

Si dica se le seguenti afferamzioni sono vere o false. Si motivino le proprie risposte, anche citando precisamente risultati contenuti negli appunti del corso.

Sia data una funzione lineare $l : \mathbb{R}^m \to \mathbb{R}^n$.

1. Se $m = n$, allora $\dim \ker l = 0 \Leftrightarrow \dim \operatorname{Im} l = n$.
2. Se $m > n$, allora $\dim \ker l > 0$.
3. Se $m = n$ e $\dim \operatorname{Im} l = n - 2$, allora $\forall b \in \mathbb{R}^n$, $\exists x^* \in \mathbb{R}^m$ tale che $l(x^*) = b$.

Esercizio 2. (6)

Siano dati gli spazi vettoriali U e V. Si dica se le seguenti funzioni sono lineari e in tal caso se ne calcoli ker e Im. Si motivino le proprie risposte, anche citando precisamente risultati contenuti negli appunti del corso.

1. $l_1 : U \to V$, $\forall u \in U$, $l_1(u) = v^*$, dove v^* e'un fissato e diverso da zero vettore di V.
2. $l_2 : U \to V$, $\forall u \in U$, $l_2(u) = 0$.
3. $l_2 : U \to U$, $\forall u \in U$, $l_3(u) = u$.

Esercizio 3. (6)

Discutere al variare di $a \in \mathbb{R}$ il sistema

$$\begin{bmatrix} a & 1-a & a-2 & -a \\ -a & a & 2a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2a+1 \\ 3a+1 \end{bmatrix}$$

Esercizio 4. (6)

Dato l'insieme

$$S = \cap_{n \in \mathbb{N} \setminus \{0\}} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right),$$

si dica se:

1. $0 \in S$,
2. $0 \in Int\ S$,
3. 1 e' un punto di accumulazione di S.

Esercizio 5. (6)

Siano date le funzioni

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = \begin{pmatrix} e^x + y \\ e^y + x \end{pmatrix}$$

$$g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto g(x)$$

$$h : \mathbb{R} \to \mathbb{R}^2, \qquad h(x) = f(x, g(x))$$

Si assume che g e' di classe C^2.

a. Si dica cosa e' il differenziale di h;

b.se possibile, si calcoli il differenziale di h nel punto 0;

c.se possibile si calcoli la derivata direzionale di nel punto 0 nella direzione 1.

Esercizio 6. (6)

Si consideri la funzione

$$\pi : \mathbb{R}_{++} \to \mathbb{R}, \qquad \pi(y) = py - c(y)$$

con $c : \mathbb{R}_{++} \to \mathbb{R}$ di classe C^2 e tale che $\forall y \in \mathbb{R}_{++}$, $c'(y) > 0$ and $c''(y) > 0$, e $p \in \mathbb{R}_{++}$ (π e' una funzione di profitto e c e' una funzione di costo).

Assumendo che il massimo globale di π esista e sia unico, si scriva l'equazione che lo caratterizza e si spieghi perche' tale equazione ha come soluzione il massimo globale. Come varia la soluzione al variare di p?

Compito di
Matematica per le Applicazioni Economiche 2
15 giugno 2010

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (6)

Si dica se le seguenti afferamzioni sono vere o false. Si motivino le proprie risposte, anche citando precisamente risultati contenuti negli appunti del corso.

Sia data una funzione lineare $l : \mathbb{R}^m \to \mathbb{R}^n$.

1. Se $m = n$, allora $\dim \ker l = 0 \Leftrightarrow \dim \operatorname{Im} l = n$.
2. Se $m > n$, allora $\dim \ker l > 0$.
3. Se $m = n$ e $\dim \operatorname{Im} l = n - 2$, allora $\forall b \in \mathbb{R}^n$, $\exists x^* \in \mathbb{R}^m$ tale che $l(x^*) = b$.

Esercizio 2. (6)

Siano dati gli spazi vettoriali U e V. Si dica se le seguenti funzioni sono lineari e in tal caso se ne calcoli ker e Im. Si motivino le proprie risposte, anche citando precisamente risultati contenuti negli appunti del corso.

1. $l_1 : U \to V$, $\forall u \in U$, $l_1(u) = v^*$, dove v^* e'un fissato e diverso da zero vettore di V.
2. $l_2 : U \to V$, $\forall u \in U$, $l_2(u) = 0$.
3. $l_2 : U \to U$, $\forall u \in U$, $l_3(u) = u$.

Esercizio 3. (6)

Si dica per quali valori di $a \in \mathbb{R}$, il seguente sistema ammette una, nessuna o infinite soluzioni:

$$\begin{bmatrix} a & 1-a & a-2 & -a \\ -a & a & 2a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2a+1 \\ 3a+1 \end{bmatrix}$$

Esercizio 4. (6)

Dato l'insieme

$$S = \cap_{n \in \mathbb{N} \setminus \{0\}} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right),$$

si dica se:

1. $0 \in S$,
2. $0 \in Int\ S$,
3. 1 e' un punto di accumulazione di S.

Esercizio 5. (6)
Siano date le funzioni

$$f:\mathbb{R}^2\to\mathbb{R}^2,\qquad f(x,y)=\begin{pmatrix} e^x+y\\ e^y+x\end{pmatrix}$$

$$g:\mathbb{R}\to\mathbb{R},\quad x\mapsto g(x)$$

$$h:\mathbb{R}\to\mathbb{R}^2,\qquad h(x)=f(x,g(x))$$

Si assuma che g e' di classe C^2.
a. Si scriva h in forma espicita, cioe', in una forma in cui non appaia la funzione f.
b. Si calcoli il Jacobiano di h usando quando ottenuto al punto a. e usando la formula relativa alle funzioni composte.

Esercizio 6. (6)
Siano date le funzioni, $f,g:\mathbb{R}_{++}\to\mathbb{R}_{++}$ di classe C^2 e tali che $\forall x\in\mathbb{R}_{++}$,

$$f'(x)>0,\quad f''(x)<0,\qquad g'(x)<0,\quad g''(x)<0.$$

Sia inoltre data la funzione

$$h:\mathbb{R}_{++}\to\mathbb{R},\qquad h:x\mapsto(g\circ f)(x).$$

Assumendo che la funzione h ammette massimo globale, si dica se esiste una equazione che lo caratterizza, e in caso affermativo si scriva tale equazione.
Si assuma che tale equazione sia

$$\phi(x,a)=0$$

con ϕ di classe C^2 e tale che per qualsiasi valore di x e a sia $\frac{\partial\phi(x,a)}{\partial x}>0$ e $\frac{\partial\phi(x,a)}{\partial a}<0$. Qual e' l'effetto di una variazione di a sul valore massimo.

Compito di
Matematica per le Applicazioni Economiche 2
8 luglio 2010

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Esercizio 1. (6)
Siano dati $a \in \mathbb{R}$ e la funzione

$$l_a : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x, y, z) \mapsto \begin{pmatrix} ax - ay + z \\ x + (a+1)\,y + (a-1)\,z \end{pmatrix}.$$

Si calcolino gli insiemi dei valori di $a \in \mathbb{R}$ tali che:
a. $\dim \ker l_a = 0$;
b. $\dim \ker l_a = 1$;
b. $\dim \ker l_a = 2$

Esercizio 2. (6)
Sia V lo spazio vettoriale di tutte le funzioni $f : \mathbb{R} \to \mathbb{R}$. Si dica se i seguenti sottoinsiemi di V sono sottospazi vettoriali di V:
1. L'insieme delle funzioni derivabili;
2. l'insieme delle funzioni _non_ continue.

Esercizio 3. (6)
Si discuta il seguente sistema rispetto al parametro $b \in \mathbb{R}$:

$$\begin{cases} x - 2y + bz & = & 2 \\ x - 3y & = & 3 \\ bx - y & = & 1 \end{cases}$$

Esercizio 4. (6)
Siano dati $p_1, p_2 \in \mathbb{R}$, $w \in \mathbb{R}_{++}$. Si dica se il seguente insieme (di bilancio) e' chiuso e/o limitato:

$$\left\{(x_1, x_2) \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq w\right\}$$

Esercizio 5. (6)
Date le funzioni $g, h : \mathbb{R} \to \mathbb{R}_{++}$ di classe C^2, si calcoli la matrice Jacobiana di

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x, y, z) = \left(\tfrac{g(x)}{h(z)}, \qquad g(h(x)) + xy, \qquad \ln(g(x) + h(x))\right)$$

Esercizio 6. (6)
Siano dati (i parametri) $a, b \in \mathbb{R}$ e le seguenti funzioni (rispettivamente, di domanda e di offerta) di classe C^2 :

$$d : \mathbb{R}_{++} \to \mathbb{R}_{++}, \qquad p \mapsto d(p, a),$$

$$s : \mathbb{R}_{++} \to \mathbb{R}_{++}, \qquad p \mapsto s(p, b).$$

Si assuma che $\forall a, b \in \mathbb{R}$, esista una unica soluzione $p^*(a, b)$ della equazione (cioe', un cosidetto prezzo di equilibrio sul mercato)

$$d(p, a) = s(p, b).$$

Si dica sotto quali condizioni e' possibile calcolare l'effetto di una variazione di a sulla unica soluzione $p^*(a, b)$.

Esercizio 1.

$$\dim \ker l_a = 3 - \operatorname{rank} \begin{bmatrix} a & -a & 1 \\ 1 & a+1 & a-1 \end{bmatrix}$$

$\det \begin{bmatrix} a & -a \\ 1 & a+1 \end{bmatrix} = 2a + a^2$.Dunque se $a \in \mathbb{R} \setminus \{-2, 0\}$, rank $[l_a] = 2$.
Se $a = -2$,

$$\operatorname{rank} \begin{bmatrix} -2 & 2 & 1 \\ 1 & -1 & -3 \end{bmatrix} = 2,$$

Se $a = 0$,

$$\operatorname{rank} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 2$$

Dunque $\forall a \in \mathbb{R}$, $\dim \ker [l_a] = 1$.

Compito di
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7 settembre 2010

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Esercizio 1. (6)
Si dica se esistono valori di $a \in \mathbb{R}$ per i quali la seguente funzione lineare

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x_1, x_2, x_3) \mapsto \begin{pmatrix} ax_1 + 2x_2 + x_3 \\ ax_2 + x_3 \end{pmatrix}$$

e' a. iniettiva b. suriettiva c. invertibile.

Esercizio 2. (6)
Si dimostri che $\{x := (x_i)_{i=1}^n \in \mathbb{R} : \forall i \in \{1, ..., n\} .x_i \geq 0\}$ non e' un sottospazio vettoriale di $\mathbb{R}^n$.

Esercizio 3. (6)
Discutere al variare di $a \in \mathbb{R}$ il sistema

$$\begin{cases} x + ay - z = 0 \\ 2x - y - az = a - 2 \\ x - 3y - z = 0 \end{cases}$$

Esercizio 4. (6)
Si dica se esiste una soluzione al sequente problema:
Sia

$$C = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + x \leq 0, \qquad x \geq 0, \qquad y \geq 0\right\}.$$

Si trovi $(x^*, y^*) \in C$ tale che $\forall (x, y) \in C,\ f(x^*, y^*) \geq f(x, y)$, dove f e' la composizione di funzioni continue.

Esercizio 5. (6)
Date le funzioni $g, h : \mathbb{R} \to \mathbb{R}_{++}$ di classe C^2, si calcoli la matrice Jacobiana di

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x, y, z) = \left(e^{f(x)} + y, \qquad g(h(x)) - z\ , \qquad \tfrac{1}{g(y)+h(z)}\right)$$

Esercizio 6. (6)
Si consideri la seguente equazione:

$$I(r, a) \quad = \quad S(r, b)$$

dove $I, S : \mathbb{R}^2 \to \mathbb{R}$ sono funzioni di classe C^2 (I, S, r stanno per Investment, saving e interest rate). Si assuma che per ogni valore di a, b esiste una soluzione r della equazione, e che le derivate parziali di ciascuna funzione rispetto a ciascuna variabile sono diverse da zero. Si dica qual e' l'effetto di una variazione di a sul valore della soluzione.

Compito di
Matematica per le Applicazioni Economiche 2
25 gennaio 2011

Avete due ore e mezzo a disposizione. Il numero accanto a ogni esercizio indica il punteggio ottenibile in caso di risposta corretta. Spiegate con molta cura le vostre risposte.

Exercise 1. (7)
Say if the following statement is true or false.
Let V and U vector spaces on $\mathbb{R}$, W a vector subspace of U and $l \in \mathcal{L}(V,U)$. Then $l^{-1}(W)$ is a vector subspace of V.

Exercise 2. (7)
Say for which values of $\alpha, \beta \in \mathbb{R}$,the system below admits one, none or infinite solutions.

$$A(\alpha,\beta) \cdot x = b(\alpha,\beta)$$

where

$$A(\alpha,\beta) := \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \\ 2 & 2 \\ \alpha+1 & \alpha+\beta \end{bmatrix}, \qquad b(\alpha,\beta) := \begin{bmatrix} \beta \\ \alpha \\ 2 \\ \alpha+\beta \end{bmatrix}.$$

Exercise 3. (7)
Given $n \in \mathbb{N}$ and $n \geq 2$,find a basis for $\ker l$ and $\operatorname{Im} l$ if $l : \mathbb{R}^n \to \mathbb{R}^{n+1}$,

$$l(x) = (x, x_1)$$

where, as usual, $x := (x_i)_{i=1}^n \in \mathbb{R}^n$.

Exercise 4. (7)
Let the following differentiable functions be given:

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad (x_1,x_2,x_3) \mapsto f(x_1,x_2,x_3)$$

$$g : \mathbb{R}^3 \to \mathbb{R} \qquad (x_1,x_2,x_3) \mapsto g(x_1,x_2,x_3)$$

$$a : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1,x_2,x_3) \mapsto \begin{pmatrix} f(x_1,x_2,x_3) \\ g(x_1,x_2,x_3) \\ x_1 \end{pmatrix}$$

$$b : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (y_1,y_2,y_3) \mapsto \begin{pmatrix} g(y_1,y_2,y_3) \\ f(y_1,y_2,y_3) \end{pmatrix}$$

Compute the directional derivate of the function $b \circ a$ in the point $(0,0,0)$ in the direction $(1,1,1)$.

Exercise 5. (7) Let the following C^2 functions be given:

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad (x,y) \mapsto f(x,y)$$

$$g : \mathbb{R}^2 \to \mathbb{R} \qquad (x,y) \mapsto g(x,y).$$

Assume that $\forall (x,y) \in \mathbb{R}^2$,

$$\tfrac{\partial f(x,y)}{\partial x} > 0 \qquad \tfrac{\partial f(x,y)}{\partial y} < 0 \qquad \tfrac{\partial g(x,y)}{\partial x} > 0 \qquad \tfrac{\partial g(x,y)}{\partial y} < 0$$

and that $\forall k \in \mathbb{R}$, the level curves

$$f(x,y) + g(x,y) - k = 0$$

are well defined.

Say under which condition it is possible to compute the slope of a level curve in the point (x_0,y_0). Say if its sign is not ambigous.

Sketch of the solutions.
Esercizio 1. (7)
Say if the following statement is true or false.
Let V and U be vector spaces on $\mathbb{R}$, W a vector subspace of U and $l \in \mathcal{L}(V,U)$. Then $l^{-1}(W)$ is a vector subspace of V.
Solution.
We want to show that
1. $0 \in l^{-1}(W)$,
2. $\forall \alpha, \beta \in \mathbb{R}$ and $v^1, v^2 \in l^{-1}(W)$ we have that $\alpha v^1 + \beta v^2 \in l^{-1}(W)$.
1.

$$l(0) \overset{l\in\mathcal{L}(V,U)}{=} 0 \overset{W \text{ vector space}}{\in} W$$

,
2.Since $v^1, v^2 \in l^{-1}(W)$,

$$l\left(v^1\right), l\left(v^2\right) \in W. \tag{3}$$

Then

$$l\left(\alpha v^1 + \beta v^2\right) \overset{l\in\mathcal{L}(V,U)}{=} \alpha l\left(v^1\right) + \beta l\left(v^2\right) \overset{(a)}{\in} W$$

where (a) follows from (3) and the fact that W is a vector space.
Esercizio 2. (7)
Say for which values of $\alpha, \beta \in \mathbb{R}$,the system below admits one, none or infinite solutions.

$$A(\alpha,\beta) \cdot x = b(\alpha,\beta)$$

where

$$A(\alpha,\beta) \equiv \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \\ 2 & 2 \\ \alpha+1 & \alpha+\beta \end{bmatrix}, \qquad b(\alpha,\beta) \equiv \begin{bmatrix} \beta \\ \alpha \\ 2 \\ \alpha+\beta \end{bmatrix}.$$

Solution.
$\approx$ means equivalent to.

$$\begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 0 & \alpha \\ 2 & 2 & 2 \\ \alpha+1 & \alpha+\beta & \alpha+\beta \end{bmatrix} \approx \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 0 & \alpha \\ 1 & 1 & 1 \\ 1 & \alpha+\beta & \beta \end{bmatrix}$$

.

$$\det \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 0 & \alpha \\ 1 & 1 & 1 \end{bmatrix} = \alpha\beta - \alpha = \alpha(\beta - 1).$$

1. If $\alpha \neq 0 \wedge \beta \neq 1$, then the system has no solutions.
2. If $\alpha = 0$, then we have

$$\begin{bmatrix} 1 & 0 & \beta \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & \beta & \beta \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & \beta \\ 1 & 1 & 1 \\ 0 & \beta & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 0 & \beta \\ 1 & 1 & 1 \\ 0 & \beta & 0 \end{bmatrix} = \beta(\beta - 1)$$

2i. If $\alpha = 0 \wedge \beta \neq 0 \wedge \beta \neq 1$, then the system has no solutions.
2 ii. If $\alpha = 0 \wedge \beta = 0$,then we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and the system has a unique solution.

2 iii. If $\alpha = 0 \wedge \beta = 1$,then we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and the system has a unique solution.
3. If $\alpha \neq 0 \wedge \beta = 1$, then we have

$$\begin{bmatrix} 1 & \alpha & 1 \\ \alpha & 0 & \alpha \\ 1 & 1 & 1 \\ 1 & \alpha+1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & \alpha & 1 \\ \alpha & 0 & \alpha \\ 1 & 1 & 1 \\ 0 & \alpha & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & \alpha & 1 \\ \alpha & 0 & \alpha \\ 0 & 1-\alpha & 0 \\ 0 & \alpha & 0 \end{bmatrix}$$

whose rank is at most 2, since the first and the third columns are equal. Then, since

$$\det \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix} = -\alpha^2 \neq 0,$$

the system has a unique solution.
Summarizing,
if $(\alpha \neq 0 \wedge \beta \neq 1) \vee (\alpha = 0 \wedge \beta \neq 0 \wedge \beta \neq 1)$,then the system has no solutions; otherwise, the system has a unique solution.

Esercizio 3. (7)
Given $n \in \mathbb{N}$ and $n \geq 2$,find a basis for $\ker l$ and $\operatorname{Im} l$ if $l : \mathbb{R}^n \to \mathbb{R}^{n+1}$,

$$l(x) = (x, x_1)$$

where, as usual, $x := (x_i)_{i=1}^n \in \mathbb{R}^n$.
Solution.

$$[l] = \begin{bmatrix} I_n \\ 10...0 \end{bmatrix}_{(n+1)\times n}.$$

Then

$$\dim \operatorname{Im} l = \operatorname{rank}\ [l] = n,$$

and a basis of $\operatorname{Im} l$ is the set of column vectors of $[l]$. Moreover, from the Dimension Theorem, $\dim \ker l = 0$.

Esercizio 4. (7)
Siano date le seguenti funzioni tutte differenziabili

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad (x_1, x_2, x_3) \mapsto f(x_1, x_2, x_3)$$

$$g : \mathbb{R}^3 \to \mathbb{R} \qquad (x_1, x_2, x_3) \mapsto g(x_1, x_2, x_3)$$

$$a : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) \mapsto \begin{pmatrix} f(x_1, x_2, x_3) \\ g(x_1, x_2, x_3) \\ x_1 \end{pmatrix}$$

$$b : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (y_1, y_2, y_3) \mapsto \begin{pmatrix} g(y_1, y_2, y_3) \\ f(y_1, y_2, y_3) \end{pmatrix}$$

$$b \circ a() = \begin{pmatrix} g(y_1, y_2, y_3) \\ f(y_1, y_2, y_3) \end{pmatrix} = \begin{pmatrix} g(f(x_1, x_2, x_3), g(x_1, x_2, x_3), x_1) \\ f(f(x_1, x_2, x_3), g(x_1, x_2, x_3), x_1) \end{pmatrix}$$

Si calcoli derivata direzionale della funzione $b \circ a$ nel punto $(0, 0, 0)$ nella direzione $(1, 1, 1)$.
Solution.

$$D_x (b \circ a)(x) = D_y b(y)_{|y=a(x)} \cdot D_x a(x).$$

$$D_y b(y) = \begin{bmatrix} D_{y_1} g(y) & D_{y_2} g(y) & D_{y_3} g(y) \\ D_{y_1} f(y) & D_{y_2} f(y) & D_{y_3} f(y) \end{bmatrix}_{|y=a(x)}$$

$$D_x a(x) = \begin{bmatrix} D_{x_1} f(x) & D_{x_2} f(x) & D_{x_3} f(x) \\ D_{x_1} g(x) & D_{x_2} g(x) & D_{x_3} g(x) \\ 1 & 0 & 0 \end{bmatrix}$$

$$D_x (b \circ a)(x) =$$

$$\begin{bmatrix} D_{y_1} g(f(x), g(x), x_1) & D_{y_2} g(f(x), g(x), x_1) & D_{y_3} g(f(x), g(x), x_1) \\ D_{y_1} f(f(x), g(x), x_1) & D_{y_2} f(f(x), g(x), x_1) & D_{y_3} f(f(x), g(x), x_1) \end{bmatrix} \begin{bmatrix} D_{x_1} f(x) & D_{x_2} f(x) & D_{x_3} f(x) \\ D_{x_1} g(x) & D_{x_2} g(x) & D_{x_3} g(x) \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} D_{y_1} g & D_{y_2} g & D_{y_3} g \\ D_{y_1} f & D_{y_2} f & D_{y_3} f \end{bmatrix} \begin{bmatrix} D_{x_1} f & D_{x_2} f & D_{x_3} f \\ D_{x_1} g & D_{x_2} g & D_{x_3} g \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} D_{y_1} g \cdot D_{x_1} f + D_{y_2} g D_{x_1} g + D_{y_3} g & D_{y_1} g \cdot D_{x_2} f + D_{y_2} g \cdot D_{x_2} g & D_{y_1} g \cdot D_{x_3} f + D_{y_2} g \cdot D_{x_3} g \\ D_{y_1} f \cdot D_{x_1} f + D_{y_2} f \cdot D_{x_1} g + D_{y_3} f & D_{y_1} f \cdot D_{x_2} f + D_{y_2} f \cdot D_{x_2} g & D_{y_1} f \cdot D_{x_3} f + D_{y_2} f \cdot D_{x_3} g \end{bmatrix}.$$

...

Exercise 5. (7) Let the following C^2 functions be given:

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \mapsto f(x, y)$$

$$g : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \mapsto g(x, y).$$

Assume that $\forall (x, y) \in \mathbb{R}^2$,

$$\frac{\partial f(x,y)}{\partial x} > 0 \qquad \frac{\partial f(x,y)}{\partial y} < 0 \qquad \frac{\partial g(x,y)}{\partial x} > 0 \qquad \frac{\partial g(x,y)}{\partial y} < 0$$

and that $\forall k \in \mathbb{R}$, the level curves

$$f(x, y) + g(x, y) - k = 0$$

are well defined.

Say under which condition it is possible to compute the slope of a level curve in the point (x_0, y_0). Say if its sign is not ambigous.

Solutions.

$$\frac{dy}{dx} = -\frac{\frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y} + \frac{\partial g(x,y)}{\partial y}} = -\frac{(+)}{(-)}$$

$$a : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) \mapsto \begin{pmatrix} f(x_1, x_2, x_3) \\ g(x_1, x_2, x_3) \\ x_1 \end{pmatrix}$$

$$b : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (y_1, y_2, y_3) \mapsto \begin{pmatrix} g(y_1, y_2, y_3) \\ f(y_1, y_2, y_3) \end{pmatrix}$$

Compito di
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22 febbraio 2011

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Exercise 1. (7)
Show the following fact.
Proposition. Let a matrix $A \in \mathbb{M}(n,n)$, with $n \in \mathbb{N}\backslash\{0\}$ be given. The set

$$\mathcal{C}_A := \{B \in \mathbb{M}(n,n) : BA = AB\}$$

is a vector subspace of $\mathbb{M}(n,n)$ (with respect to the field $\mathbb{R}$).

Exercise 2. (7)
Let the following objects be given: $n \in \mathbb{N}\backslash\{0\}$, $A \in \mathbb{M}(n,n)$, $b, c \in \mathbb{M}(n,1)$, $0 \in \mathbb{M}(1,n)$, $a, d \in \mathbb{M}(1,1)$ .Assume that A has full rank n. Let the following linear systems in the unknown (x,y) with $x := (x_i)_{i=1}^n \in \mathbb{R}^n$ and $y \in \mathbb{R}$ be given:

$$\begin{bmatrix} A & b \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

Say for which values of α, b, c, d,the system admits one, none or infinite solutions.
Exercise 3. (7)
Given $n \in \mathbb{N}$ and $n \geq 3$, $a \in \mathbb{R}$, let the following function be given:

$$l_a : \mathbb{R}^n \to \mathbb{R}^3, x := (x_i)_{i=1}^n \mapsto \begin{cases} \sum_{i=1}^n x_i \\ ax_1 + x_2 \\ x_2 + x_3 \end{cases}.$$

a. Compute $\dim \operatorname{Im} l_a$ and $\dim \ker l_a$ (as a function of a);
b. If $a = 0$,find a basis for $\ker l_a$ and $\operatorname{Im} l_a$.

Exercise 4. (7)
Let the following differentiable functions be given:

$$f:\mathbb{R}^3_{++}\to\mathbb{R}^2_{++} \qquad (x_1,x_2,x_3)\mapsto\begin{pmatrix} x_1+\ln x_2\\ \cos x_3\end{pmatrix}$$

$$g:\mathbb{R}^2_{++}\to\mathbb{R}^2_{++} \qquad (y_1,y_2)\mapsto\begin{pmatrix} y_1+y_2\\ y_1\cdot y_2\end{pmatrix}$$

$$h:\mathbb{R}^2_{++}\to\mathbb{R}, \qquad (z_1,z_2)\mapsto z_1\cdot z_2.$$

Compute
a. $\phi := h\circ g\circ f$,
b. $D\phi(x)$,
c. The directional derivative of ϕ in the point $(1,1,1)$ in the direction $(1,1,1)$.
Exercise 5. (7) Let the following C^2 function be given:

$$u:\mathbb{R}^2\to\mathbb{R} \qquad (x,l)\mapsto u(x,l),$$

and assume that $\forall(x,l)\in\mathbb{R}^2$, $D_x u(x,l)>0$ and $D_l u(x,l)<0$.
Let the following function be given:

$$F:\mathbb{R}^3\times\mathbb{R}^2_{++}\to\mathbb{R}^3$$

$$(x,l,\lambda,p,w)\mapsto\begin{pmatrix} D_x u(x,l)-\lambda p\\ D_l u(x,l)-\lambda w\\ wl-px\end{pmatrix}.$$

Consider the solutions (x,l,λ,p,w) to the system

$$F(x,l,\lambda,p,w)=0. \tag{4}$$

Carefully state under which conditions it is possible to study the effect of changes of the parameters (p,w) on the the variables (x,l,λ) solution to system (4). Do not compute any determinant. Show the procedure to compute those effects.

If it can help you, observe that system (4) characterize the solutions to the problem: for given (p,w),

$$\max_{(x,l)\in\mathbb{R}^2} u(x,l) \quad \text{s.t.} \quad wl-px=0$$

Solutions.
Exercise 1.
1. $0 \in \mathbb{M}(n,n) : A0 = 0A = 0$.
2. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall B, B' \in \mathcal{C}_A$,

$$(\alpha B + \beta B')\, A = \alpha BA + \beta B' A = \alpha AB + \beta AB' = A\alpha B + A\beta B' = A\,(\alpha B + \beta B')\,.$$

Exerxcise 2.
If $a \neq 0$,then there is a unique solution. If $a = 0$,the augmented matrix becomes:

$$\begin{bmatrix} A & b & c \\ 0 & 0 & d \end{bmatrix}.$$

Therefore, if $d \neq 0$,then the system has no solutions. If $d = 0$,the system has infinite solutions.
Exercise 3. (7)
a.

$$[l_a] = \begin{bmatrix} 1 & 1 & 1 & 1 & ... & 1 \\ a & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 1 & 0 & ... & 0 \end{bmatrix}$$

Since

$$\text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} 0 = \text{rank} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 3$$

then

$$\forall a \in \mathbb{R}, \dim \operatorname{Im} l_a = 3 \text{ and } \dim \ker l_a = 0.$$

A basis of $\operatorname{Im} l_a$ is given by the column vectors of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Exercise 4. (7)
Let the following differentiable functions be given:

$$f : \mathbb{R}^3_{++} \to \mathbb{R}^2_{++} \qquad (x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 + \ln x_2 \\ \cos x_3 \end{pmatrix}$$

$$g : \mathbb{R}^2_{++} \to \mathbb{R}^2_{++} \qquad (y_1, y_2) \mapsto \begin{pmatrix} y_1 + y_2 \\ y_1 \cdot y_2 \end{pmatrix}$$

$$h : \mathbb{R}^2_{++} \to \mathbb{R}, \qquad (z_1, z_2) \mapsto z_1 \cdot z_2.$$

Compute
a. $\phi := h \circ g \circ f : \mathbb{R}^3_{++} \to \mathbb{R}$,

$$\phi\,(x_1, x_2, x_3) = (h \circ g) \begin{pmatrix} x_1 + \ln x_2 \\ \cos x_3 \end{pmatrix} = h \begin{pmatrix} x_1 + \ln x_2 + \cos x_3 \\ (x_1 + \ln x_2) \cdot \cos x_3 \end{pmatrix} = (x_1 + \ln x_2 + \cos x_3) \cdot (x_1 + \ln x_2) \cdot \cos x_3.$$

b. ...
c. ...

Matematica per le applicazioni economiche 2
June 10th, 2011

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

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Exercise 1. (7)
Let a full rank matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be given. Say for which values of $k \in \mathbb{R}$, the following linear system has solutions.

$$\begin{bmatrix} 0 & 0 & a_{11} & a_{12} & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 \\ 1 & k & 0 & 0 & k+2 \\ 1 & 0 & 2 & 3 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -3 \end{bmatrix}$$

Exercise 2. (7)
Let $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ and

$$l : \mathbb{R}^{2n} \to \mathbb{R}^n, \quad l(x) = (x, x),$$

be given. Compute a basis for $\ker l$ and a basis for $\operatorname{Im} l$.

Exercise 3. (7)
Let a vector space V of dimension n and two vector subspaces (of V) U and W be given. Define

$$Z = \{z \in V : \exists u \in U \text{ and } \exists w \in W \text{ such that } z = u + w\}.$$

Show that Z is a vector subspace of V.

Exercise 4. (7)
Let the following functions and parameters be given:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad x \mapsto f(x_1, x_2)$$

$$g : \mathbb{R} \to \mathbb{R}, \quad p \mapsto g(p)$$

such that f and g are C^2. Consider the following (non inear) system:

$$f(x_1, x_2) + g(p) - 7 = 0$$

Say under which conditions it is possible to describe the effect of changes of p on x_1 and x_2 satisfying the above system.

Exercise (7)
Let the following C^2 functions be given.

$$h : \mathbb{R}^2 \to \mathbb{R}, \qquad (y_1, y_2) \mapsto h(y_1, y_2)$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (y_1, y_2) \mapsto \begin{pmatrix} h(y_1, y_2) \\ y_1 + y_2 \end{pmatrix}$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x_1, x_2) \mapsto \begin{pmatrix} e^{3x^2} \\ (\cos x_1)^2 \end{pmatrix}$$

Compute the direction derivative of $g \circ f$ in the point $(0, 1)$ in the direction $(1, 1)$. (Hint: compute $g \circ f$.)

Matematica per le applicazioni economiche 2
July 7th, 2011

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

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Exercise 1. (7)
Let the following full rank matrices

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be given. Say for which values of $k\in\mathbb{R}$, the following linear system has solutions.

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \\ 0 & a_{11} & a_{12} & 0 & 0 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}=\begin{bmatrix} k \\ 1 \\ 2 \\ 3 \\ k \end{bmatrix}$$

Esercizio 2. (7)
Consider the following Proposition contained in Section 8.1 in the class Notes:
Proposition $\forall v\in V$,

$$[l]_{\mathbf{v}}^{\mathbf{u}}\cdot[v]_{\mathbf{v}}=[l\,(v)]_{\mathbf{u}} \tag{5}$$

Verify the above equility in the case in which
a.

$$l:\mathbb{R}^2\to\mathbb{R}^2,\quad (x_1,x_2)\mapsto\begin{pmatrix} x_1+x_2 \\ x_1-x_2 \end{pmatrix}$$

b. the basis $\mathbf{v}$ of the domain of l is

$$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

c. the basis $\mathbf{u}$ of the codomain of l is

$$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix},\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\},$$

d.

$$v=\begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Esercizio 3. (7)
Using the definition of directional derivative, compute $f'(x_0;u)$ where
a. $f:\mathbb{R}^2\to\mathbb{R}$, $(x_1x_2)\mapsto(x_1+x_2)$;
b. $x_0=u=(1,1)$.
Verify the result you obtained is the same you could have obtained using the "standard" formula.

Esercizio 4. (7)
Condider the function $f:\mathbb{R}^2_{++}\times\mathbb{R}^2\to\mathbb{R}^2$

$$(x_1,x_2,t_1,t_2)\mapsto\begin{pmatrix} e^{x_1}-\log x_2+x_1x_2-t_1+t_2 \\ \\ \log(1+x_1)+x_1^2+x_2^2-t_2-e^2+1 \end{pmatrix}.$$

Using the notation of the statement of the Implicit Function Theorem presented in Class Notes, say if that Theorem can be applied to the cases described above in the point $(x_1^0,x_2^0,t_1^0,t_2^0)=(0,e,1,1)$. If it can be applied, compute the Jacobian of g, i.e., "the effect of the changes of t_1 and t_2 on x_1 and x_2".

Esercizio 5. (7)
Complete the following proof.
Proposition. Let
$n, m \in \mathbb{N} \setminus \{0\}$ such that $m > n$, and
a vector subspace L of $\mathbb{R}^m$ such that $\dim L = n$
be given. Then, there exists $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\operatorname{Im} l = L$.
Proof. Let $\{v^i\}_{i=1}^n$ be a basis of $L \subseteq \mathbb{R}^m$. Take $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\forall i \in \{1, ..., n\}, \qquad l_2\left(e_n^i\right) = v^i,$$

where e_n^i is the i–th element in the canonical basis in $\mathbb{R}^n$. Such function does exists and, in fact, it is unique as a consequence of a Proposition in the Class Notes that we copy below:

....

Then, from the Dimension theorem

$$\dim \operatorname{Im} l = \text{...}$$

Moreover,

$$L = \text{..........} \left\{v^i\right\}_{i=1}^n \subseteq \text{..............}$$

Summarizing,

$$L \subseteq \operatorname{Im}\ l\ ,\ \dim L = n \text{ and } \dim \operatorname{Im} l \leq n,$$

and therefore

$$\dim \operatorname{Im} l = n.$$

Finally, from Proposition 179 in the class Notes since $L \subseteq \operatorname{Im}\ l$, $\dim L = n$ and $\dim \operatorname{Im} l = n$, we have that $\operatorname{Im} l = L$, as desired.

Proposition 179 in the class Notes says what follows:

.... ■

Solutions.
Esercizio 3. (6)
Let
$n, m \in \mathbb{N} \backslash \{0\}$ such that $m > n$, and
a vector subspace L of $\mathbb{R}^m$ such that $\dim L = n$
be given. Then, there exists $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\operatorname{Im} l = L$.
Proof. Let $\left\{v^i\right\}_{i=1}^n$ be a basis of $L \subseteq \mathbb{R}^m$. Take $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\forall i \in \{1, ..., n\}, l_2\left(e_n^i\right) = v^i,$$

where e_n^i is the i–th element in the canonical basis in $\mathbb{R}^n$. Such function does exists and, in fact, it is unique as a consequence of a Proposition in the Class Notes that we copy below:

Let V and U be finite dimensional vectors spaces such that $S = \left\{v^1, ..., v^n\right\}$ is a basis of V and $\left\{u^1, ..., u^n\right\}$ is a set of arbitrary vectors in U. Then there exists a unique linear function $l : V \to U$ such that $\forall i \in \{1, ..., n\}$, $l\left(v^i\right) = u^i$ - see Proposition 273, page 82.

Then, from the Dimension theorem

$$\dim \operatorname{Im} l = n - \dim \ker l \leq n.$$

Moreover, $L = \operatorname{span}\left\{v^i\right\}_{i=1}^n \subseteq \operatorname{Im} l$. Summarizing,

$$L \subseteq \operatorname{Im}\ l\ ,\ \dim L = n \text{ and } \dim \operatorname{Im} l \leq n,$$

and therefore

$$\dim \operatorname{Im} l = n.$$

Finally, from Proposition 179 in the class Notes since $L \subseteq \operatorname{Im}\ l$, $\dim L = n$ and $\dim \operatorname{Im} l = n$, we have that $\operatorname{Im} l = L$, as desired.

Proposition 179 in the class Notes says what follows: ■

Proposition. Let W be a subspace of an $n-$dimensional vector space V. Then
1. $\dim W \leq n$;
2. If $\dim W = n$, then $W = V$.

Esercizio 6. (6)

$$\begin{pmatrix} e^0 - \log e - 1 + 1 \\ \log(1+0) + 0 + e^2 - 1 - e^2 + 1 \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}$$

$$\begin{pmatrix} & x_1 & x_2 \\ e^{x_1} - \log x_2 + x_1 x_2 & e^{x_1} + x_2 & -\frac{1}{x_2} + x_1 \\ \log(1+x_1) + x_1^2 + x_2^2 & \frac{1}{1+x_1} + 2x_1 & 2x_2 \end{pmatrix}_{|(0,e,1,1)} =$$

$$\begin{pmatrix} e^{x_1} + x_2 & -\frac{1}{x_2} + x_1 \\ \frac{1}{1+x_1} + 2x_1 & 2x_2 \end{pmatrix}_{|(0,e,1,1)} = \begin{pmatrix} e^0 + e & -\frac{1}{e} \\ \frac{1}{1} & 2e \end{pmatrix} = \begin{pmatrix} 1+e & -\frac{1}{e} \\ 1 & 2e \end{pmatrix}$$

$\det \begin{pmatrix} 1+e & -\frac{1}{e} \\ 1 & 2e \end{pmatrix} = 2e + 2e^2 + \frac{1}{e} > 0$
$\frac{1}{e}\left(2e^2 + 2e^3 + 1\right)$

$$e^x - \log y + xy$$
$$\log(1+x) + x^2 + y^2$$

$\frac{d(e^x - \log y + xy)}{dx} = y + e^x$
$\frac{d(e^x - \log y + xy)}{dy} = x - \frac{1}{y}$

$\frac{d\left(\log(1+x) + x^2 + y^2\right)}{dx} = 2x + \frac{1}{x+1}$
$\frac{d\left(\log(1+x) + x^2 + y^2\right)}{dy} = 2y$

Exercise ??.
$\lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} = \lim_{h \to 0} \frac{f((1,1) + h(1,1)) - f((1,1))}{h} = \lim_{h \to 0} \frac{f((1+hu, 1+hu)) - f((1,1))}{h} == \lim_{h \to 0} \frac{2+2h-2}{h} =$

Matematica per le applicazioni economiche 2

September 13th, 2011

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (7)
Say for which values of $\alpha, \beta \in \mathbb{R}$, the following linear system has solutions.

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Exercise 2. (7)
Let V be a vector space with a basis $\mathcal{V} = \left\{v^i\right\}_{i=1}^n$, $n \geq 3$ be given. Consider the following function

$$l : V \to \mathbb{R}^2, \quad w \mapsto \left([w]_{\mathcal{V}}^1 , [w]_{\mathcal{V}}^2 \right)$$

where $[w]_{\mathcal{V}}^1 , [w]_{\mathcal{V}}^2$ denote the first and the second coordinate of the vector w with respect to the basis $\mathcal{V}$, respectively.
1. Show that l is linear;
2. Compute

$$[l]_{\mathcal{V}}^{\mathcal{E}_2} ,$$

where $\mathcal{E}_2$ is the canonical basis of $\mathbb{R}^2$.
Exercise 3. (7)
Let $x = (x_i)_{i=1}^4 \in \mathbb{R}^4$ and

$$l : \mathbb{R}^4 \to \mathbb{R}^2, \quad l\left((x_i)_{i=1}^4 \right) = (x_1, x_3) ,$$

be given. Compute a basis for $\ker l$ and a basis for $\operatorname{Im} l$.

Exercise 4. (7)
Let the following functions be given.
For any $i \in \{1, ..., m\}$,

$$g_i : \mathbb{R} \to \mathbb{R}, x_i \mapsto g_i (x_i) ;$$

$$\phi : \mathbb{R}^n \to \mathbb{R}^n, \quad (x_i)_{i=1}^n \mapsto \ (g_i (x_i))_{i=1}^n ;$$

$$f : \mathbb{R}^n \to \mathbb{R}, \quad (y_i)_{i=1}^n \mapsto f\left(\ (y_i)_{i=1}^n \right).$$

If possible compute,
1. $h := f \circ \phi$;
2. $h' (0, (1, 1, ..., 1))$;
3. $s := \phi \circ f$.

Exercise 5. (7)

Let the following function be given

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto x + y^3 - 2x^2 y - \frac{1}{8}.$$

Say if you can apply the Implicit Function Theorem to

$$f (x, y) = 0$$

around

$$(x_0, y_0) = \left(1, \frac{1}{2}\right)$$

to compute
a. the effect of "small" changes of x on y, and
b. the effect of "small" changes of y on x.

Solutions to
Exercise 1.

$$\det \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta).$$

(1) Therefore, if $\alpha \neq \beta$ and $\alpha \neq -\beta$, the system has a unique solution.
(2) If $\alpha = \beta$, the system becomes :

$$\begin{bmatrix} \beta & \beta \\ \beta & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta \\ \beta \end{bmatrix},$$

or

$$\begin{bmatrix} \beta & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta.$$

If $\beta = 0$, then $\mathbb{R}^2$ is the set of solutions.
If $\beta \neq 0$, then the set of solutions is an affine space of dimension 1.
(3) If $\alpha = -\beta$, the system becomes :

$$\begin{bmatrix} -\beta & \beta \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\beta \\ \beta \end{bmatrix},$$

or

$$\begin{bmatrix} -\beta & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\beta,$$

If $\beta = 0$, then $\mathbb{R}^2$ is the set of solutions.
If $\beta \neq 0$, then the set of solutions is an affine space of dimension 1.
Exercise2.
1. Consider

$$cr_{\mathcal{V}} : V \to \mathbb{R}^n, \qquad w \mapsto \left([w]_{\mathcal{V}}^1, ..., [w]_{\mathcal{V}}^n \right),$$

and

$$pr_{n,2} : \mathbb{R}^n \to \mathbb{R}^2, \quad (x_i)_{i=1}^n \mapsto (x_1, x_2).$$

Since the above functions are linear and $l = pr_{n,2} \circ cr_{\mathcal{V}}$, the desired result follows.
2.

$$[l]_{\mathcal{V}}^{\mathcal{E}_2} := \left[\left[l\left(v^1\right) \right]_{\mathcal{E}_2}, \left[l\left(v^2\right) \right]_{\mathcal{E}_2}, \left[l\left(v^3\right) \right]_{\mathcal{E}_2}, ..., \left[l\left(v^n\right) \right]_{\mathcal{E}_2} \right] = \left[\left[l\left(v^1\right) \right], \left[l\left(v^2\right) \right], \left[l\left(v^3\right) \right], ..., \left[l\left(v^n\right) \right] \right] = \left[\begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \end{matrix}, \begin{matrix} 0 \\ 0 \end{matrix}, \dots \right.$$

Exercise 3.

`to be done`

Exercise 4.
1.

$$h : \mathbb{R}^n \to \mathbb{R}, \quad (x_i)_{i=1}^n \mapsto f\left(\left(g_i(x_i) \right)_{i=1}^n \right).$$

2.

$$[D_x h(x)]_{1\times n} = \left[D_y f(y)_{|y=\phi(x)} \right]_{1\times n} \cdot [D_x \phi(x)]_{n\times x} =$$

$$= \left[D_{y^1} f(\phi(x)), ..., D_{y^n} f(\phi(x)) \right] \cdot \begin{bmatrix} g_1'(x_1) & & \\ & \ddots & \\ & & g_n'(x_n) \end{bmatrix} =$$

$$= \left[D_{y^1} f(\phi(x)) \cdot g_1'(x_1), ..., D_{y^n} f(\phi(x)) \cdot g_n'(x_n) \right] = \left[\left(D_{y^i} f(\phi(x)) \cdot g_i'(x_i) \right)_{i=1}^n \right]$$

Then

$$h'(0, (1, 1, ..., 1)) = \left[\left(D_{y^i} f(\phi(0)) \cdot g_i'(0) \right)_{i=1}^n \right] \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} = \sum_{i=1}^n D_{y^i} f(\phi(0)) \cdot g_i'(0)$$

Exercise 5.
$f\left(1,\frac{1}{2}\right)=1+\frac{1}{8}-2\frac{1}{2}-\frac{1}{8}=0.$
$D_x f\left(x,y\right)=2x-4xy;$
$D_x f\left(1,\frac{1}{2}\right)=2-4\frac{1}{2}=0$, therefore it is not possible to compute $\frac{dx}{dy}=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$.
$D_y f\left(x,y\right)=3y^2-2x^2;$
$D_y f\left(1,\frac{1}{2}\right)=3\frac{1}{8}-2\neq 0.$

Matematica per le applicazioni economiche 2
January 9th, 2012

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (8)
Let $l \in \mathcal{L}\left(\mathbb{R}^2, \mathbb{R}^2\right)$,

$$l\left(x_1, x_2\right) = \begin{pmatrix} x_1 \\ x_1 + 2x_2 \end{pmatrix}$$

be given. Verify that the following condition holds true:

$$\forall x = \left(x_1, x_2\right) \in \mathbb{R}^2, \qquad [l]_{\mathbf{v}}^{\mathbf{u}} \cdot [x]_{\mathbf{v}} = [l\left(x\right)]_{\mathbf{u}} \tag{6}$$

in the case in which
a. the basis $\mathbf{v}$ of the domain of l is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

b. the basis $\mathbf{u}$ of the codomain of l is

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

Exercise 2. (8)
Let V be a vector space (on $\mathbb{R}$) of finite dimension and W_1 and W_2 vector subspaces of V.Say if the following statements are true.
a. $W_1 \times W_2$ is a vector subspace of $V \times V$.
b. If $l \in \mathcal{L}\left(V \times V, V \times V\right)$, then $l\left(\{0\} \times W_2\right)$ is a vector subspace of $V \times V$.

.

Exercise 3. (8)
If possible, find a basis for the kernel and Image of the function below.

$$l : \mathbb{R}^{2n} \to \mathbb{R}^n, \qquad (x_i)_{i=1}^{2n} \mapsto (x_{2i})_{i=1}^{n}$$

Exercise 4. (8)
Using the definition of directional derivative, compute the directional derivative of $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

Exercise 5. (8)
Let the following maximization problem be given:
for given $a \in \mathbb{R}$,

$$\max_{(x,y)\in\mathbb{R}^2} u(x) + v(y) \qquad s.t. \qquad y - g(x,a) = 0,$$

where u, v and g are C^2 functions.

a. Check if the assumptions of the theorem presented in the Section 17.6."Extremum problems with equality constraints", are satisfied;

b. Using the Implicit Function Theorem, give conditions under which it is possible to compute the effect of local changes of a on the solutions of the system found in point a. above. Write the expression describing such effects.

Sketch of the Solutions.
Exercise 1.
Let $l \in \mathcal{L}\left(\mathbb{R}^2, \mathbb{R}^2\right)$,

$$l\left(x_1, x_2\right) = \left(\begin{array}{c} x_1 \\ x_1 + 2x_2 \end{array}\right)$$

be given. Verify that the following condition holds true:

$$\forall x = \left(x_1, x_2\right) \in \mathbb{R}^2, \qquad [l]_{\mathbf{v}}^{\mathbf{u}} \cdot [x]_{\mathbf{v}} = [l\left(x\right)]_{\mathbf{u}} \tag{7}$$

in the case in which
a. the basis $\mathbf{v}$ of the domain of l is

$$\left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right\},$$

b. the basis $\mathbf{u}$ of the codomain of l is

$$\left\{\left(\begin{array}{c} 1 \\ 2 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \end{array}\right)\right\},$$

$[x]_{\mathbf{v}} = x$
$l\left(x\right) = \left(\begin{array}{c} x_1 \\ x_1 + 2x_2 \end{array}\right)$

$$\alpha \left(\begin{array}{c} 1 \\ 2 \end{array}\right) + \beta \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \left(\begin{array}{c} x_1 \\ x_1 + 2x_2 \end{array}\right)$$

$$\left\{\begin{array}{lll} \alpha + \beta & = & x_1 \\ 2\alpha & = & x_1 + 2x_2 \end{array}\right.$$

$$\left\{\begin{array}{lll} \beta & = & x_1 - \left(\frac{1}{2}x_1 + x_2\right) = \frac{1}{2}x_1 - x_2 \\ \alpha & = & \frac{x_1 + 2x_2}{2} = \frac{1}{2}x_1 + x_2 \end{array}\right.$$

$[l\left(x\right)]_{\mathbf{u}} = \left(\begin{array}{c} \frac{1}{2}x_1 + x_2 \\ \frac{1}{2}x_1 - x_2 \end{array}\right)$
$[l]_{\mathbf{v}}^{\mathbf{u}} = \left[\left(\begin{array}{c} 1 \\ 1 \end{array}\right)_{\mathbf{u}}, \left(\begin{array}{c} 0 \\ 2 \end{array}\right)_{\mathbf{u}}\right] = \left[\left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right), \left(\begin{array}{c} 1 \\ -1 \end{array}\right)\right]$ as computed below:

$$\alpha \left(\begin{array}{c} 1 \\ 2 \end{array}\right) + \beta \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \left(\begin{array}{c} 1 \\ 1 \end{array}\right)$$

$$\left\{\begin{array}{lll} \alpha + \beta & = & 1 \\ 2\alpha & = & 1 \end{array}\right.$$

$$\left\{\begin{array}{lll} \beta & = & \frac{1}{2} \\ \alpha & = & \frac{1}{2} \end{array}\right.$$

$$\alpha \left(\begin{array}{c} 1 \\ 2 \end{array}\right) + \beta \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \left(\begin{array}{c} 0 \\ 2 \end{array}\right)$$

$$\left\{\begin{array}{lll} \alpha + \beta & = & 0 \\ 2\alpha & = & 2 \end{array}\right.$$

$$\left\{\begin{array}{lll} \beta & = & -1 \\ \alpha & = & 1 \end{array}\right.$$

Let's now verity the statement:

$$[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [x]_{\mathbf{v}} = [l\left(x\right)]_{\mathbf{u}} \tag{8}$$

$$\left[\begin{array}{cc} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} \frac{1}{2}x_1 + x_2 \\ \frac{1}{2}x_1 - x_2 \end{array}\right].$$

Exercise 2.

a.

WTS

1. $0 \in W_1 \times W_2$, and

2. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall u = (w_1^u, w_2^u), v = (w_1^v, w_2^v) \in W_1 \times W_2$, we have that $\alpha u + \beta v \in W_1 \times W_2$.

...

b.

$\{0\} \times W_2$ is a vector space from the argument above. Then the result follows from the fact that $\operatorname{Im} l$ is a vector space.

Exercise 3.

$[l] = \left[0, e_n^1, 0, e_n^2, ..., 0, e_n^n\right]$.

$\dim \operatorname{Im} l = \operatorname{rank}[l] = n$.

Basis of $\operatorname{Im} l$ is canonical basis.

$l(x) = 0$ if and only if

$$\begin{cases} x_2 &= 0 \\ x_4 &= 0 \\ ... & \\ x_{2n} &= 0 \end{cases}$$

Basis of $\ker l = \left\{e_{2n}^1, e_{2n}^3, ..., 0, e_{2n}^{2n.1}\right\}$

Exercise 4.

$$l(x_0; u) = \lim_{h \to 0} \frac{l(x_0 + hu) - l(x_0)}{h} = \lim_{h \to 0} \frac{l(x_0) + hl(u) - l(x_0)}{h} = l(u).$$

Exercise 5.

...

Matematica per le applicazioni economiche 2
February 6th, 2012

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (8)
Say for which values of $a \in \mathbb{R}$, the following linear system has solutions.

$$\begin{bmatrix} a-1 & a & a \\ 0 & 1 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6a \end{bmatrix}$$

For $a = 1$,if possible, write the set of solutions as an affine subspace of $\mathbb{R}^3$.

Exercise 2. (8)
Say if the following statements are true or false.
a. let V be a vector space and W a vector subspace of V. Then $V \backslash W := \{v \in V : v \notin W\}$ is not a vector subspace of V.
b. The product of linear functions is a linear function.
c. Let $\mathcal{F}$ be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Then

$$\{f \in \mathcal{F} : \forall x \in [0,1],\ f(x) = 0\}$$

is a vector subspace of $\mathcal{F}$.

Exercise 3. (8)
Let the following linear function be given

$$l : \mathbb{R}^n \to \mathbb{R}^n, \qquad (x_i)_{i=1}^n \mapsto (i \cdot x_i)_{i=1}^n,$$

with $n \in \mathbb{N}_+$.
a. Show that l is invertible,
b. compute the inverse of l;
c. Describe the kernel of the inverse and find a basis of the image of the inverse.

Exercise 4. (8)
Using the definition of differentiable functions, show that the sum of differentiable function is a differentiable function.

Exercise 5. (8)
Let the following C^2 functions be given

$$s : \mathbb{R}^2 \to \mathbb{R}, \qquad (p,a) \mapsto s(p,a),$$

$$d : \mathbb{R}^2 \to \mathbb{R}, \qquad (p,a) \mapsto d(p,a).$$

Assume that $\forall (p,a) \in \mathbb{R}^2$,

$$\frac{\partial s(p,a)}{\partial p} > 0, \qquad \frac{\partial s(p,a)}{\partial a} > 0,$$

$$\frac{\partial d(p,a)}{\partial p} < 0, \qquad \frac{\partial d(p,a)}{\partial A} < 0,$$

and that $\exists (p^*, a^*) \in \mathbb{R}^2$ such that $s(p^*.a^*) = d(p^*.a^*)$.
Say if you can study the effect of local changes of a on p around p^*.

Matematica per le applicazioni economiche 2
June 8th, 2012

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (8)
Let the following linear function be given

$$l_a : \mathbb{R}^3 \to \mathbb{R}^2, \quad l_a(x_1, x_2, x_3) = \begin{pmatrix} ax_1 + 2ax_2 + 3ax_3 \\ \\ x_1 \end{pmatrix}$$

where $a \in \mathbb{R}$. For any value of a, compute a basis of $\ker l_a$ and $\operatorname{Im} l_a$.

Exercise 2. (8)
Say if the following statements are true or false.
1. For any $n \in \mathbb{N}_+$, $l \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^{2n}\right)$ is not onto;
2. For any $n \in \mathbb{N}_+$, $l \in \mathcal{L}\left(\mathbb{R}^{2n}, \mathbb{R}^n\right)$ is not one-to-one;
3. Given $a \in \mathbb{R}$,if $l_a \in \mathcal{L}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ is such that

$$[l_a] = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix},$$

then l is an invertible function.

Exercise 3. (8)
Let $\mathcal{C}^0([0,1], \mathbb{R})$ be the set continuous functions from $[0,1]$ to $\mathbb{R}$. Consider the function

$$l : \mathcal{C}^0([0,1], \mathbb{R}) \to \mathbb{R}, \qquad l(f) = f(0).$$

1. Show that $\mathcal{C}^0([0,1], \mathbb{R})$ is a vector subspace of vector space of functions from $[0,1]$ to $\mathbb{R}$;
2. Show that l is linear;
3. Describe $\ker l$.

Exercise 4. (8)
Let the following function be given:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} \frac{x^2 y^2}{x^3 + y^3} & if \quad x \neq 0 \\ \\ y & if \quad x = 0 \end{cases}$$

Using the definition of directional derivative, if possible, compute $f'(0; u)$ for every $u \in \mathbb{R}^2$.

Exercise 5. (8)
Let the following functions and parameters be given.
$g : \mathbb{R} \to \mathbb{R}, \quad g \mapsto g(l)$ such that $\forall l \in \mathbb{R}$, $g'(l) > 0$ and $g''(l) < 0$;
$s \in \mathbb{R}$,
$w : \mathbb{R}^2 \to \mathbb{R}_+$, $(l, s) \mapsto w(l, s)$ such that $\forall (l, s) \in \mathbb{R}^2$,

$$D_l w(l, s) > 0, D_s w(l, s) > 0, D_{ll} w(l, s) > 0, \ D_{ls} w(l, s) > 0.$$

Consider the problem

$$\max_{l \in \mathbb{R}} \qquad \pi(l, s) = g(l) - w(l, s) \cdot l.$$

Assuming that the solution of that problem is characterized by the critical points of the objective function, i.e., by the equation

$$D_l \pi(l, s) = 0,$$

compute the effects of changes of s on the solution itself.

Exercise 4.
Let's compute $f'(0;u)$. If $u_1 \neq 0$.

$$\lim_{h\to 0}\frac{f(0+hu)-f(0)}{h}=\lim_{h\to 0}\frac{h^2u_1^2\cdot h^2u_2^2}{(h^3u_1^2+h^3u_2^2)\,h}=\lim_{h\to 0}\frac{u_1\cdot u_2}{u_1^2+u_2^2}=\frac{u_1\cdot u_2}{u_1^2+u_2^2}$$

If $u_1=0$,we have

$$\lim_{h\to 0}\frac{f(0+hu)-f(0)}{h}=\lim_{h\to 0}\frac{f(0,hu_2)-f(0)}{h}=\lim_{h\to 0}\frac{hu_2}{h}=u_2$$

1. Canonical form.
For given $s\in\mathbb{R}$, $\overline{L}\in\mathbb{R}_{++}$,,

$$\begin{array}{lllll}\max_{l\in\mathbb{R}} g(l)-w(l,s)\cdot l & s.t. & l & \geq 0 & \lambda_0\\ & & \overline{L}-l & \geq 0 & \lambda_1\end{array}$$

2. The set X and the functions f and g.
Obvious.
3. Existence.
$\left[0,\overline{L}\right]$ is compact.
4. Number of solutions.
The first and second derivatives of the objective function are

$$g'(l)-D_lw(l,s)\cdot l-w(l,s)$$
$$g''(l)-D_{ll}w(l,s)\cdot l-D_lw(l,s)-D_lw(l,s)\cdot l-w(l,s)<0$$

The solution is unique.
5. Necessity of K-T conditions.
Since each g_j is affine and therefore pseudo-concave, we are left with showing that there exists $l^{++}\in\mathbb{R}$ such that $g(l^{++})>>0$. Just take $l=\frac{\overline{L}}{2}$.
Therefore

$$M\subseteq S$$

6. Sufficiency of K-T conditions.
f is strictly concave and therefore pseudo-concave, and each g_j is linear and therefore quasi-concave. Therefore

$$M\supseteq S$$

7. K-T conditions.

$$\begin{array}{lllll}\max_{l\in\mathbb{R}} g(l)-w(l,s)\cdot l & s.t. & l & \geq 0 & \lambda_0\\ & & \overline{L}-l & \geq 0 & \lambda_1\end{array}$$

$$\mathcal{L}\left(l,\lambda_0,\lambda_1,;s,\overline{L}\right)=g(l)-w(l,s)\cdot l+\lambda_0 l+\lambda_1\left(\overline{L}-l\right)$$

$$\begin{cases} g'(l)-D_lw(l,s)\cdot l-w(l,s)+\lambda_0-\lambda_1 & = 0\\ \min\{l,\lambda_0\} & = 0\\ \min\left\{\overline{L}-l,\lambda_1\right\} & = 0\end{cases}$$

....

$$\frac{\partial l^*}{\partial s}=\frac{-D_{ls}w(l,s)\cdot l-D_sw(l,s)}{-\left[g''(l)-D_{ll}w(l,s)\cdot l-D_lw(l,s)-D_lw(l,s)\cdot l-w(l,s)\right]>0}$$

Matematica per le applicazioni economiche 2
July 4th, 2012

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (8)
Let the following sets be given:a.$\left\{(x,y) \in \mathbb{R}^2 : x = 0\right\}$,
b. $\left\{(x,y) \in \mathbb{R}^2 : x - y = 0\right\}$,
c. $\left\{(x,y) \in \mathbb{R}^2 : |x| - |y| = 0\right\}$,
and let the following linear function be given:

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad l(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Say if
i. the sets described above are vector subspaces of $\mathbb{R}^2$;
ii. for those sets which are vector subspaces of $\mathbb{R}^2$, compute a basis of their image through the function l.

Exercise 2. (8)
Say for which values of $k \in \mathbb{R}$, the following system admits one solution, no solutions, infinite solutions.

$$\begin{bmatrix} -4 & k & k & 1 \\ -k & 1 & k & k+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5k \\ 6 \end{bmatrix}$$

If the system admits solutions, say what is the dimension of the affine space of such solutions.

Exercise 3. (8)
Siano date le funzioni differenziabili

$$g : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto g(x,y)$$
$$h : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto h(x,y)$$
$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (t,u) \mapsto f(t,u)$$
$$h : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto f(g(x,y), h(x,y))$$

Se possibile si calcoli la derivata direzionale di h in $(0,0)$ nella direzione $(1,1)$.

Exercise 4. (8)
Let the following functions and parameters be given: $s \in \mathbb{R}$,
$g : \mathbb{R} \to \mathbb{R}$, $g \mapsto g(x)$ such that $\forall l \in \mathbb{R}$, $g'(x) > 0$ and $g''(x) < 0$;
$p : \mathbb{R}^2 \to \mathbb{R}_+$, $(x,s) \mapsto p(l,s)$ such that $\forall (x,s) \in \mathbb{R}^2$,

$$D_x p(x,s) > 0, D_s p(x,s) > 0, D_{xx} p(x,s) > 0, \; D_{xs} p(x,s) > 0.$$

Consider the problem

$$\max_{x \in \mathbb{R}} \qquad \pi(x,s) = p(x,s)\, g(x) - x.$$

Assuming that the solution of that problem is characterized by the critical points of the objective function, i.e., by the equation

$$D_x \pi(x,s) = 0,$$

compute the effects of changes of s on the solution itself

Exercise 5. (8)
Let V_1, V_2 be vector subspaces of $\mathbb{R}^n$. Say if the following set

$$V_1 \times V_2 = \left\{(v_1, v_2) \in \mathbb{R}^{2n:} : v_1 \in V_1, v_2 \in V_2\right\}$$

is a vector subspace of $\mathbb{R}^{2n}$.

Complete the following proof.

Definition. A vector subspace $V \subseteq \mathbb{R}^n$ is orthogonal to a vector subspace U of $\mathbb{R}^n$ if $\forall v \in V$ and $\forall u \in U$, $v \cdot u = 0$. Then, we say that V and U are orthogonal.

Example. Given $n = 2$, an example of orthogonal subspace is the following one.

use an additional sheet of paper

Proposition. Let $A, S \in \mathbb{N}_+$ with $S \geq A$ be given. If L is an A- dimensional vector subspace of $\mathbb{R}^S$, then there exists $l \in \left(\mathbb{R}^S, \mathbb{R}^{S-A}\right)$ such that $\ker l = L$.

Proof. Proof. Take a basis $\left\{l^1, ...l^A\right\} \subseteq \mathbb{R}^S$ of L. Define

$$C = \begin{bmatrix} l^1 \\ ... \\ l^A \end{bmatrix}_{S \times A},$$

■

where the vectors are taken to be row vectors. By definition of basis,

$$\text{rank}C = ...$$

From the Dimension Theorem,

$$\dim \ker l_C = ...$$

where

$$l_C \text{ is}$$

Let $\left\{k^1, ..., k^{S-A}\right\} \subseteq \mathbb{R}^S$ be a basis of $\ker l_C$. Then, $\forall s \in \{A, ..., S-A\}$,

$$Ck^s = ...,$$

and

$$\forall a \in 1, ..., A, \forall s \in \{1, ..., S-A\}, \ l^a \cdot k^s = 0,$$

i.e.,[1]

$$L = \left(\ker l_C\right)^{\perp}. \tag{9}$$

Define

$$M = \begin{bmatrix} k^1 \\ ... \\ k^{S-A} \end{bmatrix}_{(S-A) \times S}.$$

By definition of basis,

$$\text{rank}M = ...$$

Moreover, from the fact that $\left\{k^1, ..., k^{S-A}\right\} \subseteq \mathbb{R}^S$ is a basis of $\ker l_C$ and from (10), we have that

$$L = \left\{x \in \mathbb{R}^S : \forall s \in \{1, ..., S-A\}, \ k^s \cdot x = 0\right\} = \left\{x \in \mathbb{R}^S : M \cdot x = 0\right\} = \ker ....$$

■

.

[1] We are using the obvious fact that:
Two vector spaces V and W are orthogonal iff each each element in a basis of V is orthogonal to each element in a basis of W.

Exercise 1.
Let the following sets be given
a.$\left\{(x,y)\in\mathbb{R}^2 : x=0\right\}$;
b. $\left\{(x,y)\in\mathbb{R}^2 : x-y=0\right\}$
c. $\left\{(x,y)\in\mathbb{R}^2 : |x|-|y|=0\right\}$
and let the following linear function be given:

$$l:\mathbb{R}^2\to\mathbb{R}^2,\quad l(x,y)=\begin{pmatrix} x \\ \\ y \end{pmatrix}.$$

Say if
i. the sets described above are vector subspaces of $\mathbb{R}^2$;
ii. for those sets which are vector subspaces of $\mathbb{R}^2$compute a basis of their image through the function l.

Definition 1 *A vector subspace $V\subseteq\mathbb{R}^n$ is orthogonal to a vector subspace U of $\mathbb{R}^n$ if $\forall v\in V$ and $\forall u\in U$, $v\cdot u=0$. Then, we say that V and U are orthogonal.*

Example 2 *Given $n=2$, an example of orthogonal subspace is*

Proposition 3 *Let $A,S\in\mathbb{N}_+$ with $S\geq A$ be given. If L is an A- dimensional vector subspace of $\mathbb{R}^S$, then there eixists $l\in\left(\mathbb{R}^S,\mathbb{R}^{S-A}\right)$ such that $\ker l=L$.*

Proof. Proof. Take a basis $\left\{l^1,...l^A\right\}\subseteq\mathbb{R}^S$ of L. Define

$$C=\begin{bmatrix} l^1 \\ ... \\ l^A \end{bmatrix}_{S\times A},$$

■

where the vectors are taken to be row vectors. By definition of basis,

$$\text{rank}C=A.$$

From the Dimension Theorem,

$$\dim\mathbb{R}^S=\dim Iml_C+\dim\ker l_C$$

or

$$\dim\ker l_C=S-A.$$

Let $\{k^1, ..., k^{S-A}\} \subseteq \mathbb{R}^S$ be a basis of $\ker l_C$. Then, $\forall s \in \{A, ..., S-A\}$,

$$Ck^s = 0,$$

and

$$\forall a \in 1, ..., A, \forall s \in \{1, ..., S-A\},\ l^a \cdot k^s = 0,$$

i.e.,[2]

$$L = (\ker l_C)^{\perp}. \tag{10}$$

Define

$$M = \begin{bmatrix} k^1 \\ ... \\ k^{S-A} \end{bmatrix}_{(S-A)\times S}.$$

By definition of basis,

$$\mathrm{rank} M = S - A.$$

Moreover, from the fact that $\{k^1, ..., k^{S-A}\} \subseteq \mathbb{R}^S$ is a basis of $\ker l_C$ and from (10), we have that

$$L = \{x \in \mathbb{R}^S : \forall s \in \{1, ..., S-A\},\ k^s \cdot x = 0\} = \{x \in \mathbb{R}^S : M \cdot x = 0\} = \ker l_M.$$

[2] We are using the obvious fact that:
Two vector spaces V and W are orthogonal iff each each element in a basis of V is orthogonal to each element in a basis of W.

■

Exercise 5.
Let the following functions and parameters be given.
$g : \mathbb{R} \to \mathbb{R}, \quad g \mapsto g(x)$ such that $\forall l \in \mathbb{R},\ g'(x) > 0$ and $g''(x) < 0$;
$s \in \mathbb{R}$,
$p : \mathbb{R}^2 \to \mathbb{R}_+,\ (x, s) \mapsto p(l, s)$ such that $\forall (x, s) \in \mathbb{R}^2$,

$$D_x p(x, s) > 0, D_s p(x, s) > 0, D_{xx} p(x, s) > 0,\ \ D_{xs} p(x, s) > 0.$$

Consider the problem

$$\max_{x \in \mathbb{R}} \qquad \pi(x, s) = p(x, s)\, g(x) - x.$$

Assuming that the solution of that problem is characterized by the critical points of the objective function, i.e., by the equation

$$D_x \pi(x, s) = 0,$$

compute the effects of changes of s on the solution itself.

Matematica per le applicazioni economiche 2
September 3rd, 2012

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (8)
Let S and S' linearly independent sets of vectors in $\mathbb{R}^n$. Say it the following statements are true or false:
a. $S \cap S'$ is linearly independent;
b. $S \cup S'$ is linearly independent.

Exercise 2. (8)
Let U be a vector space an $l_1$and $l_2$belong to $\mathcal{L}(U, U)$. Say it the following statements are true or false:
a. $l_1 \circ l_2$ is linear;
b. $l_1 \cdot l_2$ is linear.

Exercise 3. (8)
Desribe the solution set of the following linear system for each $b \in \mathbb{R}$

$$\begin{cases} (b-1)x - y & = & b - 1 - z \\ x + by - z & = & b + 1 - y \\ 2x + y & = & -1 \end{cases} .$$

Exercise 4. (8)
Let $f, g; \mathbb{R} \to \mathbb{R}$ be C^2functions. Define

$$F : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto f(x + g(y)).$$

Find the Jacobian and Hessian matrices of F and say if for any $(x, y) \in \mathbb{R}.$ the following equality is true:

$$D_x F(x, y) \cdot D_{x,y} F(x, y) = D_y F(x, y) \cdot D_{x,x} F(x, y).$$

Exercise 5. (8)
Given the utility function

$$u : \mathbb{R}^2_{++} \to \mathbb{R},\ (x_1, x_2) \mapsto (x_1)^2 - \ln(x_2)^3,$$

say if the associated indifference curves are increasing and concave.

Matematica per le applicazioni economiche 2
December 19th, 20112

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (8)
Show that the following set

$$W^{\perp} = \{v \in \mathbb{R}^n : \forall w \in W, \quad v \cdot w = 0\}$$

is a vector subspace of $\mathbb{R}^n$.

Exercise 2. (8)
Let the following objects be given:
the canonical bases $\mathcal{E}_2$ of $\mathbb{R}^2$ and the basis $\mathcal{V} = \{(1,1), (1,-1)\}$ of $\mathbb{R}^2$,
the function $l \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$,

$$l(x_1, x_2) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

1. Compute

$$[l]_{\mathcal{E}_2}^{\mathcal{E}_2} \quad \text{and} \quad [l]_{\mathcal{E}_2}^{\mathcal{V}}$$

2. If possible find a basis of $\ker l$ and $\operatorname{Im} l$.

Exercise 3. (8)
For any $k \in \mathbb{R}$, say if the affine space of solutions to the following system is empty or nonempty; in the latter case, compute its dimension.

$$\begin{bmatrix} k & k-1 \\ 0 & k \\ 2 & 2k-1 \\ 0 & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Exercise 4. (8)
Let the following function be given

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x, y, z) \mapsto \begin{cases} e^x \cos z - y \\ \\ e^x \sin x - y + 1. \end{cases}$$

Showif it is possible to apply the implicit function theorem to f in $(x_0, y_0, z_0) := (0, 1, 0)$ to prove that there exists a C^1 function $(x, y) = g(z) = (g_1(z), g_2(z))$ around $z_0 = 0$ and, if possible, compute $Dg(0)$.

Exercise 5. (8)
Let the following objects be given:
$f : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$, $(k, l, x) \mapsto f(k, l, x)$.
$\gamma : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $l \mapsto \gamma(l)$
$h : \mathbb{R}^2_{++} \mapsto \mathbb{R}_{++}, (l, x) \mapsto f(\gamma(l), l, x)$.
Assuming that for any $(k, l, x) \in \mathbb{R}^3_{++}$, $Df(k, l, x) >> 0$ and $\gamma'(l) > 0$, say if the curve

$$\left\{(l, x) \in \mathbb{R}^2_{++} : h(l, x) - 1 = 0\right\}$$

is decreasing or increasing.

Exercise 6. (8)
Say if the following sets are vector subspaces of $\mathbb{M}(2,2)$.
1. $\left\{A \in \mathbb{M}(2,2) : A = A^T\right\}$;
2. $\left\{A \in \mathbb{M}(2,2) : AA^T = A^T A\right\}$;
3. $\left\{A \in \mathbb{M}(2,2) : A^2 = I\right\}$.

Exercise 7. (8)

Let the following exogenous objects be given.

parameters $b, p_b, p_g, \in \mathbb{R}_{++}$, $w > p_b \cdot b$, and

a $\mathcal{C}^2$ function $u : \mathbb{R}^2_{++} \to \mathbb{R}, (y_1, y_2) \mapsto u(y_1, y_2)$ such that $\forall (y_1, y_2) \in \mathbb{R}^2_{++}$,

$D_{y_1} u(y_1, y_2) > 0$, $D_{y_2} u(y_1, y_2) < 0$,

$D_{y_1 y_1} u(y_1, y_2) < 0$

$D_{y_2 y_2} u(y_1, y_2) < 0$

$D_{y_1 y_1} u(y_1, y_2) \cdot D_{y_2 y_2} u(y_1, y_2) - (D_{y_1 y_2} u(y_1, y_2))^2 > 0$

i. Following the steps presented in the Class Notes, analyze the following problem. For given objects introduced above,

$$\max_{(x_g, x_b) \in \mathbb{R}_{++}} u(x_g, b - x_b) \qquad s.t. \qquad \begin{array}{l} p_g x_g + p_b x_b \leq w, \\ \\ x_g \geq 0 \\ x_b \in [0, b] \end{array}$$

Exercise 8. (2)

In the problem described in the above exercise, restricting your analysis to the set of parameters for which the solution $\left(x^*_{g,} x^*_b\right)$ is such that

$p_g x^*_g - p_b x^*_b = w$ and the associated multiplier is strictly positive,

$x_g > 0$, and

$x_b \in (0, b)$,

describe the procedure to compute the effect of a change of on the solution values.

Sketch of the solutionsolutions
Exercise 1.
1. $\forall w \in W$, $0w = 0$.
2.Assume that $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v^1, v^2 \in W^{\perp}$. Then

$$\forall v \in W, \left(\alpha_1 v^1 + \alpha_2 v^2\right) w = \alpha_1 \left(v^1 w\right) + \alpha_2 \left(v^2 w\right) = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0.$$

Exercise 2.

$$[l]_{\mathcal{E}_2}^{\mathcal{E}_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

1. Compute

$$[l]_{\mathcal{E}_2}^{\mathcal{V}} = \left[\left[l\left(e_2^1\right)\right]_{\mathcal{V}}, \left[l\left(e_2^1\right)\right]\mathcal{V}\right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = I$$

2. $\operatorname{Im} l = \mathbb{R}^2$; $\ker l = \{0\}$.
Exercise 3. (8)
$\det \begin{bmatrix} k & k-1 & 1 \\ 0 & k & 2 \\ 2 & 2k-1 & 3 \end{bmatrix} = 4k - k^2 - 4 = 0$, Solution is: 2
If $k \neq 2$,no solutions.
If $k = 2$

$$\begin{bmatrix} k & k-1 & 1 \\ 0 & k & 2 \\ 2 & 2k-1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}$$

1 solution.
Exercise 4. (8)
Let the following function be given

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x, y, z) \mapsto \begin{cases} e^x \cos z - y \\ \\ e^x \sin x - y + 1. \end{cases}$$

Show that it possible to apply the implicit function theorem to f in $(x_0, y_0, z_0) := (0, 1, 0)$ to prove that there exists a C^1 function $(x, y) = g(z) = (g_1(z), g_2(z))$ around $z_0 = 0$. Compute $Dg(0)$.
The Jacobian of f is

$$\begin{bmatrix} e^x \cos z - y & -1 & -e^x \sin z \\ \\ e^x \sin x + e^x \cos x & 1 & 0 \end{bmatrix}$$

a. f is $\mathcal{C}^1$.
b. $f((x_0, y_0, z_0)) = 0$.
c. BUT

$$D_{x,y} f((x_0, y_0, z_0)) = \begin{bmatrix} e^x \cos z & -1 \\ e^x \sin x + e^x \cos x & 1 \end{bmatrix} = \begin{bmatrix} e^0 \cos 0 & -1 \\ e^0 \sin 0 + e^0 \cos 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

which does not have full rank. Therefore, we cannot apply the implicit function theorem.

Exercise 5. (8)
is decreasing or increasing.

$$\frac{dx}{dl} = -\frac{D_l h(l, x)}{D_x h(l, x)} = -\frac{D_k f(...) \cdot \gamma'(l) + D_k f(...)}{D_x f(...)} < 0.$$

Therefore the curve is decreasing.
Exercise 6. (8)

$$\max_{(x_g,x_b)\in\mathbb{R}} u\left(x_g, b-x_b\right) \qquad s.t. \qquad \begin{array}{l} w - p_g x_g - p_b x_b \geq 0 \\ x_g \geq 0 \\ x_b \geq 0 \\ b - x_b \geq 0 \end{array}$$

...

$$\begin{array}{rcc} & x_g & x_b \\ D_{y_1} u\left(x_g, b-x_b\right) & D_{11} & -D_{12} \\ -D_{y_2} u\left(x_g, b-x_b\right) & -D_{12} & D_{22} \end{array}$$

...

$$\begin{cases} D_{y_1} u\left(x_g, b-x_b\right) + \lambda_1 p_g + \lambda_2 & = 0 \\ -D_{y_2} u\left(x_g, b-x_b\right) + \lambda p_b + \lambda_3 - \lambda_4 & = 0 \\ \min\left\{\lambda_1, w - p_g x_g - p_b x_b\right\} & = 0 \\ \min\left\{\lambda_2, x_g\right\} & = 0 \\ \min\left\{\lambda_3, x_b\right\} & = 0 \\ \min\left\{\lambda_4, b - x_b\right\} & = 0 \end{cases}$$

Matematica per le applicazioni economiche 2
January 16th, 2013

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (6)
Let $\mathcal{V} = \{v^1, v^2\}$ be a basis of a vector space V and $\alpha \in \mathbb{R}\setminus\{0\}$, define $\mathcal{V}_\alpha = \{\alpha v^1, \alpha v^2\}$, which indeed is a basis of V (you do not have to show it). Given $l \in \mathcal{L}(V, V)$, write $[l]_{\mathcal{V}_\alpha}^{\mathcal{V}_\alpha}$ in terms of $[l]_{\mathcal{V}}^{\mathcal{V}}$.
Hint. Call $[l]_{\mathcal{V}}^{\mathcal{V}} := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $l(v^1) = a_{11}v^1 + a_{21}v^2$ and $l(\alpha v^1) = ...$.
Exercise 2. (9)
Given the function $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$l(x) = \left((x_{n-i})_{i=0}^{n-2}, 0 \right),$$

if possible, find a basis of $\ker l$ and $\operatorname{Im} l$.
Exercise 3. (9)
Let $\mathcal{V}_0 := \{v^1, v^2, ...v^n\}$ be a basis of a vector space V. Say if the following statements are true.
i. For any $\beta \in \mathbb{R}$, $\mathcal{V}_1 := \{v^1 + \beta v^2, v^2, ...v^n\}$ is a basis of V.
ii. For any $(\beta_i)_{i=1}^n \in \mathbb{R}^n$, $\mathcal{V}_2 := \{\beta_1 v^1, \beta_2 v^2, ..., \beta_n v^n\}$ is a basis of V.
iii. For any $(\beta_i)_{i=1}^n \in \mathbb{R}^n_{++}$, $\mathcal{V}_3 := \{\beta_1 v^1, \beta_2 v^2, ..., \beta_n v^n\}$ is a basis of V.
Exercise 4. (8)
Let the following function be given

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto 2e^{x+y} - x + y.$$

Show that it is possible to apply the implicit function theorem to $f(x, y) = 0$ in $(x_0, y_0) := (1, -1)$ to prove that there exists a C^1 function $g : B(x_0, \varepsilon) \to B(y_0, \delta)$ such that $f(x, g(x)) = 0$. Compute $g'(x_0)$ e $g''(x_0)$.
Exercise 5. (8)
Complete the following proof.
Let $\mathcal{V} = \{v^1, v^2, ..., v^n\}$ be a set of vectors in $\mathbb{R}^n$ such that for any $i, j \in \{1, ..., n\}$,

$$v^i \cdot v^j = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j. \end{cases} \tag{11}$$

Show that $\mathcal{V}$ is a basis of $\mathbb{R}^n$.
Proof.
From Proposition .... in the Class notes (write the statement:

it suffices to show that $\mathcal{V}$ is linearly independent, i.e., given $(\alpha_i)_{i=1}^n \in \mathbb{R}^n$, if

$$\sum_{i=1}^n \alpha_i v^i = ..., \tag{12}$$

then

....

Now, for any $j \in \{1, ..., n\}$, we have

$$0 \overset{(1)}{=} \left(\sum_{i=1}^n \alpha_i v^i \right) v^j \overset{(2)}{=} \sum_{i=1}^n \alpha_i v^i v^j \overset{(3)}{=} ....,$$

where (1) follows from...,

(2) follows from...,
(3) follows from... .

Exercise 6. (8)

Let the following exogenous variables (or parameters) be given: $s \in (0,1)\,, g, m \in \mathbb{R}_{++}$.
Let the following C^2 functions be given:

$$l : \mathbb{R}^2_{++} \to \mathbb{R}, \quad (y,r) \mapsto l\,(y,r)$$

and

$$i : \mathbb{R}_{++} \to \mathbb{R}, \quad r \mapsto i\,(r)\,.$$

Assume that $\forall\,(y,r) \in \mathbb{R}^2_{++}$, $D_y l\,(y,r) > 0, \quad D_r l\,(y,r) < 0$ and $i'\,(r) < 0$.
Given the following system

$$\begin{cases} s \cdot y - i\,(r) - g & = & 0 \\ \\ l\,(y,r) - m & = & 0 \end{cases}$$

(which is a basic version of the so-called IS-LM model you may have studied in Macroeconomic courses), describe very carefully the procedure to compute the effects of changes in the exogenous variable g on the (endogenous) variable y. You do not need to explicitily compute the inverse of a matrix.

Exercise 7. (8)

Following the steps presented in the Class Notes, analyze the following problem. For given $\alpha \in \mathbb{R}_{++}$,

$$\max_{(x,y)\in(-1,+\infty)^2} \log\,(1+x) + \log\,(1+y) - 10x^2 - y^2 - \alpha x \qquad s.t. \qquad \begin{array}{l} x \in [0,10] \\ \\ y \in [0,1] \end{array}$$

ii. Say if there exists values of $\alpha \in \mathbb{R}_{++}$ such that $(x.y) = (0,1)$ is a solution to the problem.

Sketch of the solutions.

Exercise 1.

Call $[l]_{\mathcal{V}}^{\mathcal{V}} := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $l\left(v^1\right) = a_{11}v^1 + a_{21}v^2$ and $l\left(\alpha v^1\right) = \alpha l\left(v^1\right) = a_{11}\left(\alpha v^1\right) + a_{21}\left(\alpha v^2\right)$; similarly $l\left(\alpha v^2\right) = \alpha l\left(v^2\right) = a_{12}\left(\alpha v^1\right) + a_{22}\left(\alpha v^2\right)$. Then

$$[l]_{\mathcal{V}_\alpha}^{\mathcal{V}_\alpha} := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [l]_{\mathcal{V}}^{\mathcal{V}}.$$

Exercise 2.

	x_1	x_2	x_3	...	x_{n-2}	x_{n-1}	x_n
x_n	0	0	0		0	0	1
x_{n-1}	0	0	0		0	1	0
x_{n-2}	0	0	0		1	0	0
...							
x_3	0	0	1		0	0	0
x_2	0	1	0		0	0	0
0	0	0	0		0	0	0

Then

$$rank\ [l] = n - 1.$$

$\dim \operatorname{Im} l = n-1$. A basis of $\operatorname{Im} l$ is the set of the last $n-1$ columns of $[l]$.
$\dim \ker l = n - (n-1) = 1$ and a basis of $\ker l$ is the vector $(1, 0, ..., 0)$.

Exercise 3.

i. True.

$0 = \alpha_1\left(v^1 + \beta v^2\right) + \alpha_2 v^2 + ... + \alpha_n v^n = \alpha_1 v^1 + (\alpha_2 + \alpha_1\beta)v^2 + ... + \alpha_n v^n \Rightarrow \alpha_1 = \alpha_2 + \alpha_1\beta = \alpha_3 = ... = \alpha_n = 0$. But then $0 = \alpha_2 + \alpha_1\beta = \alpha_2$, as desired.

ii. False.

Take $(\beta_i)_{i=1}^n = 0$.

iii. True.

$0 = \sum_{i=0}^n \alpha_i \beta_i v^i \Rightarrow \forall i \in \{1, ..., n\}, (\alpha_i\beta_i) = 0 \overset{(\beta_i)_{i=1}^n \in \mathbb{R}^n_{++}}{\Rightarrow} \forall i \in \{1, ..., n\}, \alpha_i = 0$

Exercise 4. (8)

solution.

$$g'(x) = -\frac{2e^{x+y} - 1}{2e^{x+y} + 1}$$

$$\begin{aligned} g''(x) &= -\left(\frac{2e^{x+g(x)} - 1}{2e^{x+g(x)} + 1}\right)' = \frac{-1}{\left(2e^{x+g(x)} + 1\right)^2}\left(2g'(x)\,e^{x+g(x)}\left(2e^{x+g(x)} + 1\right) - 2g'(x)\,e^{x+g(x)}\left(2e^{x+g(x)} - 1\right)\right) \\ &= \frac{-4g'(x)\,e^{x+g(x)}}{\left(2e^{x+g(x)} + 1\right)^2} \end{aligned}$$

Exercise 5.

Complete the following proof.

Let $\mathcal{V} = \left\{v^1, v^2, ..., v^n\right\}$ be a set of vectors in $\mathbb{R}^n$ such that for any $i, j \in \{1, ..., n\}$,

$$v^i \cdot v^j = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j. \end{cases} \tag{13}$$

Show that $\mathcal{V}$ is a basis of $\mathbb{R}^n$.

Proof.

From Proposition ??? in the Class notes (write the statement:

Proposition 4 *Let V be a vector space of dimension n.*

1. ...

2. *If* $S = \{u^1, ..., u^n\} \subseteq V$ *is a linearly independent set, then it is a basis of* V*;*

3. *If* span $(u^1, ..., u^n) = V$*, then* $\{u^1, ..., u^n\}$ *is a basis of* V*.*

Remark 5 *The above Proposition 4 shows that in the case of finite dimensional vector spaces, one of the two conditions defining a basis is sufficient to obtain a basis.*
)

it suffices to show that $\mathcal{V}$ is linearly independent, i.e., given $(\alpha_i)_{i=1}^n \in \mathbb{R}^n$, if

$$\sum_{i=1}^n \alpha_i v^i = \boxed{0}, \tag{14}$$

then

$$\boxed{(\alpha_i)_{i=1}^n = 0.}$$

Now, for any $j \in \{1, ..., n\}$, we have

$$0 \overset{(1)}{=} \left(\sum_{i=1}^n \alpha_i v^i\right) v^j \overset{(2)}{=} \sum_{i=1}^n \alpha_i v^i v^j \overset{(3)}{=} \alpha_j,$$

where (1) follows from $\boxed{(14)}$;
(2) follows from $\boxed{\text{properties of the scalar product}}$;
(3) follows from $\boxed{(13)}$.

Exercise 6. (8)

	y	r	g
$s \cdot y - i(r) - g$	s	$-i'(r)$	-1
$l(y,r) - m$	$\frac{\partial l(y,r)}{\partial y}$	$\frac{\partial l(y,r)}{\partial r}$	0

$\det D_{(y,r)} F(...) = s\frac{\partial l(y,r)}{\partial r} + \frac{\partial l(y,r)}{\partial y} i'(r) < 0.$

Exercise 7.

$$\max_{(x,y)\in(-1,+\infty)^2} \quad \log(1+x) + \log(1+y) - 10x^2 - y^2 - \alpha y \qquad s.t. \qquad \begin{array}{l} x \geq 0 \\ 10 - x \geq 0 \\ y \geq 0 \\ 1 - y \geq 0 \end{array}$$

$$\begin{cases} \frac{1}{1+x} - 20x - \alpha + \lambda_0 - \lambda_1 & = & 0 \\ \frac{1}{1+y} - 2y + \mu_0 - \mu_1 & = & 0 \\ \min\{x, \lambda_0\} & = & 0 \\ \min\{10 - x, \lambda_1\} & = & 0 \\ \min\{y, \mu_0\} & = & 0 \\ \min\{1 - y, \mu_1\} & = & 0 \end{cases}$$

We must have

$$\begin{cases} \min\{0, \lambda_0\} & = & 0 \\ \min\{10, \lambda_1\} & = & 0 \\ \min\{1, \mu_0\} & = & 0 \\ \min\{0, \mu_1\} & = & 0 \end{cases} \text{ or } \begin{cases} \lambda_0 & \geq & 0 \\ \lambda_1 & = & 0 \\ \mu_0 & = & 0 \\ \mu_1 & \geq & 0 \end{cases} \text{ and therefore } \begin{cases} \frac{1}{1} - \alpha + \lambda_0 & = & 0 \\ \frac{1}{2} - 2 - \mu_1 & = & 0 \end{cases}.$$

Then, $\mu_1 = \frac{1}{2} - 2 < 0$ and there is no solution to the system.

Matematica per le applicazioni economiche 2
February 20th, 2013

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (6)
1. Take for granted that $\mathbb{M}(2,2)$ is a vectors space of dimension 4 (Just assume it). Show that a basis of $\mathbb{M}(2,2)$ is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

2. Show the set of symmetric 2×2 matrices, i.e., the set

$$\mathbb{S} = \left\{ A \in \mathbb{M}(2,2) : \exists a, b, c \in \mathbb{R} \text{ such that } A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right\}$$

is a vector subspace of $\mathbb{M}(2,2)$
3. Compute $\dim \mathbb{S}$.
Exercise 2. (9)
Given the function $l \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3)$ such that

$$l((x_1, x_2, x_3, x_4)) = \begin{pmatrix} x_1 - x_2 + x_3 + x_4 \\ x_1 + 2x_3 - x_4 \\ x_1 + x_2 + 3x_3 - 3x_4 \end{pmatrix},$$

if possible, find a basis of $\ker l$ and $\operatorname{Im} l$.
Exercise 3. (9)
Let $n \in \mathbb{N}_+$ and $A \in \mathbb{M}(n,n)$ be given. Show that the following function is linear

$$l : \mathbb{M}(n,n) \to \mathbb{M}(n,n),$$

$$l : M \mapsto AM + MA$$

Exercise 4. (8)
Let the following function be given

$$f : \mathbb{R}^2_{++} \to \mathbb{R}^2, \qquad (x,y) \mapsto \begin{pmatrix} x^2 e^{2y} \\ \log(xy) \end{pmatrix}.$$

a. Say why the function f admits partial derivative in $(1,1)$ in the direction $(2,2)$.
b. Compute it.
Exercise 5. (8)
Describe the set of solution of the system below for each value of $a, b \in \mathbb{R}$:

$$\begin{bmatrix} a & b & a & b \\ b & a & b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Exercise 6. (8)
Let the following utility function be given.

$$u : \mathbb{R}^2_{++} \to \mathbb{R}, \quad (x,y) \mapsto u(x,y),$$

where x is a good and y a bad. Assume that u is $\mathcal{C}^2$ and that $\forall (x,y) \in \mathbb{R}^2_{++}$, $D_x u(x,y) > 0$ and $D_y u(x,y) < 0$ and $D^2 u(x,y) < 0$.
If possible, say if associated indifference curves are increasing and convex.
Recall that given $u : \mathbb{R}^2_{++} \to \mathbb{R}, \quad (x,y) \mapsto u(x,y)$, $\frac{du(x,g(x))}{dx} = D_x u(x, g(x)) + D_y u(x, g(x)) \cdot g'(x)$.
Exercise 7. (8)

a. Following the steps presented in the Class Notes, analyze the following problem. Let the following objects be given:

$p, w \in \mathbb{R}_{++}$,

a $\mathcal{C}^2$ function $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto g(x, y)$ such that $\forall (x, y) \in \mathbb{R}^2$, $Dg(x, y) < 0$ and $D_y g(x, y) < 0$ and $D^2 u(x, y)$ is negative definite.

$$\max_{(x,y)\in\mathbb{R}^2} py - wx \qquad s.t. \qquad g(x, y) \geq 0.$$

Do not study the problem of existence of a solution.

b. In the case in which the multiplier associated with the constraint is strictly positive, describe the strategy to compute the effect of changes in p and w on the solution (x^*, y^*) of the problem.

Sketch of the solutions.

Exercise 1. (6)

1. Define
$$B=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}.$$

From Proposition ....,it suffices to show that either B is linearly independent or span $B=\mathbb{M}(2,2)$.

2.

...

3.

It suffices to show that
$$B'=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$
is a basis of $\mathbb{S}$.

Exercise 2. (9)

i.
$$[l]=\begin{pmatrix}1&-1&1&1\\1&0&2&-1\\1&1&3&-3\end{pmatrix},$$

$$\operatorname{rank}\begin{pmatrix}1&-1&1&1\\1&0&2&-1\\1&1&3&-3\end{pmatrix}=\operatorname{rank}\begin{pmatrix}1&-1&1&1\\0&1&1&-2\\0&2&2&-4\end{pmatrix}=\operatorname{rank}\begin{pmatrix}1&-1&1&1\\0&1&1&-2\\0&0&0&0\end{pmatrix}\tag{15}$$

Then,

$\operatorname{rank}[l]=2=\dim\operatorname{Im}l$. Since $\det\begin{bmatrix}1&-1\\1&0\end{bmatrix}=1$, thena basis of $\operatorname{Im}l$ is the set of the first two columns of $[l]$.

$\dim\ker l=4-2=2$

From (15), a basis of $\ker l$ is obtained solving the system
$$\begin{cases}x_1-x_2=-x_3-x_4\\x_2=-x_3+2x_4\end{cases}$$

Choosen $(x_3,x_4)=(1,0)$, we have $x_2=-1$ and $x_1=-x_3-x_4+x_2=-1-0-1=-2$; choosen $(x_3,x_4)=(0,1)$, we have $x_2=2$ and $x_1=-x_3-x_4+x_2=0-1+2=1$

Then, a basis is
$$\left\{\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}1\\2\\0\\1\end{bmatrix}\right\}$$

Exercise 3. (9)

We want to show that $\forall\alpha,\beta$, and $\forall M,N\in\mathbb{M}(n,n)$, $l(\alpha M+\beta N)=\alpha l(M)+\beta l(N)$.

$l(\alpha M+\beta N)=A(\alpha M+\beta N)+(\alpha M+\beta N)A=\alpha AM+\beta AN+\alpha MA+\beta NA$

$l(M)=AM+MA$

$l(N)=AN+NA$

$\alpha l(M)+\beta l(N)=\alpha(AM+MA)+\beta(AN+NA)=\alpha AM+\alpha MA+\beta AN+\beta NA.$

Exercise 4. (8)
Let the following function be given

$$f : \mathbb{R}^2_{++} \to \mathbb{R}^2, \qquad (x, y) \mapsto \begin{pmatrix} x^2 e^{2y} \\ \log(xy) \end{pmatrix}.$$

a. Say why the function f admits partial derivative in $(1, 1)$ in the direction $(2, 2)$.
b. Compute it.
a.
The Jacobian is:

$$\begin{array}{ccc} & x & y \\ x^2 e^{2y} & 2xe^{2y} & 2x^2 e^{2y} \\ \log(xy) & \frac{1}{x} & \frac{1}{y} \end{array}$$

and therefore the function is C^1 :
b

$$f'((1,1);(2,2)) = \begin{bmatrix} 2xe^{2y} & 2x^2 e^{2y} \\ \frac{1}{x} & \frac{1}{y} \end{bmatrix}_{|(1,1)} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^2 & 2e \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4e^2 + 4e \\ 4 \end{bmatrix}$$

Exercise 5. (8)
Describe the set of solution of the system below for each value of $a, b \in \mathbb{R}$:

$$\begin{bmatrix} a & b & a & b \\ b & a & b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Since

$$\det \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a^2 - b^2 = (a+b)(a-b),$$

if

$$\langle a \neq b \rangle \wedge (a \neq -b)$$

the system admits solutions, and the affine space of solutions has dimension $4 - 2 = 2$.
Let's now consider the case $\neg (\langle a \neq b \rangle \wedge (a \neq -b)) = \langle a = b \rangle \vee (a = -b)$.
Case 1. $a = b$.
The system becomes

$$\begin{bmatrix} b & b & b & b \\ b & b & b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} b & b & b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b$$

If $b \neq 0$, the affine space of solutions has dimension 3.
If $b \neq 0$, the affine space of solutions has dimension 4.
Let's now consider the case $\neg (\langle a \neq b \rangle \wedge (a \neq -b)) = \langle a = b \rangle \vee (a = -b)$.
Case 2. $a = -b$.
The system becomes

$$-\begin{bmatrix} b & b & b & b \\ b & b & b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = -\begin{bmatrix} b \\ b \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} b & b & b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b$$

If $b \neq 0$, the affine space of solutions has dimension 3.
If $b \neq 0$, the affine space of solutions has dimension 4.
Summarizing, ...

Exercise 6. (8)

$$\frac{dy}{dx} = -\frac{D_x u\,(x,y)}{D_y u\,(x,y)} > 0.$$

$\frac{d^2y}{dx^2} = -\frac{d\left(\frac{D_x u(x,y(x))}{D_y u(x,y(x))}\right)}{dx} =$

$= -\frac{\left(D_{xx}u(x,y(x))+D_{xy}u(x,y(x))\cdot y'(x)\right)\cdot D_y u(x,y(x))-\left(D_{yx}u(x,y(x))+D_{yy}u(x,y(x))\cdot y'(x)\right)\cdot D_x u(x,y(x))}{(D_y u(x,y(x)))^2}$

Exercise 7. (8)

...

Matematica per le applicazioni economiche 2 - February 20th, 2013

Modalità 2

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Let $k, n \in \mathbb{N}_+$ and $l : \mathbb{R}^{n+k} \to \mathbb{R}^n$,

$$l\left((x_i)_{i=1}^{n+k}\right) = 2 \cdot (x_i)_{i=1}^{n}$$

be given. Find a basis of $\ker l$ and $\operatorname{Im} l$.
Exercise 2. (5)
Describe the cardinalities of the sets of solutions to the system below for each value of $a \in \mathbb{R}$:

$$\begin{bmatrix} a & 1 & a & 1 & a \\ 1 & a & 1 & a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Exercise 3. (4)
Given the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{x^2+y}{y+x} & \text{if} \quad y \neq -x \\ \\ 0 & \text{if} \quad y = -x \end{cases},$$

if possible, compute $f'((1,-1);(1,1))$, i.e., the directional derivative ín the point $(1,-1)$ in the point $(1,1)$.
Exercise 4. (5)
Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

the basis $\mathcal{U} = \{(1,0),(1,1)\}$ of $\mathbb{R}^2$, and the canonical basis $\mathcal{E}_2 := \{(1,0),(0,1)\}$ of $\mathbb{R}^2$, using the definition presented in the class notes, for any $x \in \mathbb{R}^2$, compute

$$l_{A,\mathcal{E}_2}^{\mathcal{U}}(x).$$

Exercise 5. (5)
Let the following function be given

$$f : \mathbb{R}^4_{++} \to \mathbb{R}^2, \qquad (x,y,u,v) \mapsto \begin{pmatrix} e^{x^2} - y^2 - u^3 + v^2 - e^4 + 8 \\ \\ \ln\left(y^2\right) - 2u^2 + 3v^4 + 5 \end{pmatrix}$$

i. Say if it is possible to apply the implicit function theorem to f in $(x_0,y_0,u_0,v_0) = (2,-1,2,1)$ to prove that there exists a C^1 function $(x,y) = g(u,v)$ in an open neighborhood of $(u_0,v_0) = (2,1)$.
ii. Write the formula to compute $D_{(u,v)}g((2,1))$.
Do not miss the exercise in next page.

Exercise 6. (6)

i. Following the steps presented in the Class Notes, analyze the following problem. For given $a, b, q \in \mathbb{R}_{++}$,

$$\min_{(x,y)\in\mathbb{R}^2} ax^2 + by^2 \qquad s.t. \qquad \begin{aligned} &x + y \leq q, \\ &x \geq 0, \\ &y \geq 0. \end{aligned}$$

ii. If possible, find an explicit solution for (x, y) and compute the effect of a change in a on the solution value of x.

Matematica per le applicazioni economiche 2 - February 20th, 2013

$\boxed{\text{Modalità 2}}$

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (4)

Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Say if the following sets are subspaces of V.

a. $W_1 = \{f \in V : f(1) = f(2) = 0\}$,

b. $W_2 = \{f \in V : f(0) = 0 \text{ and for any } x \in \mathbb{R},\ f'(x) = x\}$.

Exercise 2. (6)

Consider the function $l_a \in \mathcal{L}\left(\mathbb{R}^3, \mathbb{R}^3\right)$ such that

$$l_a(x) = M(a) \cdot x,$$

where

$$M(a) = \begin{bmatrix} a & 0 & 1 \\ a & 1 & 0 \\ 1 & 0 & a \end{bmatrix}$$

and $a \in \mathbb{R}$. For any $a \in \mathbb{R}$, find a basis of $\ker l_a$ and $\operatorname{Im} l_a$.

Exercise 3. (4)

Show the following result.

Given $x, y \in \mathbb{R}^n$, if $x \neq 0, y \neq 0$ and $xy = 0$, then $\{x, y\}$ is linearly independent.

Hint. You want to show that $\alpha x + \beta y = 0 \Rightarrow \alpha = \beta = 0$. Multiply ... by ... and ...; use the fact that $xy = 0$ ... .

Exercise 4. (5)

Given

$$T : \mathbb{R}^2 \to \mathbb{R}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the basis $\mathcal{B} = \{(1,1), 1, 2\}$ of $\mathbb{R}^2$, compute $[T]_{\mathcal{B}}^{\mathcal{B}}$.

Exercise 5. (5)

Let the following function be given

$$f : \mathbb{R}^4 \to \mathbb{R}^2, \qquad (x, y, t, z) \mapsto \begin{cases} -3 + (x+y)\,t + z \\ 2t - (1+x) \cdot e^{t(x-1)} \end{cases}$$

i. Say if it is possible to apply the implicit function theorem to f in $(x_0, y_0, t_0, z_0) := (1,1,1,1)$ to prove that there exists a C^1 function $(x, y) = g(t, z)$in an open neighborhood of $(t_0, z_0) = (1, 1)$.

ii. Write the formula to compute $D_{(t,z)}g(1,1) \in \mathbb{M}(2,2)$.

Do not miss the exercise in next page.

Exercise 6. (6)
Following the steps presented in the Class Notes, analyze the following problem.

$$\min_{(x,y,z)\in\mathbb{R}^3} \frac{1}{2}\left(x^2+y^2+z^2\right) \qquad s.t. \qquad x+y+z\leq -3,$$

Matematica per le applicazioni economiche 2 - September, 2013

$\boxed{\text{Modalità 2}}$

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (4)
Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Say if the following sets are subspaces of V.
a. $W_1 = \{f \in V : f(1) = f(2) = 0\}$,
b. $W_2 = \{f \in V : f(0) = 0 \text{ and for any } x \in \mathbb{R},\ f'(x) = x\}$.
Exercise 2. (6)
Consider the function $l_a \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$l_a(x) = M(a) \cdot x,$$

where

$$M(a) = \begin{bmatrix} a & 0 & 1 \\ a & 1 & 0 \\ 1 & 0 & a \end{bmatrix}$$

and $a \in \mathbb{R}$. For any $a \in \mathbb{R}$, find a basis of $\ker l_a$ and $\operatorname{Im} l_a$.
Exercise 3. (4)
Show the following result.
Given $x, y \in \mathbb{R}^n$, if $x \neq 0, y \neq 0$ and $xy = 0$, then $\{x, y\}$ is linearly independent.
Hint. You want to show that $\alpha x + \beta y = 0 \Rightarrow \alpha = \beta = 0$. Multiply ... by ... and ...; use the fact that $xy = 0$ ... .
Exercise 4. (5)
Given

$$T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the basis $\mathcal{B} = \{(1, 1), 1, 2\}$ of $\mathbb{R}^2$, compute $[T]_{\mathcal{B}}^{\mathcal{B}}$.
Exercise 5. (5)
Let the following function be given

$$f : \mathbb{R}^4 \to \mathbb{R}^2, \qquad (x, y, t, z) \mapsto \begin{cases} -3 + (x + y)t + z \\ \\ 2t - (1 + x) \cdot e^{t(x-1)} \end{cases}$$

i. Say if it is possible to apply the implicit function theorem to f in $(x_0, y_0, t_0, z_0) := (1, 1, 1, 1)$ to prove that there exists a C^1 function $(x, y) = g(t, z)$in an open neighborhood of $(t_0, z_0) = (1, 1)$.
ii. Write the formula to compute $D_{(t,z)}g(1, 1) \in \mathbb{M}(2, 2)$.

Do not miss the exercise in next page.

Exercise 6. (6)
Following the steps presented in the Class Notes, analyze the following problem.

$$\min_{(x,y,z)\in\mathbb{R}^3} \frac{1}{2}\left(x^2+y^2+z^2\right) \qquad s.t. \qquad x+y+z \leq -3,$$

Matematica per le applicazioni economiche 2

$\boxed{\textbf{Modalita' 2}}$

December 16th, 2013

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)

a. Show that the set of concave functions from $\mathbb{R}$ to $\mathbb{R}$ is not a vector subspace of the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$.

b. Show that the following set is a vector subspace of $\mathbb{M}(2,2)$:

$$W := \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2,2) : a + b = c + d \right\}.$$

Exercise 2. (5)

For any $a \in \mathbb{R}$, consider the function $l_a \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ such that

$$l_a(x_1, x_2, x_3) = \begin{pmatrix} (a-1)x_1 + (a-1)x_2 \\ \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

For any value of $a \in \mathbb{R}$, find a basis of $\operatorname{Im} l_a$ and $\ker l_a$.

Exercise 3. (4)

Let V be a vector space. Show that if $\{u, v\} \subseteq V$ is linearly independent, then $\{u + v, u - v\}$ is linearly independent as well.

Exercise 4. (6)

Let the following objects be given:

$u : \mathbb{R}^3 \to \mathbb{R}_{++}$, $(x_1, x_2, x_3) \mapsto u(x_1, x_2, x_3)$,

$g_1 : \mathbb{R} \to \mathbb{R}$, $l \mapsto g_1(l)$,

$g_2 : \mathbb{R} \to \mathbb{R}$, $l \mapsto g_2(l)$,

$L \in \mathbb{R}_{++}$.

Assume that

u, g_1, g_2 are C^2,

for any $(x_1, x_2, x_3) \in \mathbb{R}^3$, $D_{x_1}u(x_1, x_2, x_3) > 0, D_{x_2}u(x_1, x_2, x_3) > 0$, $D_{x_3}u(x_1, x_2, x_3) < 0$ and $D^2u(x_1, x_2, x_3) << 0$

for any $l \in \mathbb{R}$, $g_1'(l) > 0, g_1''(l) < 0$, $g_2'(l) > 0, g_2''(l) < 0$.

Following the steps presented in the class notes, discuss the following problem.

$$\max_{l \in \mathbb{R}} \quad u(g_1(l), g_2(l), l) \qquad\qquad s.t. \qquad\qquad l \in [0, L].$$

Exercise 5. (5)

Let the following utility function be given.

$$u : \mathbb{R}^2_{++} \to \mathbb{R}, \quad (x, l) \mapsto \log(x) - \log(l).$$

If possible, say if associated indifference curves are increasing and convex.

Do not miss Exercise 6 on next page.

Solve only one between Exercises 6a and 6b.
Exercise 6a. (5)
a. Show that the following set is $\mathbb{R}^2$ open

$$\left\{(x,y) \in \mathbb{R}^2 : x^2 + \sin(xy) + e^{x+y^2} > 0\right\};$$

b. Show that $(0,1) \cup \{3\}$ is not $\mathbb{R}$ open.

Exercise 6b. (5)
Given the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{x^2}{y+2x} & \text{if} \quad y \neq -2x \\ \\ x+y & \text{if} \quad y = -2x \end{cases},$$

if possible, compute $f'(0;u)$ for any $u \in \mathbb{R}^2$.

Matematica per le applicazioni economiche 2
$\boxed{\textbf{Modalita' 2}}$
January 16th, 2013

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
a. Let be $x_0 \in \mathbb{R}^n \setminus \{0\}$be given. Show that the following set is not a vector space

$$W_{x_0} := \{x \in \mathbb{R}^n : x \cdot x_0 \geq 0\}.$$

Hint: Does x_0 belong to W_{x_0}?
b. Let V and U be vector spaces and $v_0 \in V$. Show that the following set is a vector subspace of $\mathcal{L}(V, U)$:

$$L_0 = \{l \in \mathcal{L}(V, U) : l(v_0) = 0\}.$$

Exercise 2. (5)
For any $a \in \mathbb{R}$, consider the function $l_a \in \mathcal{L}\left(\mathbb{R}^4, \mathbb{R}^2\right)$ such that

$$l_a(x_1, x_2, x_3, x_4) = \begin{pmatrix} ax_1 + x_2 + ax_3 + x_4 \\ \\ x_1 + ax_2 + x_3 + ax_4 \end{pmatrix}.$$

For any value of $a \in \mathbb{R}$, find a basis of $\operatorname{Im} l_a$ and $\ker l_a$.

Exercise 3. (4)
Let a vector space V of dimension $n \in \mathbb{N}_+, n \geq 5$ and $\left\{v^1, v^2\right\}$ be given. Say if the following statements are true or false:
a. There exists $\left\{v^3, ..., v^n\right\} \subseteq V$ such that $\left\{v^1, v^2, v^3, ..., v^n\right\}$ is a basis of V;
b. There exists $\left\{v^3, ..., v^n, v^{n+1}\right\} \subseteq V$ such that $\left\{v^1, v^2, v^3, ..., v^{n+1}\right\}$ is a basis of V.

Exercise 4. (6)
Using the steps described in the Class Notes, discuss the following maximization problem.
For given $\alpha \in \mathbb{R}_{++}$,

$$\max_{(x,y)\in\mathbb{R}^2_{++}} \alpha(\log(x) + \log(y)) - x^2 - y^2 \quad s.t. \quad \begin{array}{l} x \geq 1 \\ y \geq 1 \\ x + y \leq 10 \end{array}$$

Exercise 5. (5)
Let the following function be given.

$$F : \mathbb{R}^3_{++} \to \mathbb{R}^2, \quad (x, y, t) \mapsto \begin{cases} f(x, y, t) - 1 \\ \\ x^2 + e^{xy} - \log(xy) + t^2 - (1 + e) \end{cases},$$

where $f : \mathbb{R}^3_{++} \to \mathbb{R}$ is a C^1 function such that $f(1, 1, 0) = 1$ and $Df(1, 1, 0) = (1, 1, 1)$.
Say if the solution vector (x, y) to the system $F(x, y, t) = 0$ is a C^1function in a neighborhood of $t = 0$ and compute the derivative of that function (with respect to t). Hint: Apply the implicit function theorem to $F(x, y, t) = 0$ at the point $(1, 1, 0)$.

Solve only one between Exercises 6a and 6b.
Exercise 6a. (5)
Using
a. the characterization of continuous functions in terms of open sets,
b. the characterization of continuous functions in terms of closed sets,

show that the following function is not continuous.

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} |x| & \text{if} \quad x \neq 0, \\ 2 & \text{if} \quad x = 0. \end{cases}$$

Exercise 6b. (5)
Given the function

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 + y^2,$$

using the definition show that f is differentiable in $(1, 0)$.
Hints: use the definition which involves $\lim_{u \to 0} \frac{f(x_0+u)-f(x_0)-df_{x_0}(u)}{\|u\|} = 0$; "guess" that $df_{(1,0)}(u) = Df(1,0) \cdot (u_1, u_2) = 2u_1$; recall that $\|(u_1, u_2)\|^2 = (u_1)^2 + (u_2)^2$.

Proof. Take $v \in V$. Then

1. $(g_1 \circ (f_1 + f_2))(v) = g_1((f_1 + f_2)(v)) = g_1(f_1(v) + f_2(v)) \overset{g_1\ linear}{=} g_1(f_1(v)) + g_1(f_2(v)) =$
   $= (g_1 \circ f_1)(v) + (g_1 \circ f_2)(v) = (g_1 \circ f_1 + g_1 \circ f_2)(v)$.
2. $((g_1 + g_2) \circ f_1)(v) = (g_1 + g_2)(f_1(v)) = g_1(f_1(v)) + g_2(f_1(v)) = (g_1 \circ f_1)(v) + (g_2 \circ f_1)(v) = (g_1 \circ f_1 + g_2 \circ f_1)(v)$.
3. $(k(g_1 \circ f_1))(v) = k(g_1 \circ f_1)(v) = k(g_1(f_1(v))) = (kg_1)(f_1(v)) = ((kg_1) \circ f_1)(v)$, and
   $(k(g_1 \circ f_1))(v) = k(g_1 \circ f_1)(v) = k(g_1(f_1(v))) = g_1(kf_1(v)) = g_1((kf_1)(v)) = (g_1 \circ (kf_1))(v)$.

■

6b.

$$\lim_{u \to 0} \frac{f(x_0 + u) - f(x_0) - df_{x_0}(u)}{\|u\|} = \lim_{u \to 0} \frac{f(1 + u_1, u_2) - f(1, 0) - Df(1, 0) \cdot (u_1, u_2)}{\|(u_1, u_2)\|^2} =$$

$$= \lim_{u \to 0} \frac{1 + 2u_1 + u_1^2 + u_2^2 - 1 - 2u_1}{\|(u_1, u_2)\|} = \lim_{u \to 0} \frac{u_1^2 + u_2^2}{\|(u_1, u_2)\|} = \lim_{u \to 0} \frac{\|(u_1, u_2)\|^2}{\|(u_1, u_2)\|} = \lim_{u \to 0} \|(u_1, u_2)\| = 0.$$

Matematica per le applicazioni economiche 2
$\boxed{\text{Modalita' 1}}$
February 4th, 2014

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Let V be a vector space be given.
a. Show that for any $w_0 \in V \setminus \{0\}$,

$$W_{w_0} := \{v \in V : \exists \alpha \in \mathbb{R} \text{ such that } v = \alpha w_0\}$$

is a vector subspace of V;
b. Show that $\dim W_{w_0} = 1$. Hint. Show that $\{w_0\}$ is a basis for W_{w_0}.

Exercise 2. (5)
For any $a, b \in \mathbb{R}$, consider the function $l_{a,b} \in \mathcal{L}\left(\mathbb{R}^4, \mathbb{R}^4\right)$ such that

$$l_{a,b}(x_1, x_2, x_3, x_4) = \begin{pmatrix} ax_1 \\ ax_2 \\ bx_3 \\ bx_4 \end{pmatrix}.$$

For any value of $a \in \mathbb{R}$ and $b \in \mathbb{R}$, find a basis of $\operatorname{Im} l_{a,b}$ and $\ker l_{a,b}$.
Hint. Consider the four case: i. $a \neq 0, b \neq 0$; ii. $a = 0, b = 0$; iii. $a \neq 0, b = 0$; iv. ... .

Exercise 3. (5)
a. Show that the following statement is false: given a matrix $A \in \mathbb{M}(2,2)$,

$$A \neq 0 \Rightarrow A^2 \neq 0.$$

Hint: take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

b. Show that for any $A \in \mathbb{M}(n,n)$ such that A^2, the system $Ax = 0$ has a solution $x^* \neq 0$.
c. Show that for any $A \in \mathbb{M}(n,n)$ such that A^2, rank $A < n$.

Exercise 4. (5)
Using the steps described in the Class Notes, discuss the following maximization problem. For given $p_1, p_2 \in \mathbb{R}_{++}$,

$$\max_{(x_1,x_2)\in\mathbb{R}^2} \; p_1x_1 + p_2x_2 \quad s.t. \qquad x_2 \leq -e^{x_1} + 1.$$

Do not study the existence problem. **Do not** compute solutions.

Exercise 5. (5)
Let the following function be given.

$$f:\mathbb{R}^3_{++}\to\mathbb{R},\quad (x,y,z)\mapsto e^{xyz}+\log(xy)+z-e-1,$$

Say if
a. a solution x to the system $f(x,y,t)=0$ is a C^1 function $g(y,z)$ in a neighborhood of $(y,z)=(1,1)$
and compute $D_{(y,z)}g(1,1)$;
b. a solution y to the system $f(x,y,t)=0$ is a C^1 function $h(x,z)$ in a neighborhood of $(x,z)=(1,1)$
and compute $D_{(x,z)}h(1,1)$.
Hint: Apply the implicit function theorem to $f(x,y,z)=0$ at the point $(1,1,1)$.

Solve only one between Exercises 6a and 6b.

Exercise 6a. (5)
Let $a\in\mathbb{R}_{++}$ be given.
a. Show that $\cap_{n=1}^{+\infty}\left(a-\frac{1}{n},a+\frac{1}{n}\right)=\{a\}$;
b. Compute $\text{Int}(\{a\})$, $\mathcal{F}(\{a\})$ and $D(\{a\})$, *without proving your statements.*

Exercise 6b. (5)
A function $f:\mathbb{R}^2\to\mathbb{R},\ x\mapsto f(x)$ is homogenous of degree $n\in\mathbb{N}_+$ if

$$\text{for every } x=(x_1,x_2)\in\mathbb{R}^2 \text{ and every } a\in\mathbb{R}_+,\ f(ax_1,ax_2)=a^n f(x_1,x_2).$$

Show that if f is homogenous of degree n and $f\in C^1\left(\mathbb{R}^2,\mathbb{R}\right)$, then

$$\text{for every } x=(x_1,x_2)\in\mathbb{R}^2,\qquad x_1\cdot D_{x_1}f(x_1,x_2)+x_2\cdot D_{x_2}f(x_1,x_2)=nf(x_1,x_2).$$

Hint. Differentiate both sides of

$$f(ax_1,ax_2)=a^n f(x_1,x_2)$$

with respect to a and then replace a with 1.

Lemma 6 *Let $v, v' \in V$. Then*

i. $v \in W \Leftrightarrow v + W = W$, *and*

ii. $v + W = v' + W \underset{\not\Rightarrow}{\Leftarrow} v, v' \in W$.

Proof.

i. $[\Rightarrow]$ We want to show that if $v \in W$, then a) $v + W \subseteq W$ and b) $W \subseteq v + W$.

a) Since $v \in W$ and W is a vector space, for any $w' \in W$, $v + w' \in W$, i.e., the desired result.

b) Take $w \in W$. We want to find $w' \in W$ such that $w = v + w'$. Take $w' = w - v$; $w' \in W$, since $w, v \in W$. Then

$$w = v + w' = v + (w - v) \in v + W,$$

as desired.

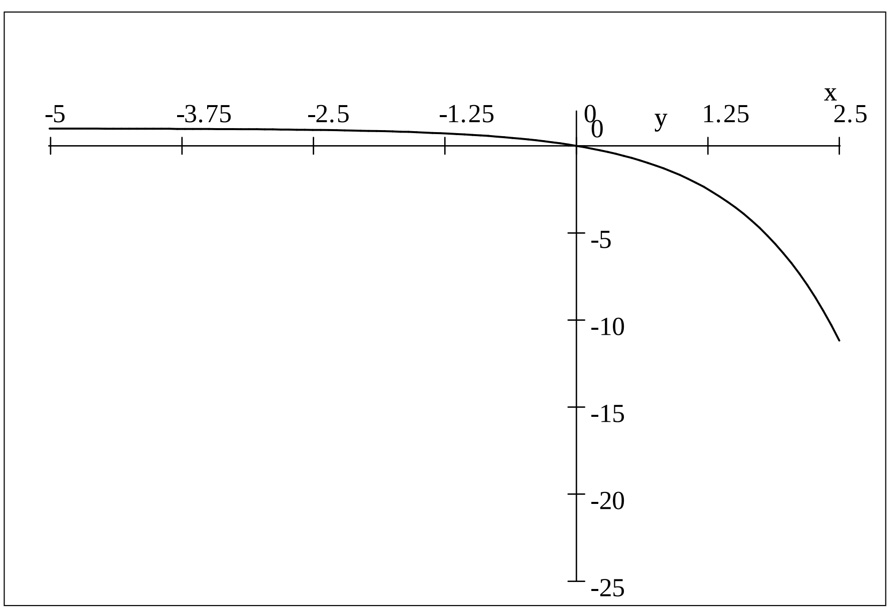
$[\Leftarrow]$ Since $0 \in W$, $v + 0 \in v + W = W$ by assumption.

■

Exercise 5. (5)
Let the following function be given.

$$f : \mathbb{R}^3_{++} \to \mathbb{R}, \quad (x, y, z) \mapsto e^{xyz} + \log(xy) + z - e - 1,$$

$f(x, y, z) = e^{xyz} + \log(xy) + z - e - 1$
$f(1, 1, 1) = 0$
$D_x f(x, y, z) = yze^{xyz} + \frac{1}{x}$
$D_y f(x, y, z) = xze^{xyz} + \frac{1}{y}$
$D_z f(x, y, z) = xye^{xyz} + 1$
$D_{(y,z)} g(1, 1) = -\frac{1}{yze^{xyz} + \frac{1}{x}} \left(xze^{xyz} + \frac{1}{y}, xye^{xyz} + 1 \right)$
$D_{(x,z)} h(1, 1) = -\frac{1}{xze^{xyz} + \frac{1}{y}} \left(yze^{xyz} + \frac{1}{x}, xye^{xyz} + 1 \right)$
$-e^x + 1$



Matematica per le applicazioni economiche 2

$\boxed{\textbf{Modalita' 2}}$

June 17th, 2014

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
a. Find a basis of $\mathbb{R}^3$which contains the following set

$$S = \{(1,2,5), (0,1,2)\}.$$

b. Say if the following set is a vector subspace of $\mathbb{M}(n,n)$:

$$S = \left\{ A := [a_{ij}]_{i,j \in \{1,\dots,n\}} \in \mathbb{M}(n,n) : \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = 0 \right\}.$$

Exercise 2. (5)
Find a basis of $\operatorname{Im} l$ and $\ker l$ if

$$l : \mathbb{R}^{k+1} \to \mathbb{R}^{k-1}, \qquad (x_1, x_2, \dots x_{k-1}, x_k, x_{k+1}) \mapsto (x_1, x_2, \dots x_{k-2}, 0).$$

Exercise 3. (5)
Given the set

$$W = \left\{ (x,y,z) \in \mathbb{R}^3 : x = y \right\},$$

a. show that W is a vector space,
b. find a basis of W,
c. say what is the dimension of W.

Exercise 4. (5)
Using the steps described in the Class Notes, discuss the following maximization problem.
For given $\alpha \in (0,1)$,

$$\begin{array}{llll} \min_{(x_1,x_2)\in\mathbb{R}^2} \; x_1 + \alpha x_2 & \text{s.t.} & x_1 \cdot x_2 & \geq 100 \\ & & x_1 & \geq 0 \\ & & x_2 & \geq 0 \end{array}$$

Exercise 5. (5)
Let the following function be given.

$$f : \mathbb{R}^3_{++} \times (0,1) \times \mathbb{R}_{++} \to \mathbb{R}^3,$$

$$f(x_1, x_2, \lambda; \alpha, u) = \begin{cases} -x_1 - \frac{\lambda}{x_1} \\ x_2 - \alpha \frac{\lambda}{x_2} \\ \log(x_1) + \alpha \log(x_2) - u \end{cases}$$

Assuming that $f(x_{10}, x_{20}, \lambda_0; \alpha_0, u_0) = 0$; say if it possible to study the effect of changes of α on (x_1, x_2, λ) assuming that $f(x_1, x_2, \lambda; \alpha, u) = 0$.

See next page.

Solve only one between Exercises 6a and 6b.

Exercise 6a. (5)
Provide an example of a subset of $\mathbb{R}$ which is open and bounded. You have to show your statements.

Exercise 6b. (5)
If possible compute the directional derivative $f'(0;\ u)$ for any $u \in \mathbb{R}^3$, if f is defined as follows

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = \begin{cases} \frac{x \cdot y + z}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0. \end{cases}$$

	x_1	x_2	λ
$-x_1 - \frac{\lambda}{x_1}$	-1	0	$-\frac{1}{x_1}$
$x_2 - \alpha \frac{\lambda}{x_2}$	0	1	0
$\log(x_1) + \alpha \log(x_2) - \underline{u}$	$-\frac{1}{x_1}$	$-\alpha \frac{\lambda}{x_2}$	0

Matematica per le applicazioni economiche 2
$\boxed{\textbf{Modalita' 2}}$
July 7th, 2014

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
a.
Let A, B be $n \times n$ matrices. Show that if $\det A \neq 0$ and $AB = 0$, then $B = 0$.
b. Show that the following statement is **false**: if $A \in \mathbb{M}(2,2)$, $AB = 0$ and

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then $A = 0$.

Exercise 2. (5)
Find a basis of $\operatorname{Im} l$ and $\ker l$ if

$$l : \mathbb{R}^5 \to \mathbb{R}^5, \qquad (x_1, x_2, x_3, x_4, x_5) \mapsto (x_5, x_4, x_3, 0, x_1 + x_2).$$

Exercise 3. (5)
Say if the following set is a vector space

$$\left\{ A \in \mathbb{M}(2,2) : \ \exists x, y \in \mathbb{R} \text{ such that } A = \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \right\}$$

Exercise 4. (5)
Using the steps described in the Class Notes, discuss the following maximization problem.
For given $\alpha \in (0,1)$,

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \ x_1 + \alpha x_2 \qquad \text{s.t.} \qquad \begin{array}{lcl} x_1 \cdot x_2 & \geq & 100 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

Exercise 5. (5)
Suppose that the demand and supply functions of a given good are described by the following functions

$$\text{demand function}: \qquad d : \mathbb{R}^2_{++} \to \mathbb{R} \qquad (p, \alpha) \mapsto d(p, \alpha),$$

$$\text{supply function}: \qquad s : \mathbb{R}^2_{++} \to \mathbb{R} \qquad (p, \beta) \mapsto s(p, \beta),$$

where p is the price of the good and α and β are exogenous variables.
Assume that d and s are C^1 functions and that there exists an equilibrium price $p^e \in \mathbb{R}_{++}$ at $(\alpha_0, \beta_0) \in \mathbb{R}^2_{++}$, i.e.,

$$d(p^e, \alpha_0) = s(p^e, \beta_0).$$

Assume also that $\forall \alpha, \beta, p \in \mathbb{R}_{++}$, $Dd(p, \alpha) << 0$ and $Ds(p, \beta) >> 0$.
Is it possible to compute the effect of a change in α on the associated equilibrium price in a sufficiently small neighborhood of (p^e, α_0, β_0).

See next page.

Solve only one between Exercises 6a and 6b.

Exercise 6a. (5)
Provide an example of a subset of $\mathbb{R}$ which is closed and unbounded. You have to show your statements.

Exercise 6b. (5)
Let the following C^1 functions be given.

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad x := (x_i)_{i=1}^n \mapsto f(x).$$

$$g_i : \mathbb{R} \to \mathbb{R} \qquad t \mapsto g_i(t) \qquad\qquad \text{for any } i \in \{1, 2, ..., n\}$$

Defined

$$h : \mathbb{R} \to \mathbb{R} \qquad t \mapsto (g_1(t), ..., g_i(t), ..., g_n(t)),$$

if possible, compute $h'(t)$.

Matematica per le applicazioni economiche 2

$\boxed{\textbf{Modalita' 2}}$

December 9, 2014

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Define

$$MC(2) := \left\{ A \in \mathbb{M}(2,2) : \exists a, b \in \mathbb{R} \text{ such that } A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\}.$$

Show that
a. $MC(2)$ is a vector subspace of $\mathbb{M}(2,2)$;
b. $\dim MC(2) = 2$;

Exercise 2. (5)
Find a basis of $\operatorname{Im} l$ and $\ker l$ if, given $n \in \mathbb{N}$, $n > 5$,

$$l : \mathbb{R}2n \to \mathbb{R}^n, \qquad (x_i)_{i=1}^{n-1} \mapsto \left((x_{i+1} + x_n)_{i \in \{1,\dots,n-2\}}, x_{n+2} \right)$$

Exercise 3. (5)
Discuss the following system for all values of $k \in \mathbb{R}$,

$$\begin{bmatrix} k & 1 & 0 \\ k & k & 1 \\ k & 0 & k \\ 1 & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Exercise 4. (5)
Given $a, b \in \mathbb{R}_{++}$, following the steps presented in the Class Notes, analyze the following problem.

$$\min_{(x,y)\in\mathbb{R}^2} ax + by \qquad s.t. \qquad bx^2 + ay^2 \leq 1$$

Say for which values of $(a, b) \in \mathbb{R}^2_{++}$, if any, the associated solution solution (x^*, y^*) is such that
i. $x^* < 0$;
ii. $y^* < 0$.

Exercise 5. (5)
Let the following function be given

$$h : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x_1, x_2, t) \mapsto \begin{cases} f(x_1, x_2, t) - 1 \\ \\ x_1 + x_2 - t \end{cases}$$

Assume that f is C^1, $f((1,1,2)) = 1$ and $D\, f((1,1,2)) = (1,0,1)$.
i. Say if it is possible to apply the implicit function theorem to h in $(\overline{x}_1, \overline{x}_2, \overline{t}) := (1,1,2)$ to prove that there exists a C^1 function $(x_1, x_2) = \varphi(t) = (\varphi_1(t), \varphi_2(t))$ in a open neighborhood of $\overline{t} = 2$.
ii. If possible, compute $D\varphi(2) \in \mathbb{R}^2$.

Solve only one between Exercises 6a and 6b.
Exercise 6a. (5)
Say if the following statement is true or false.
If x is a boundary point for $S \subseteq \mathbb{R}^n$, then it is an accumulation point for S.

Exercise 6b. (5)
Assume that the following functions are C^1,

$$g_1 : \mathbb{R} \to \mathbb{R}, \qquad l \mapsto g_1(l),$$

$$g_2 : \mathbb{R} \to \mathbb{R}, \qquad k \mapsto g_2(k),$$

$$f : \mathbb{R}^4 \to \mathbb{R}, \qquad (x_1, x_2, x_3, x_4) \mapsto f(x_1, x_2, x_3, x_4).$$

Defined

$$h : \mathbb{R}^2 \to \mathbb{R}, \quad (l, k) \mapsto f(g_1(l), g_2(k), l, k),$$

compute $Dh(l, k)$.

Matematica per le applicazioni economiche 2
$\boxed{\textbf{Modalità 2}}$
January 9th 2015

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Defined

$$V_1 = \{(x, y) \in \mathbb{R} : x = 2a \text{ and } y = a \text{ for some } a \in \mathbb{R}\}$$
$$V_2 = \{(x, y) \in \mathbb{R} : x = \ b \text{ and } y = b \text{ for some } a \in \mathbb{R}\}$$

Say if

$$V_1, \qquad V_2, \qquad V_1 \cap V_2, \qquad V_1 \cup V_2$$

are vector subspaces of $\mathbb{R}^2$.
Exercise 2. (5)
Find a basis of $\operatorname{Im} l$ and $\ker l$ for the linear functions l_1 and l_3 defined below.

$$l_1 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) \mapsto (0, x_1, x_2),$$
$$l_2 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x, y, z) \mapsto (x, y, 0),$$
$$l_3 = l_2 \circ l_1.$$

Exercise 3. (5)
For any value $k \in \mathbb{R}$, let $S(k)$ be the solution set of the following linear system:

$$\begin{bmatrix} 1 & k & 0 & k & k \\ 0 & 1 & k & 0 & k \\ 1 & 0 & k & 2k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\det \begin{bmatrix} 1 & k & k \\ 0 & 1 & 0 \\ 1 & 0 & 2k \end{bmatrix} = k$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For any $k \in \mathbb{R}$, say if $S(x)$ is empty, and, if it is not empty, compute its dimension.

Exercise 4. (5)
Let the following objects be given: $a \in \mathbb{R}_{++}$, $f, g : \mathbb{R} \to \mathbb{R}$ of class C^2 and such that $\forall x \in \mathbb{R}$, $f'(x) > 0, g'(x) > 0, f''(x) < 0, g''(x) < 0$.
Following the steps presented in the Class Notes, analyze the problem below. For given $a \in \mathbb{R}_{++}$,

$$\min_{(x,y)\in\mathbb{R}^2} x^2 - 8x + ye^{y-3} \qquad s.t. \qquad \begin{array}{l} x + y \le a \\ x \ge 0 \\ y \ge 0 \end{array}$$

Exercise 5. (5)
Let $a \in \mathbb{R}_{++}$ and the following function be given

$$f : \mathbb{R}^3_{++} \to \mathbb{R}^3, \qquad (x, y, \mu) \mapsto \begin{cases} -2x - 8 - \mu \\ -e^{y-3} - ye^{y-3} - \mu \\ a - x - y \end{cases}$$

Assume that there exists $(\overline{x}, \overline{y}, \overline{\mu}, \overline{a}) \in \mathbb{R}^3_{++}$ such that $f(\overline{x}, \overline{y}, \overline{z}, \overline{a}) = 0$.

Carefully check the assumptions needed to apply the implicit function theorem to f in $(\overline{x}, \overline{y}, \overline{z}, \overline{a})$ to prove that there exists a C^1 function $(x, y, \mu) = \varphi(a)$ in a open neighborhood of $\overline{a}$.

Solve only one between Exercises 6a and 6b.
Exercise 6a.(5)
Say if the following statement is true or false.
The maxization problem described below admits a solution.
Let the continuous functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}^n$ be given;

$$\min_{x \in \mathbb{R}^n} f(x) \qquad \text{s.t.} \qquad g(f(x)) \leq 0 \text{ and } \|x\| \leq 1.$$

Exercise 6b. (5)
If possible, compute $f'(0; u)$ for any $(u_1, u_2) \in \mathbb{R}^2$ for the function defined below.

$$f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto \begin{cases} \frac{x^2+y^2}{y-x} & if \quad y \neq x, \\ \\ x^2 & if \quad y = x. \end{cases}$$

Matematica per le applicazioni economiche 2
$\boxed{\textbf{Modalità 2}}$
January 9th 2015

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Defined

$$V_1 = \{(x, y) \in \mathbb{R} : x = 2a \text{ and } y = a \text{ for some } a \in \mathbb{R}\}$$
$$V_2 = \{(x, y) \in \mathbb{R} : x = \ b \text{ and } y = b \text{ for some } a \in \mathbb{R}\}$$

Say if

$$V_1, \qquad V_2, \qquad V_1 \cap V_2, \qquad V_1 \cup V_2$$

are vector subspaces of $\mathbb{R}^2$.

Exercise 2. (5)
Find a basis of $\operatorname{Im} l$ and $\ker l$ for the linear functions l_1 and l_3 defined below.

$$l_1 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) \mapsto (0, x_1, x_2),$$
$$l_2 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x, y, z) \mapsto (x, y, 0),$$
$$l_3 = l_2 \circ l_1.$$

Exercise 3. (5)
For any value $k \in \mathbb{R}$, let $S(k)$ be the solution set of the following linear system:

$$\begin{bmatrix} 1 & k & 0 & k & k \\ 0 & 1 & k & 0 & k \\ 1 & 0 & k & 2k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For any $k \in \mathbb{R}$, say if $S(x)$ is empty, and, if it is not empty, compute its dimension.

Exercise 4. (5)
Following the steps presented in the Class Notes, analyze the problem below. For given $a \in \mathbb{R}_{++}$,

$$\min_{(x,y)\in\mathbb{R}^2} x^2 - 8x + ye^{y-3} \qquad s.t. \qquad \begin{array}{l} x + y \le a \\ x \ge 0 \\ y \ge 0 \end{array}$$

Exercise 5. (5)
Let $a \in \mathbb{R}_{++}$ and the following function be given

$$f : \mathbb{R}^3_{++} \to \mathbb{R}^3, \qquad (x, y, \mu) \mapsto \begin{cases} -2x - 8 - \mu \\ -e^{y-3} - ye^{y-3} - \mu \\ a - x - y \end{cases}$$

Assume that there exists $(\overline{x}, \overline{y}, \overline{\mu}, \overline{a}) \in \mathbb{R}^3_{++}$ such that $f(\overline{x}, \overline{y}, \overline{z}, \overline{a}) = 0$.

Carefully check the assumptions needed to apply the implicit function theorem to f in $(\overline{x}, \overline{y}, \overline{z}, \overline{a})$ to prove that there exists a C^1 function $(x, y, \mu) = \varphi(a)$ in a open neighborhood of $\overline{a}$.

Solve only one between Exercises 6a and 6b.
Exercise 6a.(5)
Say if the following statements are true or false. For any $n \in \mathbb{N}$,

$$S \subseteq \mathbb{R}^n \text{ is compact } \Leftrightarrow S \subseteq \mathbb{R}^n \text{ is convex.}$$

Exercise 6b. (5)
Using the definition, say if the following function is differentiable in $(0, 0)$

$$f : \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x + 2y.$$

Hint: Knowing that f is C^1, "guess" the expression for $df_{(0,0)}$

Matematica per le applicazioni economiche 2

$\boxed{\textbf{Modalità 2}}$

February 6th,2015

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Define $\mathcal{C}^0(\mathbb{R},\mathbb{R})$ as the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and given $g^* \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$ define also

$$I = \left\{f \in \mathcal{C}^0(\mathbb{R},\mathbb{R}) : \exists h \in \mathcal{C}^0(\mathbb{R},\mathbb{R}) \text{ such that } f = g^* h\right\} = \left\{f = g^* h \text{ for some } h \in \mathcal{C}^0(\mathbb{R},\mathbb{R})\right\}.$$

Show that I is a vector space.

Exercise 2. (5)
Let the following linear function be given: $l : \mathbb{R}^4 \to \mathbb{R}^4$,

$$l(x_1,x_2,x_3,x_4) = \begin{pmatrix} ax_1 \\ ax_2 \\ bx_3 \\ bx_4 \end{pmatrix},$$

where $a,b \in \mathbb{R}$. Find $\dim \ker l$ and $\dim \operatorname{Im} l$ for any value of $a,b \in \mathbb{R}$.
Hint. Analize the following cases. i. $a \neq 0, b \neq 0$; ii. $a = 0, b \neq 0$; iii. $a \neq 0, b = 0$; iv. $a = 0, b = 0$.

Exercise 3. (5)
For any value $a \in \mathbb{R}$, let $S(a)$ be the solution set of the following linear system:

$$\begin{bmatrix} a & 2a+1 & a+2 \\ 1 & a+1 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}.$$

For any $a \in \mathbb{R}$, say if $S(a)$ is empty, and, if it is not empty, compute its dimension.

Exercise 4. (5)
Following the steps presented in the Class Notes, analyze the problem below. For given $p \in \mathbb{R}_{++}$,

$$\min_{(x,l)\in\mathbb{R}^2} u(x) - f(l) - pl \qquad s.t. \qquad \begin{array}{l} l \in [0,8] \\ x \leq g(l) \\ x \geq 0, \end{array}$$

where f, g, u are $\mathcal{C}^2$ functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\begin{array}{ll} u' > 0, & u'' < 0, \\ f' > 0, & f'' > 0, \\ g' > 0 & g'' < 0. \end{array}$$

Do not discuss the existence problem.

Exercise 5. (5)
Let the following function be given

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x,y,a) \mapsto \begin{cases} x + y - a \\ 2x + y - 3a \end{cases}$$

1. Solve explicitely the system $f(x,y,a) = 0$ in the unknowns x, y as a function of the parameter a. Compute the derivatives of the solutions with respect to a.

2. Carefully check the assumptions needed to apply the implicit function theorem to f to get the same values of the derivatives found in 1. above.

Solve only one between Exercises 6a and 6b.
Exercise 6a.(5)
Say if the following statements are true or false. For any $n \in \mathbb{N}$,

$$S \subseteq \mathbb{R}^n \text{ is closed } \Leftrightarrow S \subseteq \mathbb{R}^n \text{ is bounded.}$$

Exercise 6b. (5)
If possible, compute the directional derivative of the following function

$$f : \mathbb{R}^2_{++} \to \mathbb{R}^2, \ (x, y) \mapsto \begin{pmatrix} x^2 + xy \\ \\ x \log\left(y^3\right) \end{pmatrix}.$$

in $(1, 1)$ in the direction $(1, 1)$.

Matematica per le applicazioni economiche 2
$\boxed{\textbf{Modalità 2}}$
June 11, 2015

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)

Let W_1 and W_2 be subspaces of a vector space V. We say V is the direct sum of $W_1$and W_2, and we write $V = W_1 \oplus W_2$, if

for any $v \in V$, there exists a **unique** $(v^1, v^2) \in W_1 \times W_2$ such that $v = v^1 + v^2$.

Let the following subspaces of $\mathbb{R}^3$ be given.

$$U = \{(x, y, z) \in \mathbb{R}^3 : z = 0\},$$

i.e., the xy plane;

$$W = \{(x, y, z) \in \mathbb{R}^3 : x = 0\},$$

i.e., the yz plane;

$$Z = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\},$$

i.e., the z axis; $L = \{(k, k, k) : k \in \mathbb{R}\}$.

Verify that

a. $\mathbb{R}^3 = U \oplus Z$.

b. $\mathbb{R}^3$ is not the direct sum of U and W.

Exercise 2. (5)

Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that the following set is a vector subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$:

$$\mathcal{W} := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : \forall x \in \mathbb{R},\ f(-x) = f(x)\}.$$

Exercise 3. (5)

For any value $a \in \mathbb{R}$, let $S(a)$ be the solution set of the following linear system:

$$\begin{bmatrix} a & a & 0 & 0 \\ 1 & 2 & a & 1-a \\ a & 2a & 4a & 4a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ a \\ 1 \end{bmatrix}.$$

For any $a \in \mathbb{R}$, say if $S(a)$ is empty, and, if it is not empty, compute its dimension.

Exercise 4. (5)

Given

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

a. compute $A^3 := A \cdot A \cdot A$;

b. find a basis of $\ker l$ and $\operatorname{Im} l$, where l

Exercise 4. (5)

Following the steps presented in the Class Notes, analyze the problem below. For given $a, w \in \mathbb{R}_{++}$,

$$\min_{(x,y)\in\mathbb{R}_{++}\times\mathbb{R}} \quad y + a \cdot \log x \qquad s.t. \qquad \begin{array}{l} y \geq 0 \\ x + y \leq w \end{array}$$

Do not miss exercises on next page.

Exercise 5. (5)
Let the following (production) function be given:

$$f:\mathbb{R}^2_{++}\rightarrow\mathbb{R},\ (x,y)\mapsto\left(\delta x^{-\rho}+(1-\delta)\,y^{-\rho}\right)^{-\frac{1}{\rho}},$$

where $\rho\geq 1$, $\delta\in(0,1)$.
Compute the slope the level curve (isoquant) $f(x,y)=k$ with $k\in\mathbb{R}_{++}$.

Solve only one between Exercises 6a and 6b.
Exercise 6a. (5) $*$
Say if the following statements are true or false. For any $n\in\mathbb{N}$,

$$S\subseteq\mathbb{R}^n \text{ is open } \Leftrightarrow\ S\subseteq\mathbb{R}^n \text{ is convex.}$$

Exercise 6b. (5) $*$
Let the following differentiable functions be given

$$g:\mathbb{R}\rightarrow\mathbb{R},\ t\mapsto g(t)$$

$$h:\mathbb{R}\rightarrow\mathbb{R},\ t\mapsto h(t)$$

$$f:\mathbb{R}^2\rightarrow\mathbb{R},\ (x,y)\mapsto f(x,y)$$

$$F:\mathbb{R}\rightarrow\mathbb{R},\ t\mapsto f(g(t),h(t))$$

Using the chain rule,compute $F'(t)$.

Any vector $(a,b,c) \in \mathbb{R}^3$ can be written uniquely as follows:

$$(a,b,c) = (a,b,0) + (0,0,c)$$

where $(a,b,0) \in U$ and $(0,0,c) \in Z$.

Example 7 ii. $\mathbb{R}^3 = U \oplus L$.

Any vector $(a,b,c) \in \mathbb{R}^3$ can be written uniquely as follows:

$$(a,b,c) = (a-c, b-c, 0) + (c,c,c)$$

where $(a,b,0) \in U$ and $(0,0,c) \in Z$.

The projections p_U and p_L of V into U and L respectively are.

$$p_U(a,b,c) = (a-c, b-, 0) \quad \textit{and} \quad p_L(a,b,c) = (c,c,c)$$

iii. $\mathbb{R}^3 \neq U \oplus W$.

Since any vector in $\mathbb{R}^3$ can be written as a sum of a vector in U and a vector in W, $\mathbb{R}^3 = U+W$. However, $\mathbb{R}^3$ is not the direct sum of U and W since such sums are not unique: for instance, $(1,2,3) = (1,1,0) + (0,1,1) = (1,3,0) + (0,-1,3)$.

Exercise 3.

$\det \begin{bmatrix} a & a & 0 \\ 1 & 2 & a \\ a & 2a & 4a \end{bmatrix} = 4a^2 - a^3 = a^2(4-a)$

Case $a = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

no solution

Case $a = 4$

$$\begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 2 & 4 & -3 \\ 4 & 8 & 16 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}.$$

infinite solutions

Matematica per le applicazioni economiche 2
$\boxed{\textbf{Modalità 2}}$
July, 2015

You have two hours and a half to complete the exam. The number next to each exercise tells you the score you get in case of correct answer. Explain carefully your answers.

Exercise 1. (5)
Let V be the vector space of differentiable function from $\mathbb{R}$ to $\mathbb{R}$.For given $a \in \mathbb{R}$, say if the following set is a vector subspace of V.

$$\{f \in V : \forall x \in \mathbb{R},\ f'(x) = a \cdot f(x)\}.$$

Exercise 2. (5)
Let the following vector subspace of $\mathbb{R}^3$ be given (you do not need to show it is a subspace):

$$W = \left\{(x, y, x) \in \mathbb{R}^3 : x = 0\right\}.$$

Say if the following sets are a basis of W:
i. $\{(0,1,0), (0,0,1)\}$;
ii. $\{(1,1,0), (1,0,1)\}$;
iii. $\{(0,1,2), (0,2,4)\}$.

Exercise 3. (5)
Find a basis of $\ker l$ and of $\operatorname{Im} l$ if l is the linear function defined below. Given $n \in \{1, 2, ...\} := \mathbb{N}$, $n \geq 5$ and $x = (x_1, ..., x_i, ..., x_n)$,

$$f : \mathbb{R}^n \to \mathbb{R}^n$$
$$f(x) = (x_1 + x_2 + x_3, x_1 + x_2, x_1, 0, ..., 0)_{i=1}^n$$

Exercise 4. (5)
Let A and B be two $n \times n$ matrices with $n \geq 3$. Given the system

$$\begin{bmatrix} A & B \end{bmatrix} x = b,$$

where, obviously, $x \in \mathbb{R}^{2n}$ and $b \in \mathbb{R}^n$, describe its solutions set in the cases described below.
a. $\det A = 7$;
b. rank $A = n$;
c. $A = 0$ and rank $B = n - 1$.

Exercise 4. (5)
i. Following the steps presented in the Class Notes, analyze the following problem. For given $a \in \mathbb{R}_{++}$,

$$\max_{(x,y) \in \mathbb{R}^2_{++}} \log x + a \log y \qquad s.t. \qquad g(x, y) \geq 0,$$

where g is a C^2 concave function such that $g(1,1) > 0$. Do not analyze the existence of solution problem.

Exercise 5. (5)
Let the following function be given

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad (x, y, z) \mapsto x + x^2yz - e^{3z^2} - y - 1 + e^3$$

i. Say if it is possible to apply the implicit function theorem to f in $(x_0, y_0, z_0) = (1, 1, 1,)$ to prove that there exists a C^1 function $x = g(y, z)$ in an open neighborhood of $(y_0, z_0) = (1, 1)$.
ii. Write the formula to compute $D_{(y,z)}g((1, 1))$.
iii. Say if it is possible to apply the implicit function theorem to f in $(x_0, y_0, z_0) = (1, 1, 1,)$ to prove that there exists a C^1 function $y = h(x, z)$ in an open neighborhood of $(x_0, z_0) = (1, 1)$.
Do not miss the exercise on next page.

Solve only one between Exercises 6a and 6b.

Exercise 6a. (5)

Using the characterization of continuous functions in terms of closed set, verify that the following function is not continuous:

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 0 \\ \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Exercise 6b. (5)

Using the definition of directional derivative, compute $f'((x_0, y_0); (u_1, u_2))$, where for given $a \in \mathbb{R}$,

$$f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto ax + y^2,$$

$(x_0, y_0) = (0, 0)$ and $(u_1, u_2) = (1, 1)$.

Compute that derivative using theorems you know.

Exercise 5. (5)
Let the following function be given

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, z) \mapsto x + x^2yz - e^{3z^2} - y - 1 + e^3$$

i. Say if it is possible to apply the implicit function theorem to f in $(x_0, y_0, z_0) = (1, 1, 1,)$ to prove that there exists a C^1 function $x = g(y, z)$ in an open neighborhood of $(y_0, z_0) = (1, 1)$.

ii. Write the formula to compute $D_{(y,z)}g((1, 1))$.

iii. Say if it is possible to apply the implicit function theorem to f in $(x_0, y_0, z_0) = (1, 1, 1,)$ to prove that there exists a C^1 function $y = h(x, z)$ in an open neighborhood of $(x_0, z_0) = (1, 1)$.

$f(x, y, z) = x + x^2yz - e^{3z^2} - y - 1 + e^3$
$f(1, 1, 1) = e^{3\times 1} - e^3 = 0$
$\frac{\partial f(x,y,z)}{\partial x} = 2xyz + 1$
$\frac{\partial f(x,y,z)}{\partial y} = x^2z - 1$
$\frac{\partial f(x,y,z)}{\partial z} = x^2y - 6ze^{3z^2}$