Notes for Mathematics 1
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Chapter 1

Some properties of functions

Definition 1 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is increasing on $A \subseteq S$ if for any $x_1, x_2 \in S$ such that $x_1 < x_2$, we have that $f(x_1) \leq f(x_2)$. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is increasing if it is increasing on S.*

Example 2 *$f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x + 7$ is increasing: $x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 7 < 3x_2 + 7$.*

Example 3 *$g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$ is not increasing: we want to show that $\exists x_1, x_2 \in S$ such that $x_1 < x_2$ and $f(x_1) > f(x_2)$. Take $x_1 = -1$ and $x_2 = 0$; then $f(-1) = 1 > 0 = f(0)$.*

Example 4 *$h : \mathbb{R} \to \mathbb{R},\ h(x) = 1$ is increasing.*

Definition 5 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is strictly increasing on $A \subseteq S$ if for any $x_1, x_2 \in S$ such that $x_1 < x_2$ we have that $f(x_1) < f(x_2)$. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is strictly increasing if it is strictly increasing on S.*

Remark 6 *If a function is strictly increasing, then it is increasing, but not viceversa.*

Example 7 *$g : \mathbb{R} \to \mathbb{R},\ g(x) = x^3$ is strictly increasing. $h : \mathbb{R} \to \mathbb{R},\ h(x) = 1$ is increasing, but not strictly increasing.*

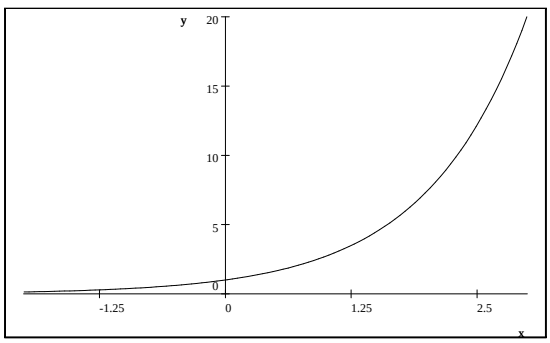
Definition 8 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is decreasing on $A \subseteq S$ if $x_1, x_2 \in S$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is decreasing if it is decreasing on S.*

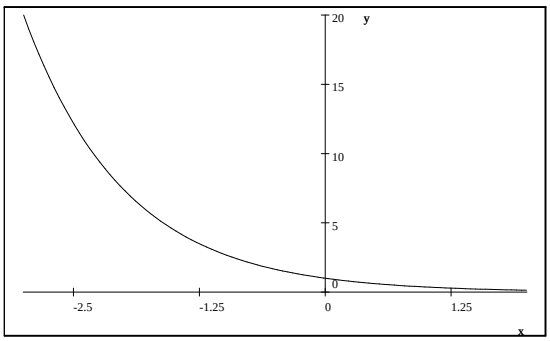
Definition 9 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is strictly decreasing on $A \subseteq S$ if $x_1, x_2 \in S$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is strictly decreasing if it is strictly decreasing on S.*

Definition 10 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is monotone on $A \subseteq S$ if it is increasing or decreasing on $A \subseteq S$.*

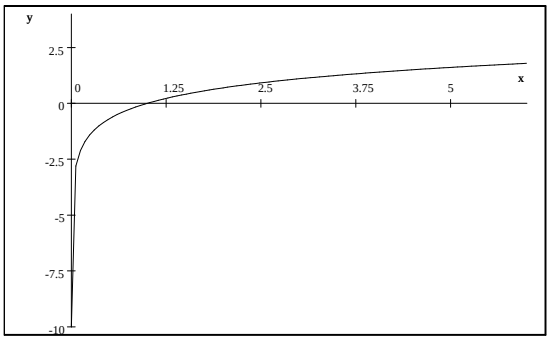
Remark 11 *f is increasing $\Leftrightarrow -f$ is decreasing. Indeed, f is increasing if and only if for any $x_1, x_2 \in S$ such that $x_1 < x_2$, we have that $f(x_1) \leq f(x_2)$ and therefore, multiplying both sides by -1, we get $-f(x_1) \geq -f(x_2)$.*

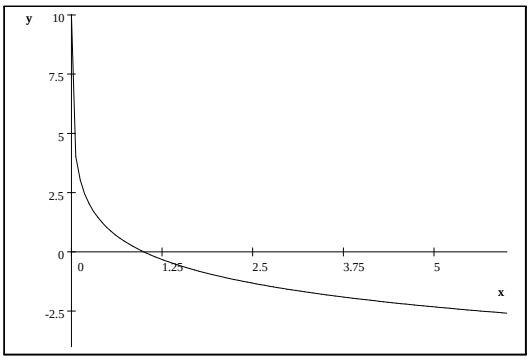
Remark 12 *As shown in previous classes, we have what follows.*
$f : \mathbb{R} \to \mathbb{R},\ x \mapsto a^x$ is strictly increasing if $a > 1$ and it is strictly decreasing if $a \in (0, 1)$;





$f : \mathbb{R}_{++} \to \mathbb{R}$, $x \mapsto \log_a x$ *is strictly increasing if* $a > 1$ *and it is strictly decreasing if* $a \in (0,1)$.





Exercise 13 *Given arbitrary functions* $f, g : S \subseteq \mathbb{R} \to \mathbb{R}$, *show that*

1. f *increasing,* g *increasing* $\Rightarrow f + g$ *increasing;*

2. a. f *increasing ,* $\alpha > 0 \Rightarrow \alpha f$ *increasing; b.* f *increasing,* $\alpha < 0 \Rightarrow \alpha f$ *decreasing;*

3. a. f *increasing,* g *increasing,* $f > 0, g > 0 \Rightarrow fg$ *increasing; b.* f *increasing,* g *increasing,* $f < 0, g < 0 \Rightarrow fg$ *increasing ;*

4. f *increasing and* $f > 0 \Rightarrow \frac{1}{f}$ *decreasing.*

Solutions

...

3a. By assumption, taken $x_1 < x_2$, *we have* $f(x_1) \leq f(x_2)$ *and* $g(x_1) \leq g(x_2)$. *Therefore, since* f *and* g *are strictly positive, we get*

$$f(x_1) \cdot g(x_1) \leq f(x_2) \cdot g(x_1) \tag{1.1}$$

and

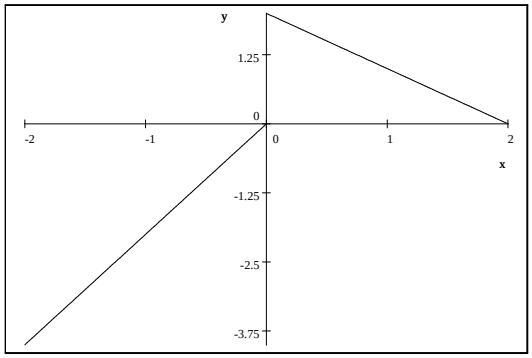
$$f(x_2) \cdot g(x_1) \leq f(x_2) \cdot g(x_2). \tag{1.2}$$

Then, from (1.1) *and* (1.2), *we get the desired result.*

...

Remark 14 *$f : S \subseteq \mathbb{R} \to \mathbb{R}$ is strictly increasing or strictly decreasing, then f is one-to-one and therefore it is invertible. The opposite implication is false, as shown by the function f:$(-\infty, 2] \to \mathbb{R}$,*

$$\begin{cases} 2x & \text{if} \quad x < 0 \\ -x + 2 & \text{if} \quad 0 \leq x \leq 2 \end{cases}$$



Recall the following definitions.

Definition 15 *A nonempty set $A \subseteq \mathbb{R}$ is bounded above if it admits an upper bound k_1 and it is bounded below if it admits a lower bound k_2.*

Definition 16 *A nonempty set $A \subseteq \mathbb{R}$ is bounded if it is bounded above and below.*

Definition 17 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is bounded above if the set $\operatorname{Im}(f)$ admits an upper bound k_1 and it is bounded below if $\operatorname{Im}(f)$ admits a lower bound k_2.*

A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is bounded if it is bounded above and below.

Definition 18 *Given a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$, we define*

$$\sup f = \sup \operatorname{Im}(f)$$

and

$$\max f = \max \operatorname{Im}(f).$$

$\max f$ is called global maximum value for f. A point $x \in S$ such that $f(x) = \max f$ is called a global maximum point for f.

Definition 19 *Given a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$, we define*

$$\inf f = \inf \operatorname{Im}(f)$$

and

$$\min f = \min \operatorname{Im}(f).$$

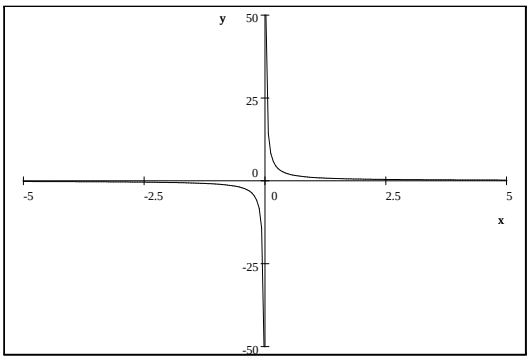
$\min f$ is called global minimum value for f. A point $x \in S$ such that $f(x) = \min f$ is called a global minimum point for f.

Remark 20 *Given a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$, $\max f$ may or may not exists. If it exists, then it is unique. Observe that if $\max f$ exists, there exist at least one global maximum point for f.*

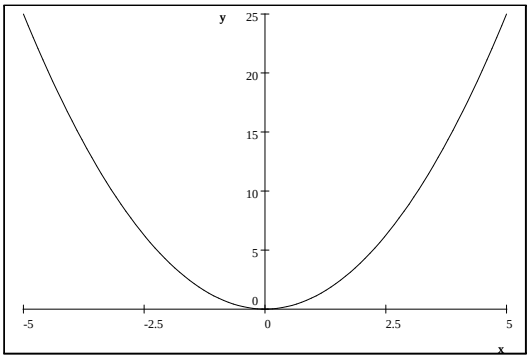
Example 21

f	$\sup f$	$\inf f$	$\max f$	$\min f$	*set of max points*	*set of min points*
$f_1 : \mathbb{R}\setminus\{0\} \to \mathbb{R},\ f_1(x) = \frac{1}{x}$	$+\infty$	$-\infty$	*no*	*no*	$\varnothing$	$\varnothing$
$f_2 : \mathbb{R} \to \mathbb{R},\ f_2(x) = x^2$	$+\infty$	0	*no*	0	$\varnothing$	$\{0\}$
$f_3 : \left[-2\pi, \frac{3}{2}\pi\right) \to \mathbb{R},\ f_3(x) = \sin x$	1	-1	1	-1	$\left\{-\frac{3}{2}\pi, \frac{1}{2}\pi\right\}$	$\left\{-\frac{\pi}{2}\right\}$
$f_4 : \mathbb{R} \to \mathbb{R},\ f_4(x) = 1$	1	1	1	1	$\mathbb{R}$	$\mathbb{R}$

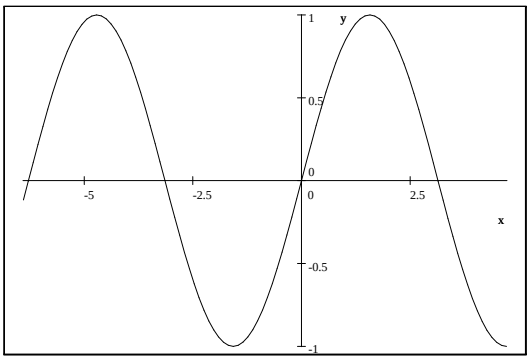
$f_1(x) = \frac{1}{x}$



$f_2(x) = x^2$



$f_3(x) = \sin x$



Definition 22 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is even if*
a. $x \in S \Rightarrow -x \in S$, i.e., S is symmetric with respect to the origin, and
b. $\forall x \in S$, $f(x) = f(-x)$.

Remark 23 *In words, f is even if graph (f) is symmetric with respect to the vertical axis. Indeed, if $x \in S$, then $(x, f(x)), (-x, f(-x)) \in graph(f)$, but $(-x, f(-x)) = (-x, f(x))$.*

Remark 24 *If f is even, then it is enough to study $f_{|S \cap \mathbb{R}_+}$ and then flip the graph so obtained symmetrically with respect to the vertical axis.*

Definition 25 *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is odd if*
a. $x \in S \Rightarrow -x \in S$, i.e., S is symmetric with respect to the origin, and
b. $\forall x \in S$, $f(x) = -f(-x)$.

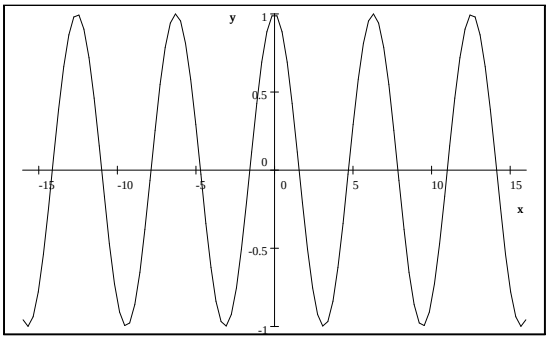
Remark 26 *In words, f is odd if graph (f) is symmetric with respect to the origin. Indeed, if $x \in S$, then $(x, f(x)), (-x, f(-x)) \in graph(f)$, ma $(-x, f(-x)) = (-x, -f(x))$).*

Remark 27 *If f is even, then it is enough to study $f_{|S \cap \mathbb{R}_+}$ and then flip the graph so obtain symmetrically with respect to the origin of the axes.*

Example 28

f	*even*	*odd*
$f_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f_1(x) = \frac{1}{x}$	*no*	*yes*
$f_2 : \mathbb{R} \to \mathbb{R},\ f_2(x) = x^2$	*yes*	*no*
$f_3 : [-2\pi, 2\pi) \to \mathbb{R},\ f_3(x) = \sin x$	*no*	*yes*
$f_4 : \mathbb{R} \to \mathbb{R},\ f_4(x) = \cos x$	*yes*	*no*

$f_4(x) = \cos x$



Remark 29 *$f : \mathbb{R} \to \mathbb{R}$ is even and odd if and only if $f = 0$,as verified below. By assumption, $f(x) = f(-x) = -f(-x)$. Then $f(-x) = 0 = f(x)$.*

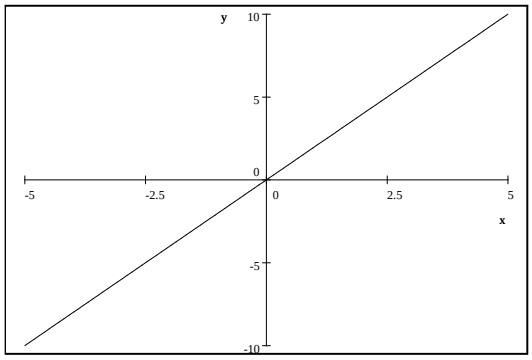
Chapter 2

Limits

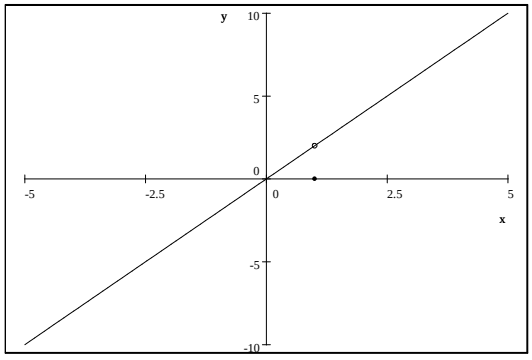
2.1 Definitions

Let's start with some motivation for the search of a rigorous definition of the limit of a function $f : \mathbb{R} \to \mathbb{R}$. We want to study what happens to $f(x)$ when x gets close to x_0. What is the answer in the following cases?

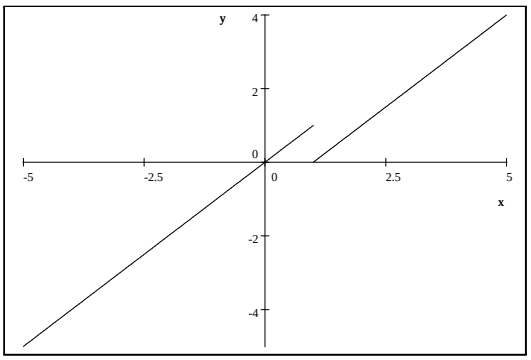
1. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$; $x_0 = 1$.



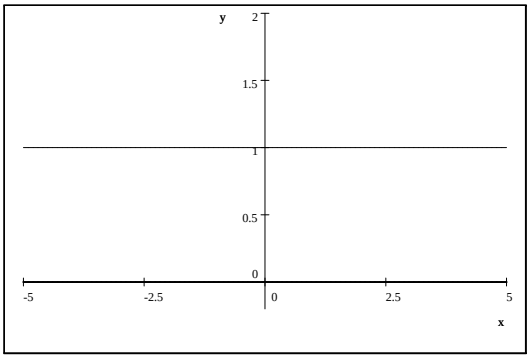
2. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 2x & \text{if} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$; $x_0 = 1$.



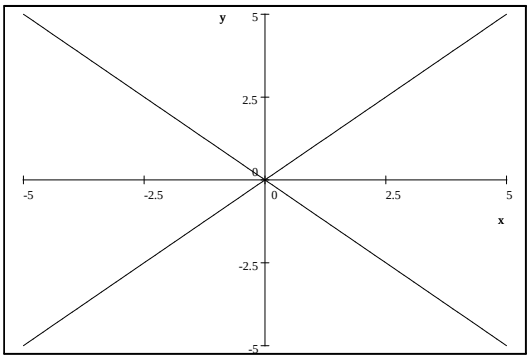
3. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} x & \text{if} \quad x \leq 1 \\ x - 1 & \text{if} \quad x > 1 \end{cases}$; $x_0 = 1$.



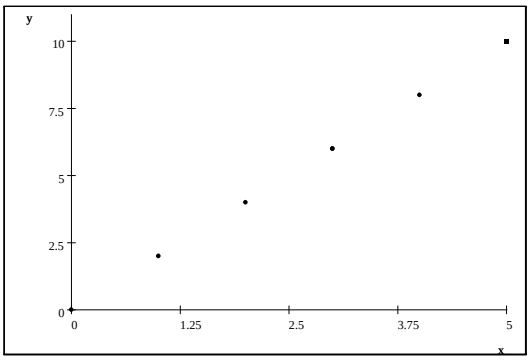
4. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ 1 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$; $x_0 = 0$.



5. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ -x & \text{if} \quad x \notin \mathbb{Q} \end{cases}$; $x_0 = 0$.

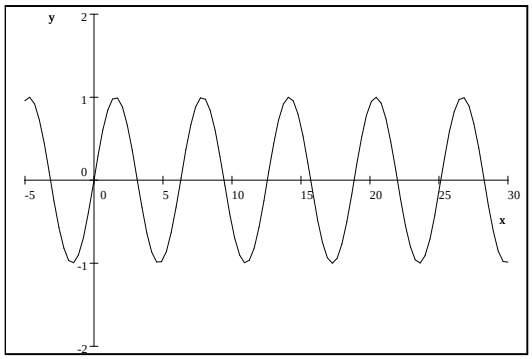


6. $f : \mathbb{N} \to \mathbb{R}$, $x \mapsto 2x$; $x_0 = 3$.

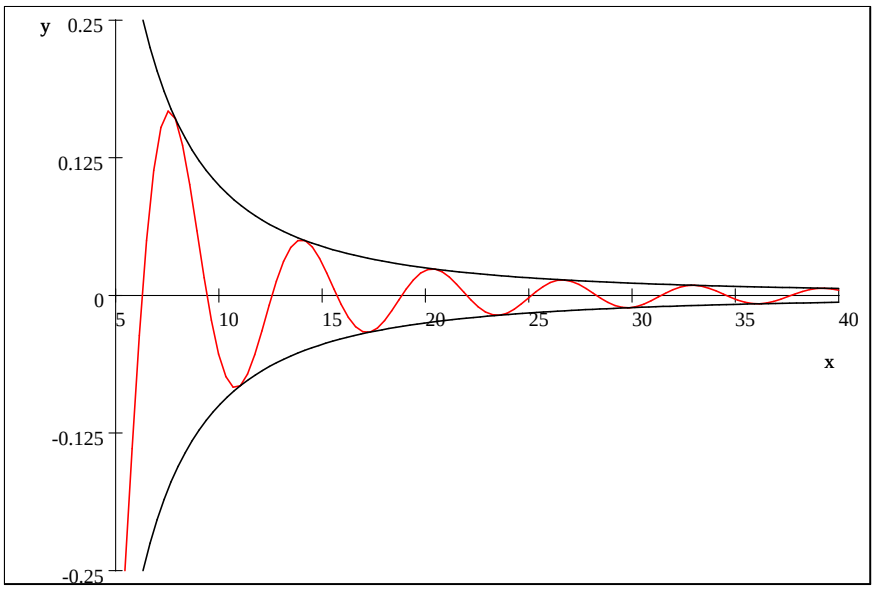


We can also try to understand what happens to $f(x)$ when x becomes very large. What is your answer in the following cases?

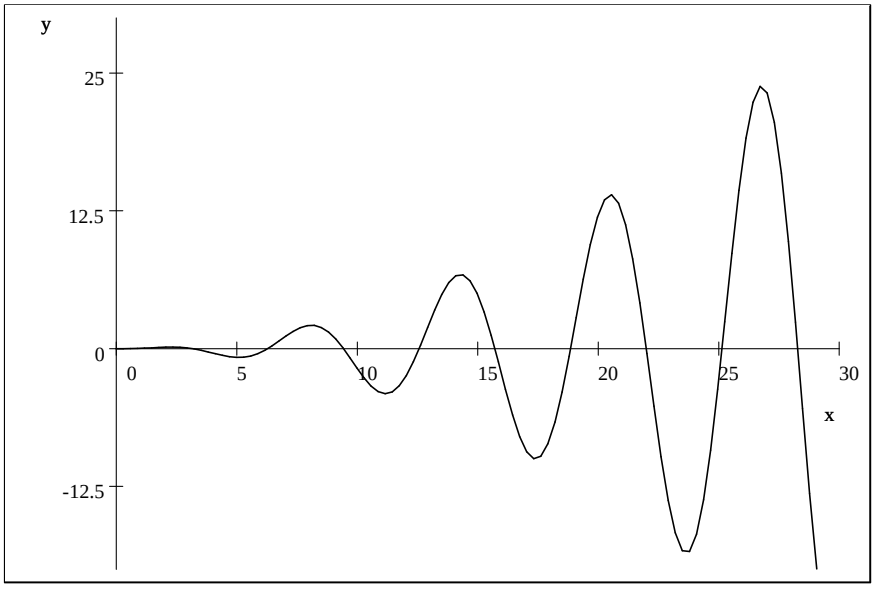
7. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin x$.



8. $f : \mathbb{R}_{++} \to \mathbb{R}$, $x \mapsto \frac{\sin x}{x^2}$



9. $f : \mathbb{R}_{++} \to \mathbb{R}$, $x \mapsto x^2 \cdot \sin x$



In what follows, we formalize the above discussion using the concept of limit. A preliminary definition is needed. Its role will be explained in Remark 34 below.

Definition 30 *$x \in \mathbb{R}$ is an accumulation point for $S \subseteq \mathbb{R}$ if any open ball centered at, or neighborhood of, x contains points of S different from x, i.e., if*

$$\forall r \in \mathbb{R}_{++}, \quad (S \backslash \{x\}) \cap B(x, r) \neq \varnothing$$

The set of accumulation points of S is denoted by $D(S)$ and it is called the Derived set of S.

Definition 31 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. We say that $l \in \mathbb{R}$ is a limit for f as x tends to x_0 and we write*

$$\lim_{x \to x_0} f(x) = l$$

if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } |f(x) - l| < \varepsilon. \tag{2.1}$$

Remark 32 *Since the neighborhood of, or the ball around, $z \in \mathbb{R}$ with radius $r \in \mathbb{R}_{++}$ is*

$$B(z, r) := \{x \in \mathbb{R} : |x - z| < r\} = (z - r, z + r),$$

the statement in (2.1) is equivalent to the statements below:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \cap B(x_0, \delta) \setminus \{x_0\}, \text{ then } f(x) \in B(l, \varepsilon).$$

Remark 33 *Observe that, in general, δ depends upon ε. For that reason, we sometimes write δ_ε in the place of δ. Moreover, we use the following convention. If*

$$x \in S \cap B(x_0, \delta_\varepsilon) \setminus \{x_0\} \Rightarrow f(x) \in B(l, \varepsilon),$$

then

$$x \in S \cap B\left(x_0, \delta_{g(\varepsilon)}\right) \setminus \{x_0\} \Rightarrow f(x) \in B(l, g(\varepsilon)),$$

i.e., taken $g(\varepsilon) > 0$, we define $\delta_{g(\varepsilon)} > 0$ as the radius of that neighborhood of x_0 whose points (in S and different from x_0) have images in a neighborhood of l of radius $g(\varepsilon)$. The above convention is applied very often in the following Sections.

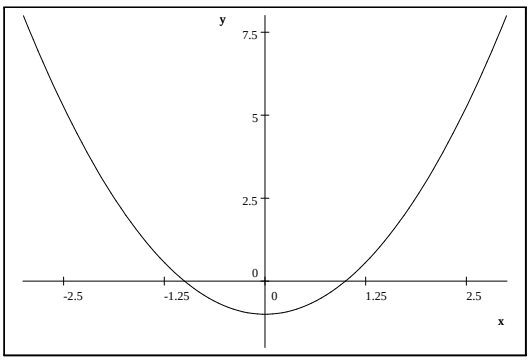
Remark 34 *The reason to require x_0 to be an accumulation point of S is the following one. To study the behavior of $f(x)$ when x gets closer to x_0, we have to be able to get close to x_0 remaining in the domain of f. For that reason, no matter how close we are to x_0, we need to be able to find a point in the domain of f, i.e., it must be the case that*

$$\forall r \in \mathbb{R}_{++}, B(x_0, r) \cap S \setminus \{x_0\} \neq \varnothing,$$

which is just the definition of x_0 being an accumulation point for S.

Example 35 *Let's verify that*

$$\lim_{x \to 0} \left(x^2 - 1\right) = -1.$$



We want to show that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } 0 < |x - 0| < \delta, \text{ then } \left|x^2 - 1 - (-1)\right| < \varepsilon,$$

i.e.,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } 0 < |x| < \delta, \text{ then } \left|x^2\right| < \varepsilon.$$

To find the required δ, it is convenient to start from the desired conclusion, i.e., $\left|x^2\right| < \varepsilon$.

$$\left|x^2\right| < \varepsilon \ \Leftrightarrow \ x^2 < \varepsilon \Leftrightarrow x^2 - \varepsilon < 0 \Leftrightarrow \ x \in \left(-\sqrt{\varepsilon}, + - \sqrt{\varepsilon}\right).$$

Therefore,

$$|x| < \sqrt{\varepsilon} \Leftrightarrow \ x \in \left(-\sqrt{\varepsilon}, +\sqrt{\varepsilon}\right) \Rightarrow \left|x^2\right| < \varepsilon.$$

Then, choosing $\delta = \sqrt{\varepsilon}$, we have

$$|x| < \delta \Rightarrow \left|x^2\right| < \varepsilon,$$

as desired.

Remark 36 *The basic steps in trying to verify the condition characterizing the definition of limits are the following ones. Let $f : S \subseteq \mathbb{R} \to \mathbb{R}$ be given.*

1. solve the inequality $|f(x) - l| < \varepsilon$, i.e., the system

$$\begin{cases} f(x) > l - \varepsilon \\ \\ f(x) < l + \varepsilon; \end{cases}$$

let A be the set of such solutions.

2. find $\delta > 0$ small enough in order to have $(x_0 - \delta, x_0 + \delta) \cap S \subseteq \setminus \{x_0\} \subseteq A$.

To solve a system of inequalities could be very difficult. Some help could be obtained considering the system

$$\begin{cases} |f(x) - l| < \varepsilon \\ \\ x \in (x_0 - \alpha, x_0 + \alpha) \quad \texttt{or} \quad |x - x_0| < \alpha \end{cases}, \tag{2.2}$$

where $\alpha \in \mathbb{R}_{++}$ has to be chosen. Call $A(\alpha)$ the set of solutions to system (2.2). We can then proceed similarly to what done above, i.e., 1. we compute $A(\alpha)$, and 2. find $\delta > 0$ small enough in order to have$(x_0 - \delta, x_0 + \delta) \cap S \subseteq \setminus \{x_0\} \subseteq A(\alpha)$.

Remark 37 *Still in other words, we have to proceed as follows. Start from the inequality*

$$|f(x) - l| < \varepsilon;$$

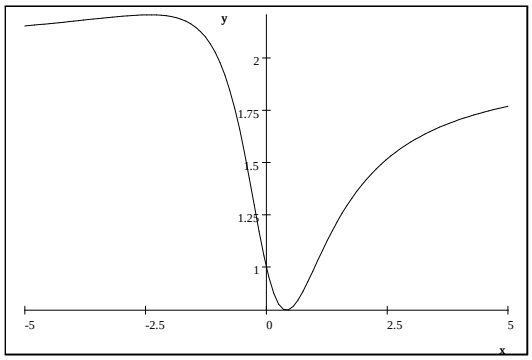
Find an inequality

$$|x| < g(\varepsilon)$$

which implies the desired inequality; set $\delta = g(\varepsilon)$ (be sure that $g(\varepsilon) > 0$).

Example 38 *We want to show that*

$$\lim_{x \to 0} \frac{2x^2 - x + 1}{x^2 + 1} = 1.$$



We want then to show that $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $\forall x \in \mathbb{R}$

$$|x| < \delta \Rightarrow \left| \frac{2x^2 - x + 1}{x^2 + 1} - 1 \right| < \varepsilon.$$

Observe that

$$\left| \frac{2x^2 - x + 1}{x^2 + 1} - 1 \right| = \left| \frac{2x^2 - x + 1 - x^2 - 1}{x^2 + 1} \right| = \left| \frac{x^2 - x}{x^2 + 1} \right| = \frac{|x| \cdot |x - 1|}{x^2 + 1}$$

Then, we can choose $\alpha = 1$, getting the following specification of system (2.2):

$$\begin{cases} \frac{|x| \cdot |x-1|}{x^2+1} < \varepsilon \\ \\ |x| < 1 \end{cases}. \tag{2.3}$$

Now, observe that $|x| < 1$, i.e., $-1 < x < 1$, implies that $|x - 1| < 2$; moreover, $x^2 + 1 > 1$. Then,

$$\frac{|x| \cdot |x - 1|}{x^2 + 1} < 2 |x| .$$

Therefore, the set of solutions to system (2.3) is contained the set of solutions to the following system

$$\begin{cases} 2 |x| < \varepsilon \\ \\ |x| < 1 \end{cases}. \tag{2.4}$$

Then it is enough to choose $\delta = \min \left\{ 1, \frac{\varepsilon}{2} \right\}$.

Example 39 *Let's now show that*

$$\lim_{x \to 0} \left(x^2 - 1 \right) \neq 1,$$

i.e.,

$$\neg \left\langle \ \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if and } 0 < |x - 0| < \delta, \text{ then } \left| \left(x^2 - 1 \right) - 1 \right| < \varepsilon, \ \right\rangle$$

or

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \ 0 < |x| < \delta \text{ and } \left| \left(x^2 - 1 \right) - 1 \right| \geq \varepsilon,$$

or

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0 \ \exists \overline{x} \in \mathbb{R} \text{ such that } \overline{x} \in (-\delta, +\delta) \setminus \{0\} \quad \text{and } \overline{x}^2 - 1 \notin (1 - \varepsilon, 1 + \varepsilon).$$

Take $\varepsilon = \frac{1}{2}$. Then if $\delta > 1$, take $\overline{x} = 1$ and observe that $f(\overline{x}) = 0 \notin \left(\frac{1}{2}, \frac{3}{2} \right)$; if $\delta \leq 1$, take $\overline{x} = \frac{\delta}{2} \in (-\delta, +\delta)$ and observe that $f(\overline{x}) = \frac{\delta^2}{4} - 1 < 0$ and therefore $f(\overline{x}) \notin \left(\frac{1}{2}, \frac{3}{2} \right)$.

We can also study the case in which x gets closer to x_0 just from the left or from the right of x_0.

Definition 40 *Given $S \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}$, we define*

$$S_{x_0}^{+} := \{ x \in S : x > x_0 \}$$

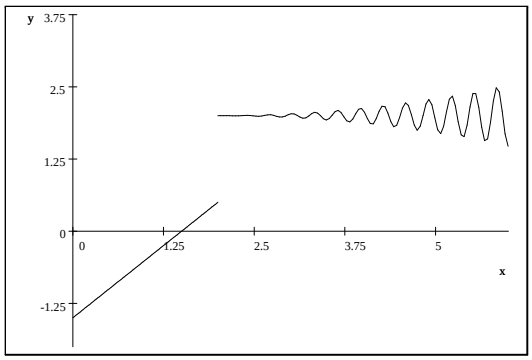
and

$$S_{x_0}^{-} := \{ x \in S : x < x_0 \}.$$

Definition 41 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D\left(S_{x_0}^{+} \right)$ be given*[1]*. We say that $l \in \mathbb{R}$ is the right-hand side limit of f at x_0 and we write*

$$\lim_{x \to x_{0+}} f(x) = l$$

[1] If $x_0 \in D\left(S_{x_0}^{+} \right)$, we can say that x_0 is a"right" accumulation point for S.



if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < x - x_0 < \delta, \text{ then } |f(x) - l| < \varepsilon.$$

or

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } x_0 < x < x_0 + \delta, \text{ then } |f(x) - l| < \varepsilon.$$

Similar definition can be given for the left-hand side limit.

Definition 42 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D\left(S_{x_0}^-\right)$ be given*[2]*. We say that $l \in \mathbb{R}$ is the left-hand side limit of f at x_0 and we write*

$$\lim_{x \to x_0^-} f(x) = l$$

if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < x_0 - x < \delta, \text{ then } |f(x) - l| < \varepsilon.$$

or

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } x_0 - \delta < x < x_0, \text{ then } |f(x) - l| < \varepsilon.$$

Remark 43 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D\left(S_{x_0}^+\right) \cap D\left(S_{x_0}^-\right)$ be given . Then,*

$$\left\langle \lim_{x \to x_0} f(x) = l \right\rangle \Leftrightarrow \left\langle \lim_{x \to x_0^-} f(x) = l \quad \wedge \quad \lim_{x \to x_0^+} f(x) = l \right\rangle.$$

The above result follows from the very definitions of limits and the fact that $x_0 \in D\left(S_{x_0}^+\right) \cap D\left(S_{x_0}^-\right)$ implies $x_0 \in D(S)$.

Remark 44 *We have given the definition of*

$$\lim_{x \to x_0} f(x) = l$$

in the case in which $x_0, l \in \mathbb{R}$. We now want to present the definitions in the cases in which $x_0, l \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 45 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. We say that $+\infty$ is a limit for f as x tends to x_0 and we write*

$$\lim_{x \to x_0} f(x) = +\infty$$

if

$$\forall k > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) > k,$$

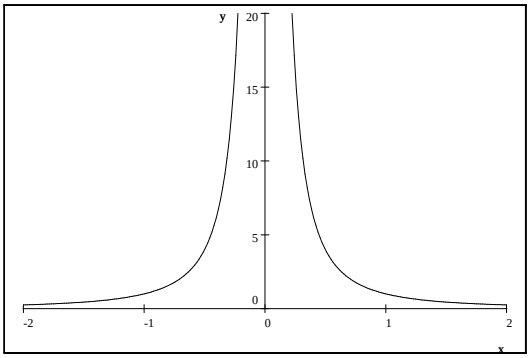
or

$$\forall k > 0 \ \exists \delta > 0 \text{ such that if } x \in S \cap B(x_0, \delta) \setminus \{x_0\}, \text{ then } f(x) \in (k, +\infty).$$

[2] If $x_0 \in D\left(S_{x_0}^-\right)$, we can say that x_0 is a"left" accumulation point for S.

Example 46 *Let's show that*

$$\lim_{x\to 0}\frac{1}{x^2}=+\infty.$$



We want to show that

$$\forall k>0\ \exists\delta>0 \text{ such that if } x\in\mathbb{R}\setminus\{0\} \text{ and } 0<|x|<\delta, \text{ then } \tfrac{1}{x^2}>k.$$

$\frac{1}{x^2}>k \Leftrightarrow x^2-\frac{1}{k}<0$ *and* $x\neq 0 \Leftrightarrow x\in\left(-\frac{1}{\sqrt{k}},+\frac{1}{\sqrt{k}}\right)\setminus\{0\}$. *Then it is enough to choose* $\delta=\frac{1}{\sqrt{k}}$.

Consistently with Remark 44, we have 9 possible definitions of limits, summarized in the following table.

$x_0\downarrow$ \ $l\rightarrow$	$\in\mathbb{R}$	$+\infty$	$-\infty$
$\in\mathbb{R}$	1	2	3
$+\infty$	4	5	6
$-\infty$	7	8	9

. (2.5)

Definition 31 corresponds to Case 1 in the table and Definition 45 to Case 2. Before presenting the other definitions, we want to describe an intuitive, informal, mnemonic definition which covers all possible cases. That definition and an associated simple graph will allow to give a rigorous definition for all cases described in Table (2.5).

In the presented Definitions (i.e., Definitions 31, 45 and also Definitions 41, 42) two expressions were crucial: "accumulation point" and "neighborhood of a point". It is useful to specialize those terms in the case of $x_0, l\in\{-\infty,+\infty\}$.

1. Given $S\subseteq\mathbb{R}$, we say that $+\infty$, respectively, $-\infty$, is an accumulation point for S, if S is unbounded above, respectively, below.

2. A neighborhood of $+\infty$ is the interval $(k,+\infty)$ for any $k\in\mathbb{R}$; a neighborhood of $-\infty$ is the interval $(-\infty,k)$ for any $k\in\mathbb{R}$.

We can then give the following general informal description of limit.

Definition 47 ***(informal)*** *Let a function* $f: S\subseteq\mathbb{R}\to\mathbb{R}$ *and a* $x_0\in\mathbb{R}\cup\{-\infty,+\infty\}$ *be an accumulation point for* S. *We say that* $l\in\mathbb{R}\cup\{-\infty,+\infty\}$ *is a limit for* f *as* x *tends to* x_0 *and we write*

$$\lim_{x\to x_0} f(x)=l$$

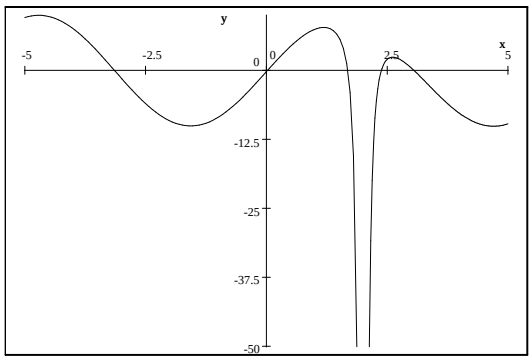
if

for any neighborhood $B(l)$ *of* l, *there exists a neighborhood* $B(x_0)$ *of* x_0 *such that if* $x\in(B(x_0)\cap S)\setminus\{x_0\}$, *then* $f(x)\in B(l)$. (2.6)

Remark 48 *Definition of left-side and right-side limits are given in a similar way.*

Using the above informal definition, we can then deal, for example, with Case 3 in Table 2.5 and say that

$$\lim_{x\to x_0} f(x)=-\infty \qquad \text{with } x_0\in\mathbb{R},$$



means

for any neighborhood $B(-\infty)$ of $-\infty$, there exists a neighborhood $N(x_0)$ such that if $x \in (N(x_0) \cap S) \setminus \{x_0\}$, then $f(x) \in B(-\infty)$,

or

for any interval $(-\infty, k)$, there exists $\delta > 0$ such that if $x \in (B(x_0, \delta) \cap S) \setminus \{x_0\}$, then $f(x) \in (-\infty, k)$.

Formally, we then have

Definition 49 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. We say that $-\infty$ is a limit for f as x tends to x_0 and we write*

$$\lim_{x \to x_0} f(x) = -\infty$$

if

$$\forall k > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) < k,$$

or

$$\forall k > 0 \ \exists \delta > 0 \text{ such that if } x \in (B(x_0, \delta) \cap S) \setminus \{x_0\}, \text{ then } f(x) < k.$$

Let's also give the definition of limit corresponding to entry 4 in Table 2.5.

Definition 50 *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded above be given. We say that l is a limit for f as x tends to $+\infty$ and we write*

$$\lim_{x \to +\infty} f(x) = l$$

if

$$\forall \varepsilon > 0 \ \exists k > 0 \text{ such that if } x \in S \text{ and } x > k, \text{ then } |f(x) - l| < \varepsilon,$$

or

$$\forall \varepsilon > 0 \ \exists k > 0 \text{ such that if } x \in (k, +\infty) \cap S \text{ then } |f(x) - l| < \varepsilon,$$

or

$$\forall \varepsilon > 0 \exists k > 0 \text{ such that } f((k, +\infty) \cap S) \subseteq B(l, \varepsilon).$$

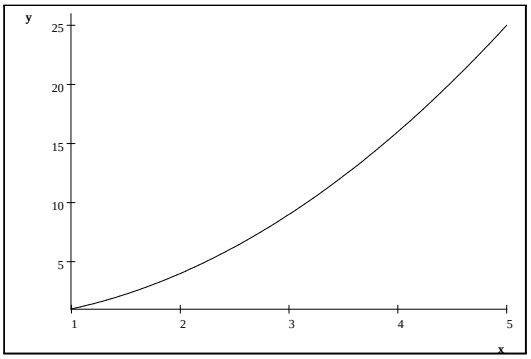
Remark 51 *We urge the reader to write the remaining 5 cases of Table 2.5. Below you can find the formal definitions.*

Definition 52 *(Definition 5 in the table) Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded above be given. We write*

$$\lim_{x \to +\infty} f(x) = +\infty$$

if

$$\forall k_1 > 0 \ \exists k_2 > 0 \text{ such that if } x \in S \text{ and } x > k_2, \text{ then } f(x) > k_1.$$

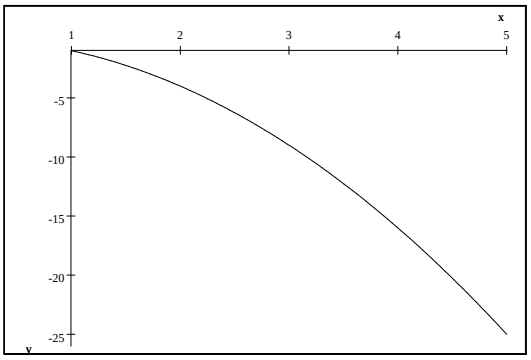


Definition 53 *(Definition 6 in the table) Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded above be given. We write*

$$\lim_{x \to +\infty} f(x) = -\infty$$

if

$$\forall k_1 < 0 \ \exists k_2 > 0 \text{ such that if } x \in S \text{ and } x > k_2, \text{ then } f(x) < k_1.$$



Definition 54 *(Definition 7 in the Table) Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded below be given. We say that*

$$\lim_{x \to -\infty} f(x) = l$$

if

$$\forall \varepsilon > 0 \quad \exists k < 0 \text{ such that if } x \in S \text{ and } x < k, \text{ then } |f(x) - l| < \varepsilon.$$

Definition 55 *(Definition 8 in the Table) Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded below be given. We say that*

$$\lim_{x \to -\infty} f(x) = +\infty$$

if

$$\forall k_1 > 0 \quad \exists k_2 < 0 \text{ such that if } x \in S \text{ and } x < k_2, \text{ then } f(x) > k_1.$$

Definition 56 *(Definition 9 in the Table) Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded below be given. We say that*

$$\lim_{x \to -\infty} f(x) = -\infty$$

if

$$\forall k_1 < 0 \;\; \exists k_2 < 0 \text{ such that if } x \in S \text{ and } x < k_2, \text{ then } f(x) < k_1.$$

2.2 Some theorems

Theorem 57 ***(uniqueness)*** *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. If*

$$\lim_{x\to x_0} f(x) = l_1 \in \mathbb{R}, \quad \textit{and} \quad \lim_{x\to x_0} f(x) = l_2 \in \mathbb{R},$$

then

$$l_1 = l_2.$$

Proof. Main informal steps: 1. write the definitions; 2. Take $\delta^* = \min\left\{\delta_{1,\frac{\varepsilon}{2}}, \delta_{2,\frac{\varepsilon}{2}}\right\}$ and show that $\forall \varepsilon > 0,\ |l_2 - l_1| < \varepsilon$.

By assumption and by definition of limit, we have that

$$\forall \varepsilon > 0\ \exists \delta_{1,\varepsilon} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_{1,\varepsilon}, \text{ then } |f(x) - l_1| < \varepsilon_1,$$

and

$$\forall \varepsilon > 0\ \exists \delta_{2,\varepsilon} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_{2,\varepsilon}, \text{ then } |f(x) - l_2| < \varepsilon_2.$$

Take define $\delta^* = \min\left\{\delta_{1,\frac{\varepsilon}{2}}, \delta_{2,\frac{\varepsilon}{2}}\right\}$. Then, we have that

$$\text{if } x \in S \text{ and } 0 < |x - x_0| < \delta^*, \text{ then } |f(x) - l_1| < \tfrac{\varepsilon}{2} \text{ and } |f(x) - l_2| < \tfrac{\varepsilon}{2}. \tag{2.7}$$

Therefore, if $x \in S$ and $0 < |x - x_0| < \delta^*$, then

$$|l_2 - l_1| = |l_2 - f(x) + f(x) - l_1| \overset{(1)}{\leq} |l_2 - f(x)| + |f(x) - l_1| =$$

$$= |l_2 - f(x)| + |l_1 - f(x)| \overset{(2)}{<} \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

where
(1) follows from the triangle inequality, and
(2) from (2.7).
Summarizing,

$$\forall \varepsilon > 0,\ |l_2 - l_1| < \varepsilon,$$

which implies the desired result $|l_2 - l_1| = 0$. ■

Remark 58 *The above theorem holds in the case in which $x_0, l_1, l_2 \in \mathbb{R} \cup \{-\infty, +\infty\}$.*

Remark 59 *Does the above theorem hold if in the definition of limit we do not assume that $x_0 \in D(S)$?*

Proposition 60 ***(Sign permanence)*** *Let a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. If*

$$\lim_{x\to x_0} f(x) = l \in \mathbb{R}\backslash\{0\},$$

then $\exists r \in \mathbb{R}_{++}$ such that

$$\forall x \in B(x_0, r) \cap S \setminus \{x_0\}, \quad \textit{sign } f(x) = \textit{sign } l.$$

Proof. Main informal steps: 1. Distinguish the cases $l > 0$ and $l < 0$; 2. In the first case, write the definition of limit with $\varepsilon = \frac{l}{2}$.

Since $l \in \mathbb{R}\backslash\{0\}$, we have only the two cases $l > 0$ and $l < 0$. We consider in detail just the first case.

By assumption and definition of limit taking $\varepsilon = \frac{l}{2} > 0$, we have

$$\exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } |f(x) - l| < \tfrac{l}{2}.$$

But

$$|f(x) - l| < \frac{l}{2} \Leftrightarrow 0 < \frac{l}{2} = l - \frac{l}{2} < f(x) < l + \frac{l}{2} = \frac{3}{2}l,$$

and therefore summarizing

$$\exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) > 0,$$

as desired.
For the analysis of the case $l < 0$, take $\varepsilon = -\frac{l}{2} > 0$. ■

Proposition 61 *Let functions $f, g : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given.*
If
1.

$$\exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) \leq g(x),$$

2.

$$\lim_{x \to x_0} f(x) = l_1 \in \mathbb{R},$$

3.

$$\lim_{x \to x_0} g(x) = l_2 \in \mathbb{R},$$

then

$$l_1 \leq l_2.$$

Proof. Main informal steps. Write the definitions of limits in Assumption 2. and 3 in the form $l_1 - \frac{\varepsilon}{2} < f(x) < l_1 + \frac{\varepsilon}{2}$; use Assumption 1.
From Assumptions 2. and 3, we have that

$$\forall \varepsilon > 0 \ \exists \delta_{1,\varepsilon} > 0 \text{ such that if } x \in S \cap B(x_0, \delta_{1,\varepsilon}) \setminus \{x_0\}, \text{ then } l_1 - \varepsilon < f(x) < l_1 + \varepsilon, \tag{2.8}$$

and

$$\forall \varepsilon > 0 \ \exists \delta_{2,\varepsilon} > 0 \text{ such that if } x \in S \cap B(x_0, \delta_{2,\varepsilon}) \setminus \{x_0\}, \text{ then } l_2 - \varepsilon < g(x) < l_2 + \varepsilon. \tag{2.9}$$

Taken $\delta^* = \min\left\{\delta, \delta_{1,\frac{\varepsilon}{2}}, \delta_{1,\frac{\varepsilon}{2}}\right\}$,and using Assumption 1., we have that

$$\forall \varepsilon > 0, \text{ if } x \in S \cap B(x_0, \delta_{2,\varepsilon}) \setminus \{x_0\}, \text{ then } l_1 - \tfrac{\varepsilon}{2} < f(x) \leq g(x) < l_2 + \tfrac{\varepsilon}{2}.$$

and therefore

$$\forall \varepsilon > 0, \ l_1 - \tfrac{\varepsilon}{2} < l_2 + \tfrac{\varepsilon}{2},$$

or

$$\forall \varepsilon > 0, \ l_1 - l_2 < \varepsilon,$$

as desired. ■

Proposition 62 ***(Comparison among limits)*** *Let functions $f, g, h : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. If*
1.

$$\exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) \leq g(x) \leq h(x),$$

2.

$$\lim_{x \to x_0} f(x) = l \in \mathbb{R},$$

3.

$$\lim_{x \to x_0} h(x) = l,$$

then

$$\lim_{x \to x_0} g(x) = l.$$

Proof. Main informal steps: 1. Write the definition of limits in Assumption 2. and 3., using "$l - \varepsilon \leq f(x) \leq l + \varepsilon$"; 2. Use Assumption 1.
By Assumptions 2. and 3., we have

$$\begin{aligned} &\forall \varepsilon > 0 \ \exists \delta_{1,\varepsilon} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_{1,\varepsilon}, \text{ then} \\ &|f(x) - l| < \varepsilon \quad \text{or} \quad l - \varepsilon_1 \leq f(x) \leq l + \varepsilon, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\forall \varepsilon > 0 \ \exists \delta_{2,\varepsilon} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_{2,\varepsilon}, \text{ then} \\ &|h(x) - l| < \varepsilon_2 \text{ or } \ l - \varepsilon_2 \leq h(x) \leq l + \varepsilon. \end{aligned} \tag{2.11}$$

Define $\delta^* = \min\{\delta_{1,\varepsilon}, \delta_{2,\varepsilon}\}$. Then, we have that

$$\text{if } x \in S \text{ and } 0 < |x - x_0| < \delta^*, \text{ then } l - \varepsilon \overset{(1)}{\leq} f(x) \overset{(2)}{\leq} g(x) \overset{(2)}{\leq} h(x) \overset{(3)}{\leq} l + \varepsilon,$$

where
(1) follows from (2.10),
(2) follow from the inequality in Assumption 1, and
(3) from (2.11).
Summarizing,

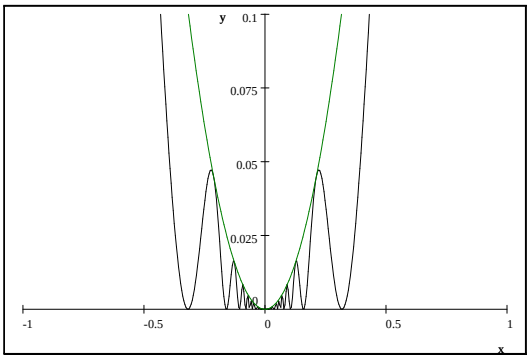
$$\forall \varepsilon > 0 \ \exists \delta^* > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta^*, \text{ then } \ l - \varepsilon \leq g(x) \leq l + \varepsilon,$$

as desired. ■

Remark 63 *In a similar way, the following results can be proven.*
Let functions $f, g : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given.
1. Assume that
(i) $\exists \delta > 0$ such that if $x \in S$ and $0 < |x - x_0| < \delta$, then $f(x) \geq g(x)$, and (ii) $\lim_{x \to x_0} g(x) = +\infty$,
then $\lim_{x \to x_0} f(x) = +\infty$.
2. Assume that
(i) $\exists \delta > 0$ such that if $x \in S$ and $0 < |x - x_0| < \delta$, then $f(x) \leq g(x)$, and (ii) $\lim_{x \to x_0} g(x) = -\infty$,
then $\lim_{x \to x_0} f(x) = -\infty$.
Indeed, we can also take $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Example 64 *Prove that*

$$\lim_{x \to 0} \left(x^2 \sin^2 \frac{1}{x} \right) = 0.$$



Since
1. $\forall x \in \mathbb{R}$,

$$0 \leq x^2 \sin^2 \frac{1}{x} \leq x^2,$$

2.

$$\lim_{x \to 0} 0 = 0 \in \mathbb{R},$$

3.

$$\lim_{x \to 0} x^2 = 0,$$

from Proposition 62, the desired result follows.

Remark 65 *Sometimes we are asked to show that a given limit does not exist. Examples.*

2.3 Operations with limits

Proposition 66 *Let functions $f, g : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. If*

1.

$$\lim_{x \to x_0} f(x) = l_1 \in \mathbb{R},$$

and

2.

$$\lim_{x \to x_0} g(x) = l_2 \in \mathbb{R}$$

then

1.

$$\lim_{x \to x_0} (f(x) \pm g(x)) = l_1 \pm l_2,$$

2.

$$\lim_{x \to x_0} (f(x) \cdot g(x)) = l_1 \cdot l_2.$$

Moreover,

3. if $\exists \delta > 0$ such that if $x \in S$ and $0 < |x - x_0| < \delta$, then $g(x) \neq 0$, and $l_2 \neq 0$, we have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}.$$

Proof. By Assumptions 1. and 2., we have

$$\forall \varepsilon > 0 \ \exists \delta_{1,\varepsilon} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_{1,\varepsilon}, \text{ then } |f(x) - l| < \varepsilon, \tag{2.12}$$

and

$$\forall \varepsilon > 0 \ \exists \delta_{2,\varepsilon} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_{2,\varepsilon}, \text{ then } |g(x) - l| < \varepsilon. \tag{2.13}$$

1. Main informal steps: Use the definitions of limits with $\frac{\varepsilon}{2}$ and the triangle inequality.
Define $\delta^* = \min\{\delta_{1,\varepsilon}, \delta_{2,\varepsilon}\}$. Then, if $x \in S$ and $0 < |x - x_0| < \delta^*$, then

$$|(f(x) + g(x)) - (l_1 + l_2)| = |[f(x) - l_1] + [g(x) - l_2]| \overset{(1)}{\leq}$$

$$\leq |f(x) - l_1| + |g(x) - l_2| \overset{(2)}{<} \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

where
(1) follows from the triangle inequality, and
(2) from (2.12) and (2.13).
Summarizing,

$$\forall \varepsilon > 0 \ \exists \delta^* > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta^*, \text{ then } |(f(x) + g(x)) - (l_1 + l_2)| < \varepsilon,$$

i.e.,

$$\lim_{x \to x_0} (f(x) + g(x)) = l_1 + l_2.$$

To show that $\lim_{x \to x_0} (f(x) - g(x)) = l_1 - l_2$, observe that

$$|(f(x) - g(x)) - (l_1 - l_2)| = |[f(x) - l_1] + [l_2 - g(x)]| \leq$$

$$\leq |f(x) - l_1| + |l_2 - g(x)| = |f(x) - l_1| + |g(x) - l_2| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

2. Main informal steps. Use the Preliminary Remark and add and subtract $l_1 \cdot g(x)$.
Preliminary Remark. Recall that

$$\forall a, b \in \mathbb{R}, \ ||a| - |b|| \leq |a - b|.$$

Observe that

$$\begin{aligned}|(f(x)\cdot g(x)-(l_1\cdot l_2))| &= |(f(x)\cdot g(x)-(l_1\cdot l_2))+l_1\cdot g(x)-l_1\cdot g(x)| = \\ &= |g(x)\cdot[f(x)-l_1]+l_1[g(x)-l_2]| \leq |g(x)|\cdot|f(x)-l_1|+|l_1|\cdot|g(x)-l_2|\end{aligned} \tag{2.14}$$

From (2.13), if $x \in B(x_0, \delta_2(\varepsilon_2 = 1)) \cap S\backslash\{x_0\}$, then

$$|g(x)| - |l_2| \leq |\ |g(x)| - |l_2|\ | \leq |g(x) - l_2| < 1,$$

where the first inequality follows from the Preliminary Remark. Then,

$$|g(x)| < |l_2| + 1. \tag{2.15}$$

Take

$$k = \max\{|l_2| + 1, |l_1|\} \tag{2.16}$$

Then if $x \in B(x_0, \delta_2(\varepsilon_2 = 1)) \cap S\backslash\{x_0\}$, from (2.14) and (2.15), we have

$$|(f(x)\cdot g(x)-(l_1\cdot l_2))| < k(|f(x)-l_1|+|g(x)-l_2|)$$

From Assumptions (2.12) and (2.13), chosen $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2k}$, we have that there exist $\delta_1\left(\frac{\varepsilon}{2k}\right)$ and $\delta_2\left(\frac{\varepsilon}{2k}\right)$ such that if

$$x \in B\left(x_0, \min\left\{\delta_2(1), \delta_1\left(\frac{\varepsilon}{2k}\right), \delta_2\left(\frac{\varepsilon}{2k}\right)\right\}\right) \cap S\backslash\{x_0\},$$

then

$$|(f(x)\cdot g(x)-(l_1\cdot l_2))| < k\left(\frac{\varepsilon}{2k}+\frac{\varepsilon}{2k}\right) = \varepsilon.$$

3. Omitted.

See Nannicini, Verdi and Vessella (2008), Theorem 7, page 148 and Barozzi and Corradi (1998), page 349. ■

Remark 67 *Proposition 66 gives information on the limit of the sum, product and quotient of functions under the assumption that the limits l_1 and l_2 of each function involved in the operation do belong to $\mathbb{R}$. What can we say if $l_1, l_2 \in \{-\infty, +\infty\}$? The following proposition studies one of the possible cases.*

Proposition 68 *Let functions $f, g : S \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in D(S)$ be given. If*

1.

$$\lim_{x\to x_0} f(x) = +\infty,$$

and

2.

$$\lim_{x\to x_0} g(x) = +\infty$$

then

$$\lim_{x\to x_0} (f(x) + g(x)) = +\infty.$$

Proof. Main informal steps: Just write assumptions and desired conclusion.

By assumptions, we know that

$$\forall k > 0\ \exists \delta_{1,k} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_1, \text{ then } f(x) > k,$$

and

$$\forall k > 0\ \exists \delta_{2,k} > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta_2, \text{ then } g(x) > k.$$

We want to show that

$$\forall k > 0\ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) + g(x) > k.$$

Taken $\delta = \min\{\delta_{1,k}, \delta_{2,k}\}$, then from the assumptions, we have that if $x \in S$ and $0 < |x - x_0| < \delta$, then $f(x) > k$ and $g(x) > k$ and therefore $f(x) + g(x) > 2k > k$, as desired. ■

To describe all possible cases which may arise when computing limits of sum, product and quotient of given functions, we need the following definition.

Definition 69 *Let function $f : S \subseteq \mathbb{R} \to \mathbb{R}$, a point $x_0 \in D(S)$ and $l \in \mathbb{R}$ be given. We write*

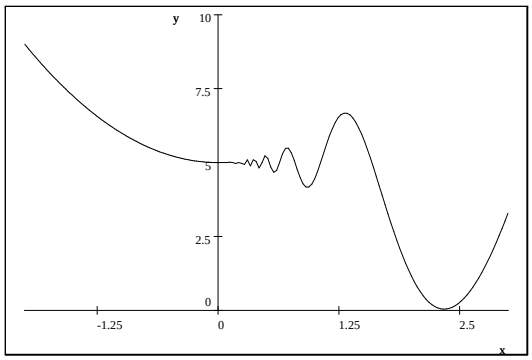
$$\lim_{x \to x_0} f(x) = l^+$$

if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } |f(x) - l| < \varepsilon \text{ and } f(x) \geq l \tag{2.17}$$

or

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } l \leq f(x) < l + \varepsilon. \tag{2.18}$$



Example 70 $\lim_{x \to 0} (\sin x)^2 + x^2 = 0^+$.

Definition 71 *We write*

$$\lim_{x \to x_0} f(x) = l^-$$

if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } |f(x) - l| < \varepsilon \text{ and } f(x) \leq l \tag{2.19}$$

or

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } 0 < |x - x_0| < \delta, \text{ then } l - \varepsilon < f(x) \leq l. \tag{2.20}$$

Remark 72 *If $\lim_{x \to x_0} f(x) = l^+$ or $\lim_{x \to x_0} f(x) = l^-$, then $\lim_{x \to x_0} f(x) = l$, but not vice versa.*

All possible cases mentioned in Remark 67 are summarized in the tables below.
Limits of the sum.

$f \to$ / $g \downarrow$	$+\infty$	$l_1 \in \mathbb{R}$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$l_2 \in \mathbb{R}$	$+\infty$	$l_1 + l_2$	$-\infty$
$-\infty$	?	$-\infty$	$-\infty$

(2.21)

Limits of the product.

$f \to$ / $g \downarrow$	$+\infty$	$l_1 > 0$	0	$l_1 < 0$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?	$-\infty$	$-\infty$
$l_2 > 0$	$+\infty$	$l_1 \cdot l_2$	0	$l_1 \cdot l_2$	$-\infty$
0	?	0	0	0	?
$l_2 < 0$	$-\infty$	$l_1 \cdot l_2$	0	$l_1 \cdot l_2$	$+\infty$
$-\infty$	$-\infty$	$-\infty$	?	$+\infty$	$+\infty$

(2.22)

Limits of the quotient.

$f \to$ / $g \downarrow$	$+\infty$	$l_1 > 0$	0	$l_1 < 0$	$-\infty$
$+\infty$	?	0	0	0	?
$l_2 > 0$	$+\infty$	$\frac{l_1}{l_2}$	0	$\frac{l_1}{l_2}$	$-\infty$
0^+	$+\infty$	$+\infty$	?	$-\infty$	$-\infty$
0^-	$-\infty$	$-\infty$	?	$+\infty$	$+\infty$
$l_2 < 0$	$-\infty$	$\frac{l_1}{l_2}$	0	$\frac{l_1}{l_2}$	$+\infty$
$-\infty$	?	0	0	0	?

(2.23)

In each table, entries in the first row indicate the value of $\lim_{x\to x_0} f(x)$ and entries in the first column indicate the value of $\lim_{x\to x_0} g(x)$. Indeed, $x_0 \in \mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$.

Each entry in the bottom right part of the table gives the result of the limit of the corresponding operation when $\lim_{x\to x_0} f(x)$is equal to the first entry of the column the considered entry belongs to and $\lim_{x\to x_0} g(x)$is equal to the first entry of the row the considered entry belongs to.

If a question mark "?" appears in an entry of the above tables, then the information contained in the corresponding row and column are not sufficient to determine the value in $\mathbb{R}^*$ corresponding to the limit under analysis. For example, consider the first question mark in the table (2.21). The exact meaning of that symbol is what follows.

Let functions $g : S \subseteq \mathbb{R} \to T \subseteq \mathbb{R}$, $f : T \to \mathbb{R}$ and points $x_0 \in D(S)$ and $y_0 \in D(T)$ be given. If

1.
$$\lim_{x\to x_0} f(x) = +\infty,$$

2.
$$\lim_{y\to x_0} g(x) = -\infty,$$

then
$$\lim_{x\to x_0} (f(x) + g(x))$$
can take different values in $\mathbb{R}^*$.

For example, taken, $f, g : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, and $x_0 = 0$,

1. if $f(x) = \frac{1}{|x|}$ and $g(x) = -\frac{1}{|x|}$, then $\lim_{x\to 0} (f(x) + g(x)) = \lim_{x\to 0} 0 = 0$;

2. if $f(x) = \frac{1}{|x|^2}$ and $g(x) = -\frac{1}{|x|}$, then $\lim_{x\to 0} (f(x) + g(x)) = \lim_{x\to 0} \left(\frac{1}{|x|^2} - \frac{1}{|x|}\right) = \lim_{x\to 0} \frac{1-|x|}{|x|^2} = +\infty$;

3. if $f(x) = \frac{1}{|x|}$ and $g(x) = -\frac{1}{|x|^2}$, then $\lim_{x\to 0} (f(x) + g(x)) = \lim_{x\to 0} \left(\frac{1}{|x|} - \frac{1}{|x|^2}\right) = -\lim_{x\to 0} \left(\frac{1}{|x|^2} - \frac{1}{|x|}\right) = -\infty$.

Other example: consider the case in which $x_0 = +\infty$. Then,

1. if $f(x) = x$ and $g(x) = -x$, then $\lim_{x\to+\infty} (f(x) + g(x)) = \lim_{x\to+\infty} 0 = 0$;

2. if $f(x) = x^2$ and $g(x) = -x$, then $\lim_{x\to+\infty} (f(x) + g(x)) = \lim_{x\to+\infty} x^2 - x = \lim_{x\to+\infty} x\left(1 - \frac{1}{x}\right) = +\infty$;

3. if $f(x) = x$ and $g(x) = -x^2$, then $\lim_{x\to+\infty} (f(x) + g(x)) = \lim_{x\to+\infty} -\left(x^2 - x\right) = -\lim_{x\to++} x\left(1 - \frac{1}{x}\right) = -\infty$.

The limits corresponding to question marks are called "indeterminate forms". The case discussed above is called an indeterminate form of the type

$$+\infty - \infty.$$

In the next chapter and in the chapter on differentiable functions, we will study some procedures to try to deal with "indeterminate forms".

Summarizing from the above tables, the indeterminate forms we met up to now are displayed below:

$$+\infty - \infty, \qquad 0 \cdot \infty, \qquad \frac{\infty}{\infty}, \qquad \frac{0}{0}.$$

Chapter 3

Sequences

3.1 Sequences in $\mathbb{R}$

3.1.1 Definitions and properties

Definition 73 *A sequence in $\mathbb{R}$ is a function $x : \mathbb{N} \to \mathbb{R}$.*

Usually, for any $n \in \mathbb{N}$, the value $x(n)$ is denoted by x_n, which is called the $n-th$ term of the sequence; the sequence is denoted by $(x_n, n \in \mathbb{N})$ or $(x_n)_{n\in\mathbb{N}}$.

Example 74 *To describe the function $f : \mathbb{N} \to \mathbb{R}$, $n \mapsto \frac{1}{n+1}$, we simply write*

$$\left(\frac{1}{n+1}, n \in \mathbb{N}\right) \quad or \quad \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}.$$

Remark 75 *The use of the above notation is motivated by the fact that the terms of a sequence have a "order" which is that one of the natural number. The symbol $\left\{\frac{1}{n+1}, n \in \mathbb{N}\right\}$ denotes the set of values that the sequence can take, i.e., the image of the function we called sequence. The two symbols denote different objects and therefore can be quite different. For example,*

$$\left((-1)^n, \quad n \in \mathbb{N}\right) = (1, -1, 1, -1, 1, ...),$$

and

$$\left\{(-1)^n, \quad n \in \mathbb{N}\right\} = \{-1, 1\}.$$

Definition 76 *$\mathbb{R}^\infty$ is the set of sequences $(x_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, $x_n \in \mathbb{R}$.*

Definition 77 *A sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ is said to be convergent to $x_0 \in \mathbb{R}$ if*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad |x_n - x_0| < \varepsilon. \tag{3.1}$$

x_0 is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ and we write

$$\lim_{n\to+\infty} x_n = x_0, \text{ or } x_n \xrightarrow{n} x_0. \tag{3.2}$$

$(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ is convergent if there exists $x_0 \in \mathbb{R}$ such that (3.1) holds. In that case, we say that the sequence converges to x_0 and x_0 is the limit of the sequence.

Example 78 *It's easy to verify by means of the definition that*

$$\lim_{n\to+\infty} \frac{1}{n} = 0.$$

Indeed, $|x_n - x_0| = \left|\frac{1}{n}\right| = \frac{1}{n}$. Since $\frac{1}{n} < \varepsilon$ whenever $n > \frac{1}{\varepsilon}$, it is enough to choose $n_0 = \frac{1}{\varepsilon}$. We have thus shown that $\forall \varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $|x_n - x_0| = \frac{1}{n} < \varepsilon$.

Proposition 79 ***(Limit uniqueness)*** *A sequence converges at most to one element of* $\mathbb{R}$*. In other words, if*

$$\lim_{n\to+\infty} x_n = l_1$$

and

$$\lim_{n\to+\infty} x_n = l_2$$

with $l_1, l_2 \in \mathbb{R}$*, then*

$$l_1 = l_2.$$

Proof. Suppose that $\lim_{n\to+\infty} x_n = l_1$ and that $\lim_{n\to+\infty} x_n = l_2$, with $l_1, l_2 \in \mathbb{R}$ and $l_1 \neq l_2$. By choosing $\varepsilon = \frac{|l_1-l_2|}{2} > 0$, we have that $\exists n_1 \in \mathbb{N}$ such that for any $n > n_1$, $|x_n - l_1| < \varepsilon$ and $\exists n_2 \in \mathbb{N}$ such that for any $n > n_2$, $|x_n - l_2| < \varepsilon$. Taking $n_0 = \max\{n_1, n_2\}$, both above inequalities hold true and using the triangle inequality we get

$$|l_1 - l_2| = |(l_1 - x_n) + (x_n - l_2)| \leq |l_1 - x_n| + |x_n - l_2| = |x_n - l_1| + |x_n - l_2| < \varepsilon + \varepsilon = |l_1 - l_2|.$$

We have found that $|l_1 - l_2| < |l_1 - l_2|$ which is absurd. ■

Definition 80 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *has limit* $+\infty$ *if*

$$\forall M > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0,\ x_n > M.$$

An analogous definition can be given in the case of a sequence converging to $-\infty$*. For both cases we can write*

$$\lim_{n\to+\infty} x_n = +\infty \quad \text{meaning} \quad \forall M > 0, \exists n_0 \text{ such that } x_n > M, \quad \forall n > n_0; \tag{3.3}$$

$$\lim_{n\to+\infty} x_n = -\infty \quad \text{meaning} \quad \forall M > 0, \exists n_0 \text{ such that } x_n < -M, \quad \forall n > n_0. \tag{3.4}$$

Proposition 81 *Given* $x_0 \in D(S)$ *and* $f : S \to T$*,*

$$\langle \lim_{x\to x_0} f(x) = l \rangle$$
$$\Leftrightarrow$$
$$\left\langle \begin{array}{l} \text{for any sequence } (x_n)_{n\in\mathbb{N}} \text{ in } S \text{ such that } \forall n \in \mathbb{N},\ x_n \neq x_0 \text{ and } \lim_{n\to+\infty} x_n = x_0, \\ \\ \lim_{n\to+\infty} f(x_n) = l. \end{array} \right\rangle$$

Remark 82 *The above Proposition holds true if* $x_0, l \in \mathbb{R} \cup \{-\infty, +\infty\}$*.*

Proof. of Proposition 81.
[⇒]
Take

$$\text{a sequence } (x_n)_{n\in\mathbb{N}} \text{ in } S \text{ such that } \forall n \in \mathbb{N},\ x_n \neq x_0 \text{ and } \lim_{n\to+\infty} x_n = x_0$$

We want to show that $\lim_{n\to+\infty} f(x_n) = l$, i.e.,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0,\ |f(x_n) - l| < \varepsilon$$

Since $\lim_{x\to x_0} f(x) = l$,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in S \ \wedge\ 0 < d(x, x_0) < \delta \Rightarrow |f(x) - l| < \varepsilon$$

Since $\lim_{n\to+\infty} x_n = x$

$$\forall \delta > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0,\ 0 \overset{(*)}{<} |x_n - x_0| < \delta$$

where $(*)$ follows from the fact that $\forall n \in \mathbb{N}$, $x_n \neq x_0$.

Therefore, combining the above results, we get

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, |f(x_n) - l| < \varepsilon$$

as desired.
[⇐]
Suppose otherwise, then

$$\begin{aligned} &\exists \varepsilon > 0 \text{ such that } \forall \delta_n = \tfrac{1}{n}, \text{ i.e., } \forall\ n \in \mathbb{N}, \exists x_n \in S \text{ such that} \\ &x_n \in S\ \wedge 0 < |x_n - x_0| < \tfrac{1}{n} \text{and } |f(x) - l| \geq \varepsilon. \end{aligned} \tag{3.5}$$

Consider $(x_n)_{n\in\mathbb{N}}$; then, from the above and from Proposition **??**, $x_n \to x_0$, and from the above (specifically the fact that $0 < |x_n - x_0|$), we also have that $\forall n \in \mathbb{N},\ \ x_n \neq x_0$. Then by assumption, $\lim_{n\to+\infty} f(x_n) = l$, i.e., by definition of limit,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that if } n > N, \text{ then } |f(x_n - l)| < \varepsilon,$$

contradicting (3.5). ■

Remark 83 *The above result is quite convenient to show that in some case limits do not exist.*

3.2 Exercises.

Section 7.15 in Nannicini, Verdi and Vessella (2008):
1,2,3,4,5,6.
Chapter 6 in Barozzi and Corradi (1989).
1,2,3,4,5,6

Chapter 4

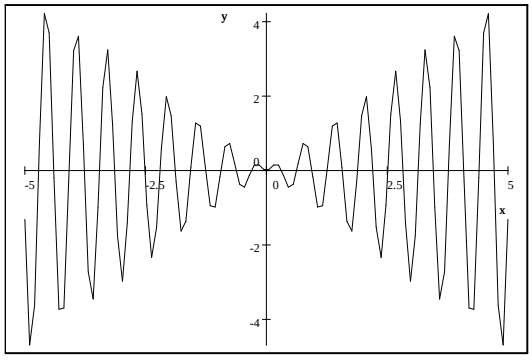
Continuous functions

4.1 Definition

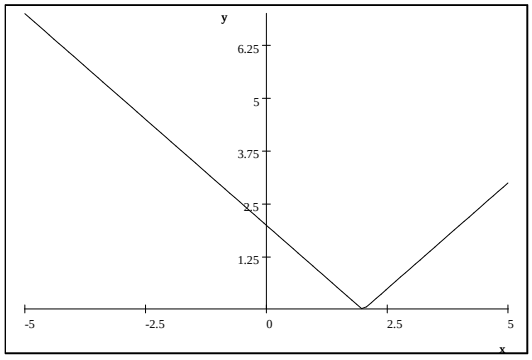
An intuitive definition of continuity for function whose domain is an interval is the following one:

"if I is an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ is continuous if its graph can be drawn without lifting your pen from paper".

1. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x \sin 10x$

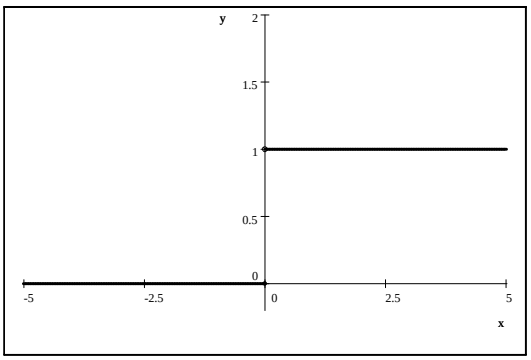


2. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x - 2|$



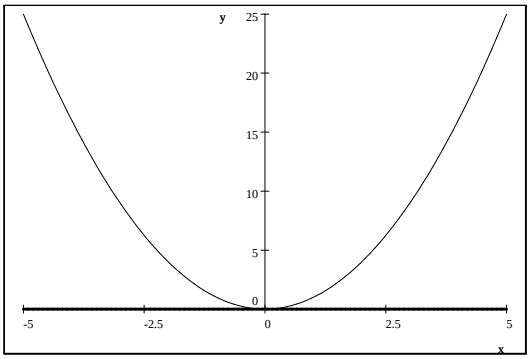
3.

$$f : \mathbb{R}, f : x \mapsto \begin{cases} 0 & if \quad x < 0 \\ 0 & if \quad x = 0 \\ 1 & if \quad x > 0 \end{cases}$$



4.

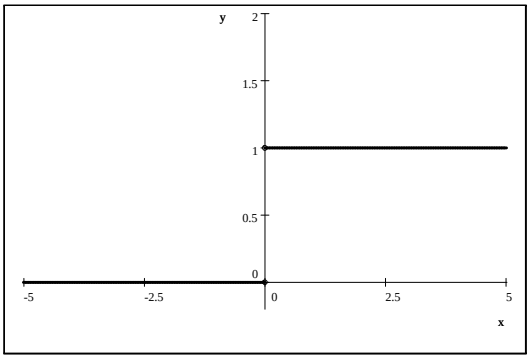
$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x^2 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$



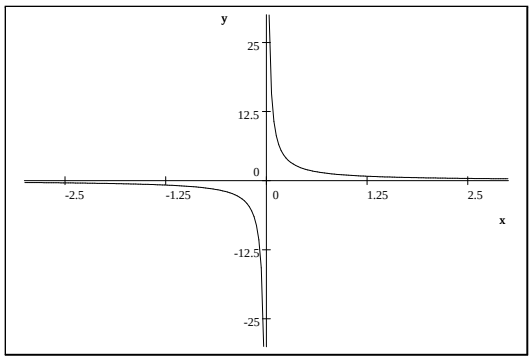
What if the domain of f is not an interval?

3.

$$f : \mathbb{R} \setminus \{0\}, \ f : x \mapsto \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x > 0 \end{cases}$$



2. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$



Definition 84 *Let $f : S \subseteq \mathbb{R} \to \mathbb{R}$ be given. We say that f is continuous in x_0 if $x_0 \in S$ and*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \text{ and } |x - x_0| < \delta, \text{ then } |f(x) - f(x_0)| < \varepsilon. \tag{4.1}$$

or,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } x \in S \cap B(x_0, \delta), \text{ then } f(x) \in B(f(x_0), \varepsilon). \tag{4.2}$$

or,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } f(S \cap B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon). \tag{4.3}$$

Remark 85 *If $x_0 \in D(S)$ (and $x_0 \in S$), then*

$$\langle \ f \text{ is continuous in } x_0 \ \rangle \Longleftrightarrow \left\langle \lim_{x \to x_0} f(x) = f(x_0) \right\rangle.$$

If $x_0 \notin D(S)$ (and $x \in S$), i.e., x_0 is an isolated point, then f is continuous in x_0, as proved below. Recall that, by definition, x_0 is an isolated point for S if 1. $x_0 \in S$, and 2. $\exists r \in \mathbb{R}_{++}$ such that $S \cap B(x_0, r) = \{x_0\}$. Then to get condition (4.2), it is enough to take $\delta = r$. Then, $x \in S \cap B(x_0, \delta)$ implies that $x = x_0$ and obviously $f(x_0) \in B(f(x_0), \varepsilon)$.

Remark 86 *If $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is continuous, $x_0 \in D(S)$ and $x \in S$, it is immediate to compute $\lim_{x \to x_0} f(x)$: by definition of continuous function, it is just $f(x_0)$, i.e.,*

$$\text{if } x_0 \in D(S) \cap S \text{ and } f \text{ is continuous in } x_0, \text{ then } \lim_{x \to x_0} f(x) = f(x_0).$$

Definition 87 *Let $f : S \subseteq \mathbb{R} \to \mathbb{R}$ be given. We say that f is continuous on $A \subseteq S$ if for any $x_0 \in A$, f is continuous in x_0. f is continuous if it is continuous on S.*

Example 88 *1.*

$$f : \mathbb{R} \setminus \{0\}, f : x \mapsto \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x > 0 \end{cases}$$

is continuous.

It is enough to show that for any $a \in \mathbb{R} \setminus \{0\}$, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a$ is continuous. We want to show that

$$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } |x - x_0| < \delta, \text{ then } |a - a| = 0 < \varepsilon,$$

which is obvious.

Observe that f is not continuous at 0 because 0 does not belong to the domain of f.

2.

$$f : \mathbb{R}, f : x \mapsto \begin{cases} 0 & if \quad x < 0 \\ 0 & if \quad x = 0 \\ 1 & if \quad x > 0 \end{cases}$$

is not continuous.

Indeed, we show that the function is not continuous in 0.

First proof. We want to show

$$not \ \langle \ \forall \varepsilon > 0 \ \exists \delta > 0 \ such \ that \ if \ |x| < \delta, \ then \ |f(x)| < \varepsilon, \ \rangle$$

i.e.,

$$\exists \varepsilon > 0 \ such \ that \ \forall \delta > 0 \ there \ exists \ \overline{x} \in \mathbb{R} \ such \ that \ |\overline{x}| < \delta \ and \ |f(\overline{x})| \geq \varepsilon.$$

Take $\varepsilon = \frac{1}{2}$*; then choose* $\overline{x} = \frac{\delta}{2}$*. Then* $\overline{x} = \frac{\delta}{2} < \delta$ *and* $f(\overline{x}) = \left|f\left(\frac{\delta}{2}\right)\right| = 1 > \frac{1}{2}$.

Second proof.

$$\lim_{x \to 0^-} f(x) = f(0) = 0 \neq 1 = \lim_{x \to 0^+} f(x).$$

3.

$$f : \mathbb{Q} \to \mathbb{R}, x \mapsto 1$$

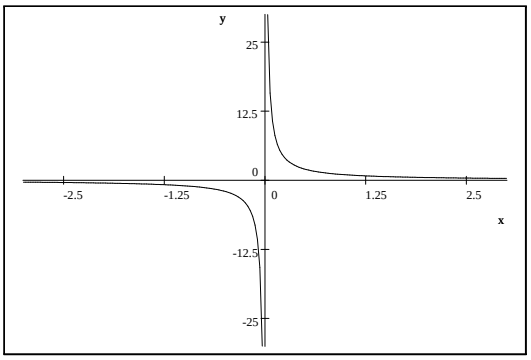
is continuous.

For any $x_0 \in \mathbb{Q}$, $\forall \varepsilon > 0 \ \exists \delta > 0$ *such that* $x \in B(x_0, \delta) \cap \mathbb{Q} \Rightarrow |f(x) - f(x_0)| = |1 - 1| = 0 < \varepsilon$.

4.

$$f : \mathbb{R} \backslash \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$$

is not continuos in zero.



5.

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

is not continuous in 0.

Proposition 89 *(sign permanence for continuous functions) If* $f : S \subseteq \mathbb{R} \to \mathbb{R}$ *is continuous in* x_0 *and* $f(x_0) \neq 0$*, then* $\exists r \in \mathbb{R}_{++}$ *such that*

$$\forall x \in B(x_0, r) \cap S \setminus \{x_0\}, \quad sign \ f(x) = sign \ l.$$

Proof. If $x_0 \notin D(S)$, then the result is obvious. If $x_0 \in D(S)$, the it is an immediate consequence of Proposition 60. ■

In what follows we try to describe the set of continuous functions.

4.2 The set of continuous functions

Proposition 90 *Let* $f, g : S \subseteq \mathbb{R} \to \mathbb{R}$ *be given. Assume that* f *and* g *are continuous in* $x_0 \in S$*, then the following statements hold true.*

1.a. $f + g : S \to \mathbb{R}$, $x \mapsto f(x) + g(x)$ *is continuous in* x_0*;*

1.b. $f - g : S \to \mathbb{R}$, $x \mapsto f(x) - g(x)$ *is continuous in* x_0*;*

2. $f \cdot g : S \to \mathbb{R}$, $x \mapsto f(x) \cdot g(x)$ *is continuous in* x_0*;*

3. if $g(x_0) \neq 0$,[1] *then* $\frac{f}{g} : S \to \mathbb{R}$, $x \mapsto \frac{f(x)}{g(x)}$ *is continuous in* x_0.

[1] The fact that $g(x_0) \neq 0$ implies that $g(x) \neq 0$ in a neighborhood of x_0.

Proof. Recall that if x_0 is an isolated point, then any function is continuous in x_0. Assume that $x_0 \in D(S)$. Then, a function $h : S \subseteq \mathbb{R} \to \mathbb{R}$ is continuous in x_0 if $\lim_{x \to x_0} h(x) = h(x_0)$. Then the desired results follows from the Proposition on the operations on limits, i.e., Proposition 66. ■

Proposition 91 *Polynomial functions are continuous functions, i.e., for any $n \in \mathbb{N}$ and for any $a_0, a_1, a_2, ..., a_n \in \mathbb{R}$, the function*

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

is continuous.

Proof. We proceed in three steps.

1. for any $n \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ is continuous;

2. for any $\alpha \in \mathbb{R}$ and for any $n \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x^n$ is continuous;

3. desired result

1. First of all observe that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^0 = 1$ is clearly continuous. We proceed by induction to show the statement for any $n \in \mathbb{N} \setminus \{0\}$.

step i. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^1$ is continuous.

It is enough to choose $\delta = \varepsilon$.

Step ii. We assume that $f_{n-1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{n-1}$ is continuous and we want to show that $f_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ is continuous. Observe that for any $x \in \mathbb{R}$, $f_n(x) = f_{n-1}(x) \cdot x$. Then, from Proposition 90.2, we get the desired result.

2. Define: for any $\alpha \in \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha$ (the constant function), $f_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ and $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x^n$. Then, since h is continuous, f_n is continuous from Step 1 and $f = h \cdot f_n$, the desired result follows from Proposition 90.2.

3. The desired results follows from Step 2 and Proposition 90.1. ■

Corollary 92 *Let $m, n \in \mathbb{N}_+$, $a_1, ..., a_m, b_1, ..., b_n \in \mathbb{R}$ with at least one b_j different from zero and $S = \left\{ x \in \mathbb{R} : \sum_{j=1}^{n} b_i x^j \neq 0 \right\}$ be given, The function*

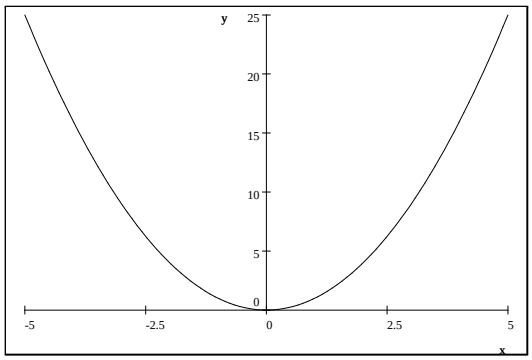
$$f : S \to \mathbb{R}, \quad x \mapsto \frac{\sum_{i=1}^{m} a_i x^i}{\sum_{j=1}^{n} b_i x^j}$$

is continuous.

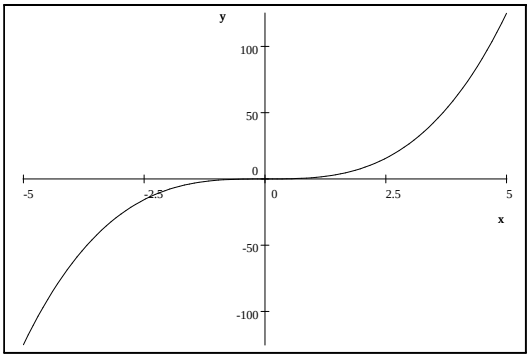
Proof. It follows from Propositions 91 and 90.3. ■

Using the properties of continuous functions and the properties of the limits summarized in Tables(2.21) - (2.23), the following results can be easily verified.

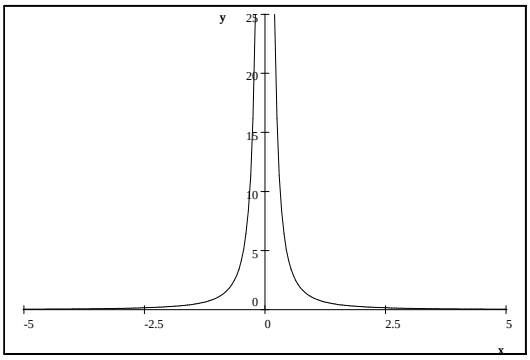
Graphs of $f : \mathbb{R} \to \mathbb{R}$, x^n with even $n \in \mathbb{N}_+$.



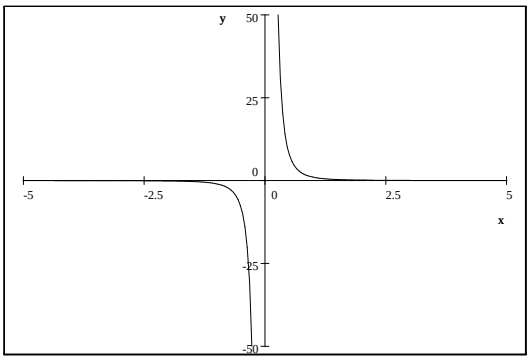
Graphs of $f : \mathbb{R} \to \mathbb{R}$, x^n with odd $nn \in \mathbb{N}_+$.



Graphs of $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $\frac{1}{x^n}$ with even $n \in \mathbb{N}_+$.



Graphs of $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $\frac{1}{x^n}$with odd $n \in \mathbb{N}_+$.



It can be verified that

1. for any $n \in \mathbb{N}_+$,
$$\lim_{x\to+\infty} x^n = +\infty, \qquad \lim_{x\to\pm\infty} \frac{1}{x^n} = 0;$$

2. for any $n \in \mathbb{N}_+$,
$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty & if \quad n \text{ is even} \\ -\infty & if \quad n \text{ is odd;} \end{cases}$$

3. for any $m \in \mathbb{N}_+$, , $a_0, a_1, ..., a_{m-1} \in \mathbb{R}$, $a_m \in \mathbb{R}\backslash\{0\}$,

$$\lim_{x\to\pm\infty}(a_0 + a_1x + ... + a_mx^m) = \lim_{x\to\pm\infty} x^m \left(\frac{a_0}{x^m} + a_1\frac{x}{x^m} + ... + a_m\frac{x^m}{x^m}\right) =$$

$$= \lim_{x\to\pm\infty} x^m \cdot \lim_{x\to\pm\infty}\left(\frac{a_0}{x^m} + a_1\frac{1}{x^{m-1}} + ... + a_m\right) = a_m \lim_{x\to\pm\infty} x^m$$

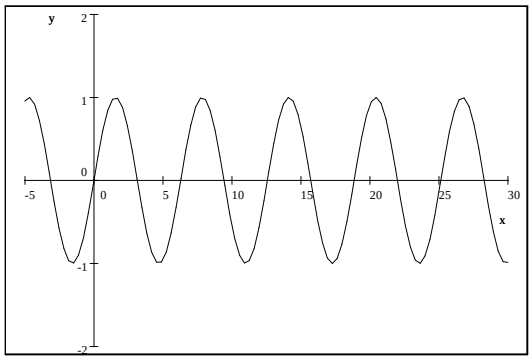
4. If the denominator of the fraction in the limit is different from zero, we have

$$\lim_{x\to\pm\infty} \frac{a_0+a_1x+...+a_mx^m}{b_0+b_1x+...+b_nx^n} = \lim_{x\to\pm\infty} \frac{\frac{x^m}{x^n}}{\frac{x^m}{x^n}} \frac{a_0\frac{1}{x^m}+a_1\frac{x}{x^m}+...+a_{m-1}\frac{x^{m-1}}{x^m}+a_m\frac{x^m}{x^m}}{b_0\frac{1}{x^n}+b_1\frac{x}{x^n}+...+b_{n-1}\frac{x^{n-1}}{x^n}+b_n\frac{x^n}{x^n}} =$$

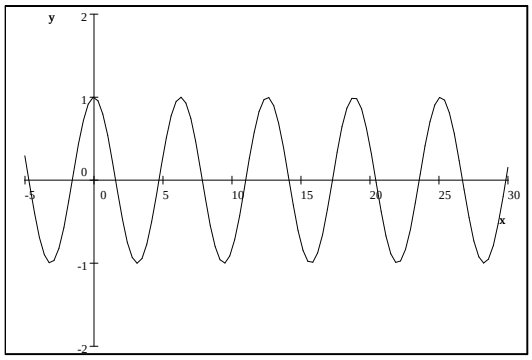
$$= \lim_{x\to\pm\infty} \frac{a_m}{b_n}x^{m-n}.$$

Proposition 93 *The functions* sin *and* cos *are continuous.*

$\sin x$



$\cos x$



Proof. Omitted. ■

Corollary 94 *Defined* $S := \{x \in \mathbb{R} : \cos x = 0\} = \left\{x \in \mathbb{R} : x = \frac{\pi}{2} + k\pi, \text{ with } k \in \mathbb{Z}\right\}$ *and* $T := \{x \in \mathbb{R} : \sin x = 0\} = \{x \in \mathbb{R} : x = k\pi, \text{ with } k \in \mathbb{Z}\}$, *the functions*

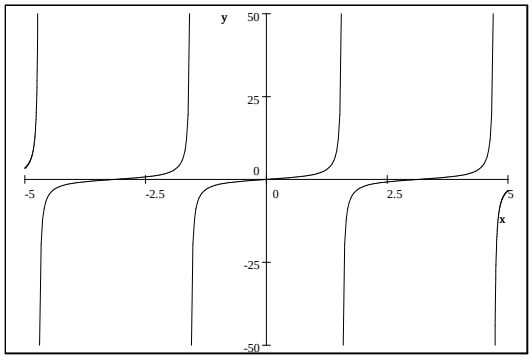
$$\tan : \mathbb{R}\backslash S \to \mathbb{R}, \quad x \mapsto \tan x := \frac{\sin x}{\cos x}$$

and

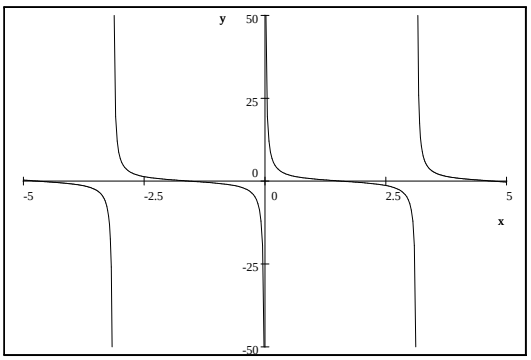
$$\cot : \mathbb{R}\backslash T \to \mathbb{R}, \quad x \mapsto \cot x := \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

are continuous.

$\tan x$



$\cot x$



Proof. It follows from Proposition 93 and Proposition 90.3. ■

Proposition 95 *Given $a \in \mathbb{R}_{++} \setminus \{1\}$, the function*

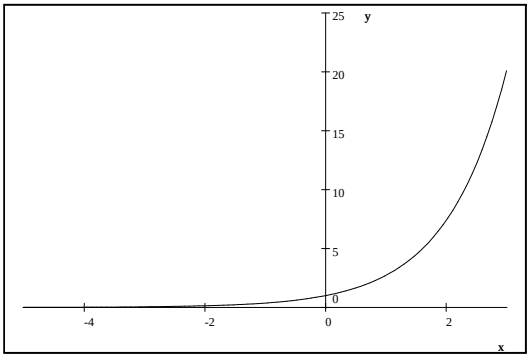
$$\exp_a : \mathbb{R} \to \mathbb{R}, \quad x \mapsto a^x$$

is continuous.

Proof. We want to show that

$$\forall x_0 \in \mathbb{R}, \ \ \forall \varepsilon > 0, \quad \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow |a^x - a^{x_0}| < \varepsilon. \tag{4.4}$$

We consider only the case $a > 1$, the analysis of case $a < 1$ being very similar.



Observe that since a^x is increasing, then

$$x \geq x_0 \quad \Rightarrow \qquad |a^x - a^{x_0}| = a^x - a^{x_0} = a^{x_0}\left(a^{x-x_0} - 1\right) = a^{x_0}\left(a^{|x-x_0|} - 1\right),$$

$$x < x_0 \quad \Rightarrow \quad |a^x - a^{x_0}| = a^{x_0} - a^x = a^x\left(a^{x_0-x} - 1\right) < a^{x_0}\left(a^{x_0-x} - 1\right) = a^{x_0}\left(a^{|x-x_0|} - 1\right).$$

Then for any $x \in \mathbb{R}$

$$|a^x - a^{x_0}| \leq a^{x_0}\left(a^{|x-x_0|} - 1\right).$$

To get (4.4), it suffices to have

$a^{x_0}\left(a^{|x-x_0|} - 1\right) < \varepsilon$ i.e., multiplying by a^{-x_0},

$a^{|x-x_0|} - 1 < \varepsilon a^{-x_0}$ and adding 1 to both sides,

$a^{|x-x_0|} < \varepsilon a^{-x_0} + 1$ and since $\log_a$ with $a > 1$ is a strictly increasing function,

$|x - x_0| < \log\left(\varepsilon a^{-x_0} + 1\right)$.

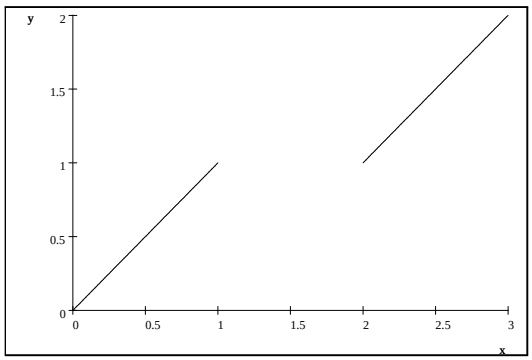
Then, observed that $\varepsilon a^{-x_0} + 1 > 1$ and therefore $\log_a\left(\varepsilon a^{-x_0} + 1\right) > 0$, it is enough to take $\delta \in (0, \log_a\left(\varepsilon a^{-x_0} + 1\right)]$. ■

Proposition 96 *Let I be an interval in $\mathbb{R}$ and $f : I \to \operatorname{Im} f$ be a strictly monotone function. Then, (f is invertible and) f^{-1} is a continuous function.*

Proof. Omitted. ■

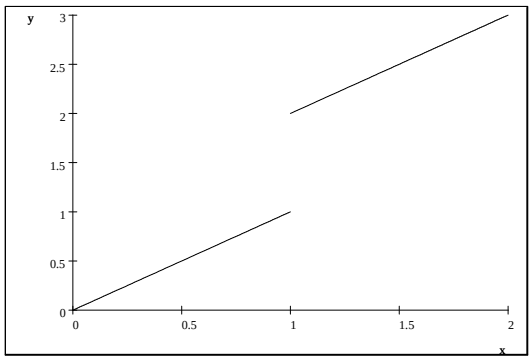
Remark 97 *The conclusion contained in the statement of the above Proposition is false if it is not assumed that the domain of f is an interval. Take*

$$f : [0,1) \cup [2,3] \to [1,2]\,, \qquad x \mapsto \begin{cases} x & if \quad x \in [0,1) \\ \\ x-1 & if \quad x \in [2,3]\,. \end{cases}$$



f is continuos and invertible, but it inverse

$$f : [1,2] \to [0,1) \cup [2,3]\,, \qquad y \mapsto \begin{cases} y & if \quad x \in [0,1) \\ \\ y+1 & if \quad x \in [1,2]\,. \end{cases}$$



it is not continuous in 1.

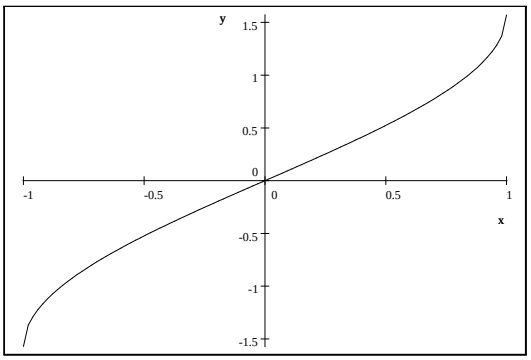
Through a restriction of domain (and codomain), the following "functions" are invertible

$$n-th\ power, \sin, \cos, \tan, \cot, \exp\,.$$

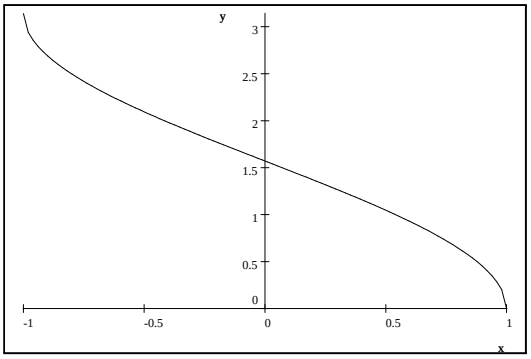
Since we have seen that those functions are continuous on well chosen intervals, then the previous proposition allows to conclude that the following functions are continuous as well:

$$n-th \text{ square root}, \arcsin, \arccos, \arctan, \operatorname{arccot},\ \log\,.$$

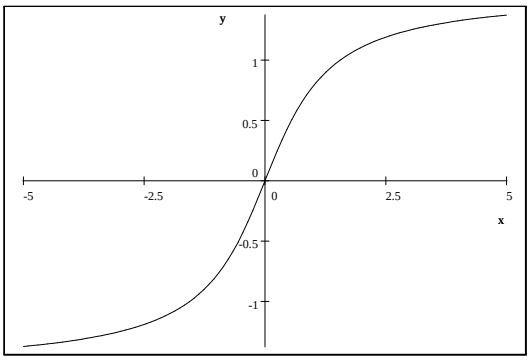
More precisely,
since $f : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto x^n$ is continuous, then $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto \sqrt[n]{x}$ is continuous;
since $f : \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \to [-1,+1]$, $x \mapsto \sin x$ is continuous, then $f^{-1} : [-1,+1] \to \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$, $x \mapsto \arcsin x$ is continuous;



since $f : [0, \pi] \to [-1, +1]$, $x \mapsto \cos x$ is continuous, then $f^{-1} : [-1, +1] \to [0, \pi]$, $x \mapsto \arccos x$ is continuous;



since $f : \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \to \mathbb{R}$, $x \mapsto \arctan x$ is continuous, then $f^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$, $x \mapsto \arcsin x$ is continuous;

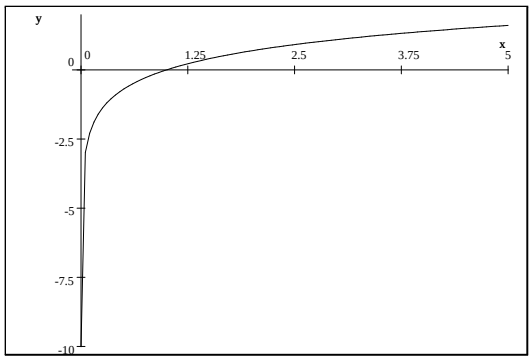


since $f : (0, \pi) \to \mathbb{R}$, $x \mapsto \operatorname{arccot} x$ is continuous, then $f^{-1} : \mathbb{R} \to (0, \pi)$, $x \mapsto \operatorname{arccot} x$ is continuous;

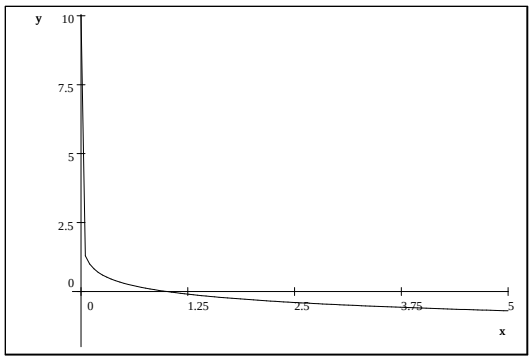
taken $a \in \mathbb{R}_{++} \setminus \{1\}$, since $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $x \mapsto a^x$ is continuous, then $f^{-1} : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $x \mapsto \log_x a$ is continuous.

Graph of $f : \mathbb{R}_{++} \to \mathbb{R}$, $f : x \mapsto \log_a x$, with $a > 1$:

$\log x$



Graph of $f : \mathbb{R}_{++} \to \mathbb{R}$, $f : x \mapsto \log_b x$, with $0 < b < 1$:[2]
$\log_{\frac{1}{a}} x = -\log_a x$



The following table summarizes some of the results we have seen above.

function	$f : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto x^n$	$f : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto \frac{1}{x^n}$
limit at $+\infty$	$+\infty$	$+\infty$
limit at $-\infty$	not defined	not defined

function	$f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin x$	$f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \cos x$	$f : \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \to \mathbb{R}$, $x \mapsto \arctan x$
limit at $+\infty$	does not exists	does not exists	$\frac{\pi}{2}$
limit at $-\infty$	does not exists	does not exists	$-\frac{\pi}{2}$

[2] Observe that if $b \in (0,1)$, then $\exists a > 1$ such that $b = \frac{1}{a}$. Then

$$\log_b x = \log_{\frac{1}{a}} x = \frac{\log_a x}{\log_a \frac{1}{a}} = -\log_a x.$$

function		$f:\mathbb{R}\to\mathbb{R}_{++}$, $x\mapsto a^x$, $a>1$		$f:\mathbb{R}_{++}\to\mathbb{R}$, $x\mapsto \log_a x$, $a>1$		$f:\mathbb{R}\to\mathbb{R}_{++}$, $x\mapsto a^x$, $a\in(0.1)$		$f:\mathbb{R}_{++}\to\mathbb{R}$, $x\mapsto \log_a x$, $a>1$
limit at $+\infty$		$+\infty$		$+\infty$		0		$-\infty$
limit at $-\infty$		0		not defined		$+\infty$		not defined

Proposition 98 *(Composition of continuous functions) Let $f:S\subseteq\mathbb{R}\to\mathbb{R}$ such that $f(S)\subseteq T\subseteq\mathbb{R}$ and $g:T\to\mathbb{R}$ be given. If f is continuous in $x_0\in S$ and g is continuous in $f(x_0)$, then $g\circ f$ is continuous in x_0.*

Proof. Main informal steps. Write the assumptions and the desired conclusion in terms of open balls.

By assumption, we have that

$$\forall\varepsilon>0\ \exists\delta_{1,\varepsilon}>0 \text{ such that if } x\in S\cap B(x_0,\delta_{1,\varepsilon}), \text{ then } f(x)\in B(f(x_0),\varepsilon), \tag{4.5}$$

and

$$\forall\varepsilon>0\ \exists\delta_{2,\varepsilon}>0 \text{ such that if } y\in T\cap B(f(x_0),\delta_{2,\varepsilon}), \text{ then } g(y)\in B(g(f(x_0)),\varepsilon). \tag{4.6}$$

We want to show that

$$\forall\varepsilon>0\ \exists\delta_{3,\varepsilon}>0 \text{ such that if } x\in S\cap B(x_0,\delta_{3,\varepsilon}), \text{ then } g(f(x))\in B(g(f(x_0)),\varepsilon). \tag{4.7}$$

From (4.5), if we choose $\varepsilon=\delta_{2,\varepsilon}$, we have that taken $\delta^*:=\delta_{1,\delta_{2,\varepsilon}}>0$, if $x\in S\cap B(x_0,\delta^*)$, then $f(x)\in B\left(f(x_0),\delta_{2,\delta_{2,\varepsilon}}\right)$. Then, chosen $\delta_{3,\varepsilon}=\delta^*$, indeed $f(x)\in B(f(x_0),\delta_{2,\varepsilon})\cap T$. Then, from (4.6), we get $g(f(x))\in B(g(f(x_0)),\varepsilon)$, as desired. ■

Proposition 99 *(limit of the composition of functions) Let the functions $g:S\subseteq\mathbb{R}\to T\subseteq\mathbb{R}$, $f:T\to\mathbb{R}$ and an accumulation point $x_0\in\mathbb{R}\cup\{-\infty,+\infty\}$ for S and an accumulation point $y_0\in\mathbb{R}\cup\{-\infty,+\infty\}$ for T be given. If*

1.

$$\lim_{x\to x_0} g(x)=y_0,$$

2.

$$\lim_{y\to y_0} f(y)=l\in\mathbb{R}$$

and

3.

$$\exists\delta>0 \text{ such that if } x\in B(x_0,\delta)\cap S\backslash\{x_0\}, \text{ then } g(x)\neq y_0, \tag{4.8}$$

then

$$\lim_{x\to x_0} f(g(x))=l.$$

Proof. Omitted. See Nannicini, Verdi and Vessella (2008), Theorem 13, page 231. ■

Remark 100 *The assumption* (4.8) *is satisfied if*

a. $y_0\in\{-\infty,+\infty\}$, or

b. g is strictly monotone in a left neighborhood of x_0 or in a right neighborhood of x_0.

Remark 101 *The above results allows to compute limits like the following one:*

$$\lim_{x\to 0}\sin\frac{x^2-3x+2\pi^2}{x^2+\pi}.$$

Since the function we want to compute the limit of is a continuous function, we have that the above limit is equal to

$$\sin\frac{0^2-3\cdot 0+2\pi^2}{0^2+\pi}=\sin 2\pi=0.$$

Remark 102 *Unless otherwise stated, we analyze continuity of a function in a point which is an accumulation point of the domain of that function.*

Remark 103 *We can use Proposition 98 to make "variable substitutions" in limit computations.*
From Proposition 98, we have that:
if 1. f is continuous in x_0, and therefore

$$\lim_{x\to x_0} f(x) = f(x_0) := y_0, \tag{4.9}$$

2. g is continuous in y_0, and therefore

$$\lim_{y\to y_0} g(y) = g(y_0), \tag{4.10}$$

then $g \circ f$ is continuous in x_0 and therefore

$$\lim_{x\to x_0} g(f(x)) = g(f(x_0)). \tag{4.11}$$

Observe that the above result does hold true also in the case in which $x_0, y_0 \in \mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$, with obvious modifications.
Then, we have that
i.

$$\lim_{x\to x_0} g(f(x)) \overset{(4.11)}{=} g(f(x_0)) \overset{(4.9)}{=} g\left(\lim_{x\to x_0} f(x)\right),$$

i.e., roughly speaking, we can bring the limit symbol inside the parenthesis;
ii.

$$\lim_{x\to x_0} g(f(x)) \overset{(4.11)}{=} g(f(x_0)) \overset{y_0:=f(x_0)}{=} g(y_0) \overset{(4.10)}{=} \lim_{y\to y_0} g(y).$$

In other words,

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to y_0} g(y),$$

where $y = f(x)$ and $\lim_{x\to x_0} f(x) = y_0$. Therefore, to try to compute $\lim_{x\to x_0} g(f(x))$, we could proceed as follows:
a. set $y = f(x)$;
b. compute $\lim_{x\to x_0} f(x)$ and call it y_0;
c. substitute $f(x)$ with y, and $x \to x_0$ with $y \to y_0$;
d. compute $\lim_{y\to y_0} g(y)$, which is just the limit we want to compute.

4.3 Some important limits

Proposition 104

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Proof. Omitted. ■

Exercise 105 *Compute* $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} =$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0} \frac{1}{1+\cos x} = \frac{1}{2}.$$

Proposition 106

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x$$

exists and it belong to the interval $(2, 3]$.

Proof. Omitted. ■

Definition 107 *The limit of the previous proposition is called Neper number and it is denoted by* e.

Example 108 *Compute*

$$\lim_{x\to+\infty} x^x = \lim_{x\to+\infty} e^{x\ln x}.$$

a. $y = f(x) = x\log x$, *and* $g(y) = e^y$;
b. $\lim_{x\to+\infty} x\log x = +\infty = y_0$;
c. $\lim_{y\to y_0} g(y) = \lim_{y\to+\infty} e^y = +\infty$.

Proposition 109 *1.*

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e;$$

2.

$$\lim_{x\to0^+}(1+x)^{\frac{1}{x}} = e;$$

3.

$$\lim_{x\to0^-}(1+x)^{\frac{1}{x}} = e.$$

Proof. 1.
Set $y = -(x+1)$ and therefore $x = -(1+y)$. Then,

$$\lim_{x\to-\infty}\left(1+\tfrac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1+\tfrac{1}{-(1+y)}\right)^{-(1+y)} = \lim_{y\to+\infty}\left(\tfrac{-1-y+1}{-(1+y)}\right)^{-(1+y)} =$$

$$= \lim_{y\to+\infty}\left(\tfrac{y}{1+y}\right)^{-(1+y)} = \lim_{y\to+\infty}\left(\tfrac{y+1}{y}\right)^{1+y} = \lim_{y\to+\infty}\left(1+\tfrac{1}{y}\right)^{1+y} =$$

$$\lim_{y\to+\infty}\left(\left(1+\tfrac{1}{y}\right)^y\cdot\left(1+\tfrac{1}{y}\right)\right) = e.$$

2. and 3.
Set $y = \frac{1}{x}$ and therefore $x = \frac{1}{y}$. Then

$$\lim_{x\to0^+}(1+x)^{\frac{1}{x}} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y = e;$$

$$\lim_{x\to0^-}(1+x)^{\frac{1}{x}} = \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y = e.$$

■

Exercise 110 *1. For any* $a\in\mathbb{R}\setminus\{0\}$, *compute* $\lim_{x\to\pm\infty}\left(\frac{a+x}{x}\right)^x$.

$$\lim_{x\to\pm\infty}\left(\frac{a+x}{x}\right)^x = \lim_{x\to\pm\infty}\left(\frac{1+\frac{x}{a}}{\frac{x}{a}}\right)^{\frac{x}{a}a} \overset{y=\frac{x}{a}}{=} \left(\lim_{x\to\pm\infty}\left(\frac{1+y}{y}\right)^y\right)^a = e^a.$$

2. For any $a\in\mathbb{R}\setminus\{0\}$, *compute* $\lim_{x\to0}(1+ax)^{\frac{1}{x}}$.

$$\lim_{x\to0}(1+ax)^{\frac{1}{x}} = \lim_{x\to0}(1+ax)^{\frac{1}{ax}a} \overset{y=ax}{=} \left(\lim_{x\to0}(1+y)^{\frac{1}{y}}\right)^a = e^a.$$

Remark 111 *Recall that, taken* $a,b\in\mathbb{R}_{++}\setminus\{1\}$ *and* $x,y\in\mathbb{R}_{++}$,*then*

1. $\log_a xy = \log_a x + \log_a y$;
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$;
3. $\log_a x^y = y\log_a x$;
4. $\log_a x = \frac{\log_b x}{\log_b a}$.

Proposition 112 *Given $a \in \mathbb{R}_{++} \setminus \{1\}$,*

1.

$$\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e,$$

and therefore $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$.

2.

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a,$$

and therefore $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

Proof. 1.

$$\lim_{x\to 0} \frac{\log_a (1+x)}{x} \overset{(1)}{=} \lim_{x\to 0} (\log_a (1+x))^{\frac{1}{x}} \overset{(2)}{=} \log_a \left(\lim_{x\to 0} (1+x)^{\frac{1}{x}} \right) = \log_a e,$$

where
(1) follows from the property of the log function, and
(2) from the continuity of the log function.
2.
Set $y = a^x - 1$ and therefore $1 + y = a^x$ and $x = \log_a (1+y)$; moreover, $x \to 0$ implies that $y \to 0$. Then,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log_a (1+y)} = \lim_{y\to 0} \frac{1}{\frac{\log_a (1+y)}{y}} = \frac{1}{\lim_{y\to 0} \left(\frac{\log_a (1+y)}{y} \right)} = \frac{1}{\log_a e} = \log a.$$

■

Proposition 113 *If $a > 1$ and $b > 0$, then*

1

$$\lim_{x\to +\infty} \frac{x^b}{a^x} = 0^+,$$

and,

2.

$$\lim_{x\to +\infty} \frac{\log_a x}{x^b} = 0^+,$$

Proof. 1.
Step 1. Taken $n \in \mathbb{N}$, then

$$\lim_{n\to +\infty} \frac{n}{a^n} = 0;$$

Step 2.

$$\lim_{x\to +\infty} \frac{x}{a^x} = 0;$$

Step 3. The desired result.
Step 1.
Since $a > 1$, there exists $d > 0$ such that $a = 1 + d$. Then, by an induction argument, it is possible to show that

$$a^n = (1+d)^n > \frac{n(n-1)d^2}{2}.$$

Then

$$0 < \frac{n}{a^n} < \frac{2n}{n(n-1)d^2}.$$

Then from the comparison theorem, $\lim_{n\to +\infty} \frac{n}{a^n} = 0$.
Step 2.
For any $x > 1$, there exists $n \in \mathbb{N}$ such that $n \leq x \leq n+1$. Therefore,

$$0 < \frac{x}{a^x} \leq \frac{n+1}{a^n} = \frac{n}{a^n} + \frac{1}{a^n}.$$

Observe that

$$\lim_{n\to+\infty}\left(\frac{n}{a^n}+\frac{1}{a^n}\right)=\lim_{n\to+\infty}\frac{n}{a^n}+\lim_{n\to+\infty}\frac{1}{a^n}=0.$$

Then, the desired result follows from the comparison theorem, again.
Step 3.
Define $c:=a^{\frac{1}{b}}$; since, by assumption, $a>1$ and $b>0$, we have that $c>1$. Then,

$$\lim_{x\to+\infty}\frac{x^b}{a^x}=\lim_{x\to+\infty}\frac{x^b}{\left(a^{\frac{1}{b}x}\right)^b}=\lim_{x\to+\infty}\left(\frac{x}{a^{\frac{1}{b}x}}\right)^b=\lim_{x\to+\infty}\left(\frac{x}{c^x}\right)^b=\left(\lim_{x\to+\infty}\frac{x}{c^x}\right)^b=0.$$

2.
Set $x=a^y$. Then $y=\log_a x$, and $\lim_{x\to+\infty}\log_a x=+\infty$. Then,

$$L:=\lim_{x\to+\infty}\frac{\log_a x}{x^b}=\lim_{y\to+\infty}\frac{y}{a^{by}}.$$

Now, using the fact that if $a>1$, $\log_a$ is a strictly increasing function, observe that $a^b>1\Leftrightarrow\log_a a^b>\log_a 1\Leftrightarrow b>0$, which is true by assumption. Then, defined $d:=a^b>1$, from the limit in part 1 of the present Proposition, we have

$$L=\lim_{y\to+\infty}\frac{y}{d^y}=0.$$

■

Corollary 114 *If $a>1$ and $b>0$, then*

$$\lim_{x\to+\infty}\frac{a^x}{x^b}=+\infty,\quad \textit{and}\quad \lim_{x\to+\infty}\frac{x^b}{\log_a x}=+\infty.$$

Example 115 *Compute*

$$\lim_{x\to0^+}x^x=\lim_{x\to0^+}e^{x\ln x}.$$

$\lim_{x\to0^+}x\log x=-\lim_{x\to0^+}\frac{(-\log x)}{\frac{1}{x}}=-\lim_{x\to0^+}\frac{\log\frac{1}{x}}{\frac{1}{x}}=-\lim_{y\to+\infty}\frac{\log y}{y}=0^-$
Therefore,

$$\lim_{x\to0^+}x^x=\lim_{x\to0^+}e^{x\ln x}=1$$

We list some important limits computed above.

	$\lim_{x\to0}\frac{\sin x}{x}=1$
	$\lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$
For any $a\in\mathbb{R}\setminus\{0\}$,	$\lim_{x\to\pm\infty}\left(\frac{a+x}{x}\right)^x=e^a$,
	$\lim_{x\to0}(1+ax)^{\frac{1}{x}}=e^a$
For any $a\in\mathbb{R}_{++}\setminus\{1\}$,	$\lim_{x\to0}\frac{\log_a(1+x)}{x}=\log_a e$,
	$\lim_{x\to0}\frac{a^x-1}{x}=\log_e a$.
For any $a>1$ and $b>0$,	$\lim_{x\to+\infty}\frac{a^x}{x^b}=+\infty$,
	$\lim_{x\to+\infty}\frac{x^b}{\log_a x}=+\infty$.

4.4 Some properties of continuous functions on closed, bounded intervals

Proposition 116 *(Nested intervals) For every $n \in \mathbb{N}$, define $I_n = [a_n, b_n] \subseteq \mathbb{R}$ such that $I_{n+1} \subseteq I_n$. Then $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$.*

Proof. By assumption,

$$a_1 \leq a_2 \leq ... \leq a_n \leq ... \tag{4.12}$$

and

$$...b_n \leq b_{n-1} \leq ... \leq b_1 \tag{4.13}$$

Then,

$$\forall m, n \in \mathbb{N}, \quad a_m < b_n$$

simply because, if $m > n$, then $a_m < b_m \leq b_n$,where the first inequality follows from the definition of interval I_m and the second one from (4.13), and if $m \leq n$, then $a_m \leq a_n \leq b_n$,where the first inequality follows from (4.12) and the second one from the definition of interval I_n.

Then $A := \{a_n : n \in \mathbb{N}\}$ is nonempty and bounded above by b_n for any n.Then $\sup A := s$ exists.

Since $\forall n \in \mathbb{N}$, b_n is an upper bound for A,

$$\forall n \in \mathbb{N}, \quad s \leq b_n$$

and from the definition of sup,

$$\forall n \in \mathbb{N}, \quad a_n \leq s$$

Then

$$\forall n \in \mathbb{N}, \quad a_n \leq s \leq b_n$$

and

$$\forall n \in \mathbb{N}, \quad I_n \neq \varnothing.$$

■

Remark 117 *The statement in the above Proposition is false if instead of taking closed bounded intervals we take either open or unbounded intervals. To see that consider $I_n = \left(0, \frac{1}{n}\right)$ and $I_n = [n, +\infty]$.*

Proposition 118 *(Bolzano- Weirstrass) If $S \subseteq \mathbb{R}$ has infinite cardinality and is bounded, then S admits at least an accumulation point, i.e., $D(S) \neq \varnothing$.*

Proof. Since S is bounded, $\exists a_0, b_0 \in \mathbb{R}$ such that $S \subseteq [a_0, b_0] := B_0$. Divide B_0 in two subinterval of equal length:

$$\left[a_0, \frac{a_0 + b_0}{2}\right] \text{ and } \left[\frac{a_0 + b_0}{2}, b_0\right]$$

Choose an interval which contains an infinite number of points in S. Call $B_1 = [a_1, b_1]$ that interval. Proceed as above for B_1. We therefore obtain a family of intervals

$$B_0 \supseteq B_1 \supseteq .... \supseteq B_n \supseteq ...$$

Observe that $lenght\ B_0 := b_0 - a_0$ and

$$\forall n \in \mathbb{N}, lenght\ B_n = \frac{b_0 - a_0}{2^n}.$$

Therefore, $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall n > N$, $lenght\ B_n < \varepsilon$.

From Proposition 116, it follows that

$$\exists x \in \cap_{n=0}^{+\infty} B_n$$

We are now left with showing that x is an accumulation point for S

$$\forall r \in \mathbb{R}_{++}, B(x, r) \text{ contains an infinite number of points.}$$

By construction, $\forall n \in \mathbb{N}$, B_n contains an infinite number of points; it is therefore enough to show that

$$\forall r \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ such that } B(x,r) \supseteq B_n.$$

Observe that

$$B(x,r) \supseteq B_n \Leftrightarrow (x-r, x+r) \supseteq [a_n, b_n] \Leftrightarrow x - r < a_n < b_n < x + r \Leftrightarrow \max\{x - a_n, b_n - x\} < r$$

Moreover, since $x \in [a_n, b_n]$,

$$\max\{x - a_n, b_n - x\} < b_n - a_n = lenght\ B_n = \frac{b_0 - a_0}{2^n}$$

Therefore, it suffices to show that

$$\forall r \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ such that } \frac{b_0 - a_0}{2^n} < r$$

i.e., $n \in \mathbb{N}$ and $n > \log_2(b_0 - a_0)$. ■

Proposition 119 *For any $x, y \in \mathbb{R}$,*

$$||x| - |y|| \leq |x - y|.$$

Problem 120 *Recall the triangle inequality),*

$$|x + y| \leq |x| + |y|, \tag{4.14}$$

and that $\forall a, b \in \mathbb{R}$

$$-b \leq a \leq b \Leftrightarrow |a| \leq b.$$

From (4.14)*, identifying x with $x - y$ and y with y, we get $\|x - y + y\| \leq |x - y| + |y|$, i.e.,*

$$|x| - |y| \leq |x - y|.$$

From (4.14) *identifying x with $y - x$ and y with x, we get $\|y - x + x\| \leq \|y - x\| + |x|$, i.e.,*

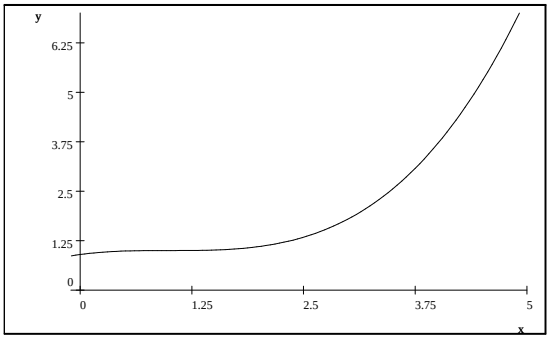
$$|y| - |x| \leq \|y - x\| = |x - y|$$

and

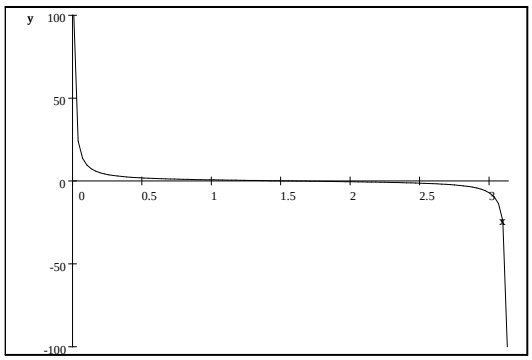
$$-|x - y| \leq |x| - |y|$$

Proposition 121 *Given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$, such that $[a, b] \subseteq S$. Then,*

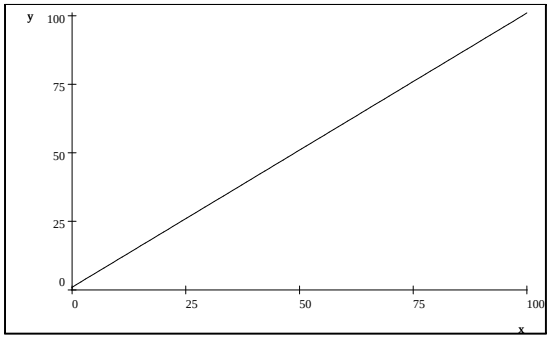
$$\langle\ f \text{ continuous on } [a,b]\ \rangle \Rightarrow \langle\ f \text{ bounded on } [a,b]\ \rangle.$$



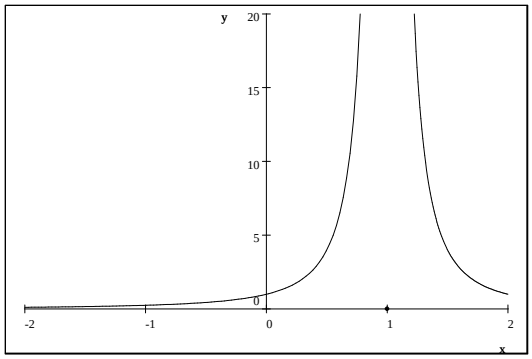
The function below is continuous, but its domain is (a, b) :



The function below is continuous, but its domain is $[a, +\infty)$



The function below is defined on an interval $[a, b]$, but the function is not continuous.



Proof. Strategy of proof. Use Bolzano-Weierstrass theorem and the continuity of f.[3]
We are going to show that f is bounded above; the proof that f is bounded below is very similar. Assume that f is not bounded above. Then,

$$\forall n \in \mathbb{N}, \exists x_n \in [a, b] \text{ such that } f(x_n) > n. \tag{4.15}$$

The set $\{x_n : n \in \mathbb{N}\}$ is then infinite and bounded (because contained in a bounded set $[a, b]$). Then, from the Bolzano - Weierstrass Theorem, $\{x_n : n \in \mathbb{N}\}$ has an accumulation point x^*.

We now show that $x^* \in [a, b]$. Suppose otherwise, i.e., $x^* \notin [a, b]$ .Then, $x^* \in \mathbb{R} \setminus [a, b]$ and therefore[4] $\exists \varepsilon > 0$ such that $B(x^*, \varepsilon) \subseteq \mathbb{R} \setminus [a, b]$ and since $\{x_n : n \in \mathbb{N}\} \subseteq [a, b]$, $B(x^*, \varepsilon) \cap \{x_n : n \in \mathbb{N}\} = \varnothing$, contradicting the fact that x^* is an accumulation point for $\{x_n : n \in \mathbb{N}\}$.

Now, since $x^* \in [a, b]$ and f is continuous on $[a, b]$, by assumption,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } x \in B(x^*, \delta) \cap [a, b] \text{ implies that } |f(x) - f(x^*)| < \varepsilon.$$

[3] The proof is taken from Anichini and Conti (...), page 188.
[4] If $x > b$, then take $\varepsilon = x - b$; if $x < a$, then take $\varepsilon = a - x$.

Since x^* is an accumulation point for $\{x_n : n \in \mathbb{N}\}$, then there exists an infinite number of points of $\{x_n : n \in \mathbb{N}\}$ which belong to $B(x^*, \delta)$ as well. Observe that since $\{x_n : n \in \mathbb{N}\} \subseteq [a, b]$, we can also say that there exists an infinite number of points of $\{x_n : n \in \mathbb{N}\}$ which belong to $B(x^*, \delta) \cap [a, b]$. Then,

$$\exists N \subseteq \mathbb{N}, \text{ with } N \text{ having infinite elements, such that } \forall n \in N, \quad |f(x_n) - f(x^*)| < \varepsilon,$$

and

$$\exists N \subseteq \mathbb{N}, \text{ with } N \text{ having infinite elements, such that } \forall n \in N, |f(x_n)| - |f(x^*)| \leq |f(x_n) - f(x^*)| < \varepsilon \tag{4.16}$$

where for first equality follows from Proposition 119.

Then, from (4.15) and (4.16), we get that

$$\exists N \subseteq \mathbb{N}, \text{ with } N \text{ having infinite elements, such that}$$

$$\forall n \in N, \quad n < f(x_n) \leq |f(x_n)| < \varepsilon + f(x^*).$$

But that is a contradiction, because $\varepsilon + f(x^*)$ is a real number and only a finite number of natural numbers is smaller than it. ■

Proposition 122 ***(Extreme value theorem for real functions of real variables)*** *Given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $[a, b] \subseteq S$. Then,*

$$\langle \ f \text{ continuous on } [a, b] \ \rangle \Rightarrow \langle \ f \text{ has a maximum and a minimum point on } [a, b] \ \rangle.$$

Proof. Strategy of proof. Show that f has a maximum point in $[a, b]$: define

$$g(x) = \left(\sup_{x \in [a,b]} f(x) \right) - f(x)$$

and show, by contradiction, that, $\exists x_0 \in [a, b]$ such that $g(x_0) = 0$, using $h(x) = \frac{1}{g(x)}$.

From the previous Proposition, f is bounded above on $[a, b]$.Therefore,

$$\exists s \in \mathbb{R} \text{ such that } \sup_{x \in [a,b]} f(x) = s.$$

Then, by definition of sup, $\forall x \in [a, b]$, $s \geq f(x)$. Define,

$$g : [a, b] \to \mathbb{R}, \qquad x \mapsto s - f(x) \geq 0.$$

Observe that if there exists $x_0 \in [a, b]$ such that $g(x_0) = 0$, then $f(x_0) = s$ and x_0 is a global point of maximum for f on $[a, b]$, and we got the desired result. Now suppose our claim is false, i.e., $\forall x \in [a, b]$, indeed $g(x) > 0$. Then,

$$h : [a, b] \to \mathbb{R}, \quad x \mapsto \frac{1}{g(x)} > 0$$

is well defined and continuous on $[a, b]$, then from the previous Proposition is bounded. Then, there exists $C \in \mathbb{R}_{++}$ such that

$$\forall x \in [a, b], \quad 0 < \frac{1}{g(x)} \leq C,$$

or

$$\forall x \in [a, b], \quad 0 < \frac{1}{s - f(x)} \leq C,$$

Then, $1 \leq Cs - Cf(x)$, and

$$\forall x \in [a, b], \quad f(x) \leq \frac{-1 + sC}{C} = s - \frac{1}{C} < s,$$

contradicting the fact that $s = \sup_{x \in [a,b]} f(x)$. ■

Proposition 123 *(existence of zeros for continuous functions) If f is continuous on $[a,b]$, and sign $f(a) \neq$ sign $f(b)$, then $\exists c \in [a,b]$ such that $f(c) = 0$.*

Proof. Strategy of proof. The desired c is $\sup\{x \in [a,b] : f(x) \leq 0\}$.
Without loss of generality, assume that $f(a) < 0$ and $f(b) > 0$. Define

$$A = \{x \in [a,b] : f(x) \leq 0\} \neq \varnothing,$$

a set which is bounded above by b. Then, $c := \sup A \in \mathbb{R}$; indeed, $c \in [a,b]$. We now want to show that $f(c) = 0$. Suppose otherwise, then either $f(c) < 0$ or $f(c) > 0$. We are going to show that in both cases, we get a contradiction.

Case 1. $f(c) < 0$.

From Proposition on sign permanence (Proposition 60)and from the continuity of f on $[a,b]$, we have that

$$\exists \delta > 0 \text{ such that } x \in (c-\delta, c+\delta) \cap [a,b] \Rightarrow f(x) > 0.$$

But, then $\forall x \in (c-\delta, c) \cap [a,b]$ [5],we have $f(x) > 0$, contradicting the definition of $c = \sup A = \sup\{x \in [a,b] : f(x) \leq 0\}$.

Case 2. $f(c) > 0$. Again from the Proposition on the sign permanence,

$$\exists \delta > 0 \text{ such that } x \in (c-\delta, c+\delta) \cap [a,b] \Rightarrow f(x) < 0.$$

Then, every $x \in \{c-\delta, c\}$ is such that $(a).\ x \in S$; $(b).\ x \geq \sup S = s$, contradicting the definition of sup . ■

Proposition 124 *(Intermediate value theorem for continuous functions) If $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is such that*
f is continuous on $[a,b] \subseteq S$,
then
$\forall k \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$, $\exists x_k \in [a,b]$ such that $f(x_k) = k$,

Proof. If $k = f(a)$ or $k = f(b)$, we are done. Then, assume that $f(a) < k < f(b)$. Define

$$g : [a,b] \to \mathbb{R}, \qquad x \mapsto f(x) - k.$$

Observe that g does satisfy the assumption of Proposition 123 on $[a,b]$. Then $\exists c \in (a,b)$ such that $g(c) = f(c) - k = 0$, as desired. ■

Proposition 125 *If f is continuous on $[a,b]$, then $f([a,b])$ is a bounded closed interval.*

Proof. Strategy: $f([a,b]) = [f(x_{\min}), f(x_{\max})]$.
From Proposition 122,

$$\exists x_{\min}, x_{\max} \in [a,b] \text{ such that } \forall x \in [a,b], \qquad f(x_{\min}) \leq f(x) \leq f(x_{\max}).$$

Then $f([a,b])$ is contained in the closed bounded interval $[f(x_{\min}), f(x_{\max})]$.
From the previous Proposition we also have that $\forall y \in [f(x_{\min}), f(x_{\max})]$, $\exists x \in [a,b]$ such that $y = f(x) \in f([a,b])$,i.e., $[f(x_{\min}), f(x_{\max})] \subseteq f([a,b])$. ■

Proposition 123 allows to formulate an algorithm, or if you want a "recipe", to solve the following problem:

Given a continuous function $f : [a,b] \to \mathbb{R}$ such that *sign* $f(a) \neq$ *sign* $f(b)$, find $x \in [a,b]$ such that

$$f(x) = 0 \tag{4.17}$$

with an error smaller than a given $\varepsilon > 0$.

[5] Observe that $(c-\delta, c) \cap [a,b] \neq \varnothing$, because $c \in (a,b)$. If that were not the case, we would have $c = a$,but then we would get $f(c) > 0 > f(a)$, a contradiction.

In other words, we want to do what follows. Let x^* be a solution to (4.17); observe that thanks to Proposition 123 such a solution does exist. The goal of our exercise is to find $\overline{x}$ such that

$$\text{Error} := |x^* - \overline{x}| < \varepsilon.$$

Bisection algorithm.
First of all check that f is continuous.
Step 0.

$$\text{Find } a_0, b_0 \in \textit{domain } f \text{ such that } a_0, b_0 \text{ and } \textit{sign } f(a_0) \neq \textit{sign } \ f(b_0).$$

To fix ideas, assume $f(a_0) < \ f(b_0)$, i.e., $f(a_0) < 0$ and $f(b_0) > 0$. The case $f(a_0) > 0$ and $f(b_0) < 0$ can be analyzed applying the algorithm to the function $-f$.

$$\text{Define } l_0 = b_0 - a_0 \text{ and } x_0 = \tfrac{a_0+b_0}{2},$$

i.e., l_0 is the length of the interval (a_0, b_0) and x_0 is the middle point of that interval. Since from Proposition 123, we know that $x^* \in (a_0, b_0)$, then $|x^* - x_0| < \frac{b_0-a_0}{2} = \frac{l_0}{2}$.[6]
Therefore,

$$\text{if } \tfrac{l_0}{2} \leq \varepsilon, \text{ choose } \overline{x} = x_0 \text{ and stop; otherwise, go to next step.}$$

Step 1.1.

Compute $f(x_0)$.
a. If $f(x_0) = 0$, then choose $\overline{x} = x^* = x_0$, and stop;
b. if $f(x_0) < 0$, define $a_1 := x_0$ and $b_1 := b_0$, and go to next step;
c. if $f(x_0) > 0$, define $a_1 := a_0$ and $b_1 := x_0$, and go to next step.

Step 1.2.

$$\text{Define } l_1 = b_1 - a_1 \text{ and } x_1 = \tfrac{a_1+b_1}{2};$$

$$\text{if } \tfrac{l_1}{2} \leq \varepsilon, \text{ choose } \overline{x} = x_1 \text{ and stop; otherwise, go to next step.}$$

Step 2.1.

Compute $f(x_1)$.
a. If $f(x_1) = 0$, then choose $\overline{x} = x^* = x_1$, and stop;
b. if $f(x_1) < 0$, define $a_2 := x_1$ and $b_2 := b_1$, and go to next step;
c. if $f(x_1) > 0$, define $a_2 := a_1$ and $b_2 := x_1$, and go to next step.

Step 2.2.

$$\text{Define } l_2 = b_2 - a_2 \text{ and } x_2 = \tfrac{a_2+b_2}{2};$$

$$\text{if } \tfrac{l_2}{2} \leq \varepsilon, \text{ choose } \overline{x} = x_2 \text{ and stop; otherwise, go to next step.}$$

Step n.

Repeat steps described in (1.1. and 1.2) or (2.1 and 2.2),

until $\frac{l_n}{2} \leq \varepsilon$ and in that case choose $\overline{x} = x_n = \frac{a_n+b_n}{2}$.

Remark 126 ***1.*** *The number n necessary to get the desired result is called the number of needed iterations. That number can be computed before starting the procedure and is equal to the solution* $\overline{n}$ *to the equation*

$$\frac{b_0 - a_0}{2^{n+1}} \leq \varepsilon.$$

[6] Indeed, if $x^* \in (a_0, b_0)$ and $x^* \geq x_0$, then $|x^* - x_0| = x^* - x_0 = x^* - \frac{a_0+b_0}{2} = \frac{2x^* - a_0 - b_0}{2} < \frac{b_0-a_0}{2}$ if and only if $2x^* - a_0 - b_0 < b_0 - a_0$, i.e., $x^* < b_0$, which is certainly true. If $x^* \in (a_0, b_0)$ and $x^* < x_0$, then $|x^* - x_0| = -x^* + x_0 = -x^* + \frac{a_0+b_0}{2} = \frac{-2x^*+a_0+b_0}{2} < \frac{b_0-a_0}{2}$ if and only if $-2x^* + a_0 + b_0 < b_0 - a_0$, i.e., $x^* > a_0$, which is certainly true as well.

in the unknown $n \in \mathbb{N}$.

2. *Proposition 123 and the above procedure nothing says about uniqueness of the solution to the equation* $f(x) = 0$.

3. *The assumption of continuity of* f *on* $[a, b]$ *is indispensable.*

Example 127 *Find a solution to the equation*

$$\frac{1}{8}x^3 + 2x = 3$$

with an error in absolute value smaller than $\frac{1}{8}$.

Step 0.

The function we want to analyze if $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{8}x^3 + 2x - 3$

$f(x) = \frac{1}{8}x^3 + 2x - 3$

Observe that $f(0) = -3 < 0$, $f(1) = -\frac{7}{8} < 0$ *and* $f(2) = 2 > 0$.

Take $a_0 = 1$ and $b_0 = 2$ and

Define $l_1 = b_1 - a_1 = 2 - 1 = 1$ and $x_1 = \frac{a_1+b_1}{2} = \frac{1+2}{2} = \frac{3}{2}$;

$\frac{l_1}{2} = \frac{1}{2} > \varepsilon$; go to next step.

Step 1.1.

$f(x_1) = f\left(\frac{3}{2}\right) = \frac{27}{64} > 0$. Then,
define $a_1 := 1$ and $b_1 := \frac{3}{2}$, and go to next step.

Step 1.2.

$l_2 = b_2 - a_2 = \frac{3}{2} - 1 = \frac{1}{2}$ and $x_2 = \frac{a_2+b_2}{2} = \frac{1}{2}\left(\frac{3}{2} + 1\right) = \frac{5}{4}$;

$\frac{l_2}{2} = \frac{1}{4} > \frac{1}{8}$; go to next step.

Step 2.1.

$f(x_2) = f\left(\frac{5}{4}\right) = \frac{125}{64}\frac{1}{8} + \frac{2\cdot 3}{5} - 3 = \frac{125+1280-1536}{512} = -\frac{131}{512} < 0$. Then,
define $a_2 := x_2 =$ and $b_2 := \frac{3}{2}$, and go to next step;

Step 2.2.

$l_3 = b_3 - a_3 = \frac{3}{2} - \frac{5}{4} = \frac{1}{4}$ and $x_3 = \frac{a_3+b_3}{2} = \frac{1}{2}\left(\frac{3}{2} + \frac{5}{4}\right) = \frac{11}{8}$;

Since $\frac{l_2}{2} = \frac{1}{8} \leq \frac{1}{8}$, we can choose $\frac{11}{8}$ and stop.

4.5 Exercises

Section 7.15 in Nannicini, Verdi and Vessella (2008):

9,13,14,15,18.

Chapter 6 in Barozzi and Corradi (1989).

7,8,9,11,12,16,17,18,19,20,21,22,23,24,26,27,28,29.

The following exercises are taken from past exams and were collected by Domenico Menicucci.

1. Per ciascuna delle seguenti funzioni (ognuna definita su $\mathbb{R}$) si individuino, se esistono, i punti di discontinuita':

$$f_1(x) = \begin{cases} 3\cos(x) - 7^{\sin(x)} & \text{se } x \leq 0 \\ \frac{\sqrt{2+x}-\sqrt{2}}{x} & \text{se } x > 0 \end{cases}$$

$$f_2(x) = \begin{cases} \frac{2x^2+14x-16}{2x^2-7x+5} & \text{se } x < 1 \\ 2 & \text{se } x = 1 \\ -4 + 5\cos(\frac{\pi}{2}x) - 2\sin(\frac{\pi}{2}x) & \text{se } x > 1 \end{cases}$$

$$f_3(x) = \begin{cases} \log(1-x) + 8^x + 3 & \text{se } x \leq 0 \\ \frac{\sqrt{1+8x}-1}{x+x^2} & \text{se } x \in (0,3) \\ \frac{2\sqrt{x-2}}{12+x-x^2} & \text{se } x \in [3,4) \cup (4,+\infty) \\ 7 & \text{se } x = 4 \end{cases}$$

2. Per ciascuna delle seguenti funzioni (ognuna definita su $\mathbb{R}$) si individuino, se esistono, i valori del parametro a, o dei parametri a e b, che rendono la funzione continua in $\mathbb{R}$:

   - (a) $f(x) = \begin{cases} 2 - ax + 3^x & \text{se } x \leq 1 \\ \frac{1}{x-1} - \frac{2}{x^2-1} & \text{se } x > 1 \end{cases}$
   - (b) $f(x) = \begin{cases} e^{ax} & \text{se } x \leq 0 \\ \log(1+x) - a^2 & \text{se } x > 0 \end{cases}$
   - (c) $f(x) = \begin{cases} ax^2 & \text{se } x \leq 0 \\ e^{ax} - 1 & \text{se } x > 0 \end{cases}$
   - (d) $f(x) = \begin{cases} -2x + b & \text{se } x \leq 1 \\ x^2 + ax + 1 & \text{se } x \in (1,2) \\ x^3 - b & \text{se } x \geq 2 \end{cases}$
   - (e) $f(x) = \begin{cases} 2e^x - a & \text{se } x \leq 0 \\ a^2 + \log(1+x) & \text{se } x > 0 \end{cases}$
   - (f) $f(x) = \begin{cases} \frac{1}{1-x} + 2a & \text{se } x < 0 \\ a^2 x - 1 & \text{se } x \in [0,1] \\ x^2 - 1 & \text{se } x > 1 \end{cases}$
   - (g) $f(x) = \begin{cases} e^x + ax & \text{se } x \leq 0 \\ x^2 - 6x & \text{se } x > 0 \end{cases}$
   - (h) $f(x) = \begin{cases} 2x^3 + 5 & \text{se } x < 0 \\ a + \cos x + 4\log(1+x) & \text{se } x \geq 0 \end{cases}$

3. Per l'equazione $\frac{1}{8}x^3 + 2x = 3$ si utilizzi il teorema di esistenza degli zeri al fine di stabilire l'esistenza di una soluzione nell'intervallo $(0, 2)$; si provi inoltre che tale intervallo contiene un'unica soluzione dell'equazione. Si impieghi l'algoritmo di bisezione per trovare un valore di x che differisce meno di $\frac{1}{16}$ dalla soluzione in $(0, 2)$.

   Per ciascuna della seguenti equazioni si dica se e' possibile utilizzare il teorema di esistenza degli zeri al fine di stabilire l'esistenza di una soluzione nell'intervallo indicato a fianco dell'equazione. Nei casi in cui questo non e' possibile, e' possibile stabilire in un altro modo l'esistenza di una soluzione nell'intervallo? In quali casi e' possibile stabilire l'esistenza di un'unica soluzione nell'intervallo?

   - (a) $2x^3 - 13x^2 + 26x = 15$; $(0, 4)$.
   - (b) $-x^3 + 3 = \log_2 x$; $(1, 2)$.
   - (c) $6\sqrt{x} = 2^x - 3$; $(1, 3)$.
   - (d) $f(x) = 0$, con $f(x) = \begin{cases} 5 - x^4 - x^5 & \text{se } x < 1 \\ 6 + 2x^4 - 5x^3 & \text{se } x \geq 1 \end{cases}$; $(0, 2)$.
   - (e) $x^3 = 1 - x^4$; $(-2, 1)$.
   - (f) $f(x) = 0$, con $f(x) = \begin{cases} -4 + \sin(\pi x) + x & \text{se } x < 2 \\ \frac{1}{2}x^3 - 3 + \frac{1}{x} & \text{se } x \geq 2 \end{cases}$; $(0, 3)$.

4. Per ciascuna delle seguenti funzioni si determini $\mathrm{Im}(f)$.

   - (a) $f : (0, 4] \to \mathbb{R}$ tale che $f(x) = \log_2 x + 5^x + 3x$;
   - (b) $f : [0, 1] \to \mathbb{R}$ tale che $f(x) = -4x^3 + (\frac{1}{2})^x$;
   - (c) $f : \mathbb{R} \to \mathbb{R}$ tale che $f(x) = \log_3(x^2 - 2x + 8)$;
   - (d) $f : \mathbb{R} \to \mathbb{R}$ tale che $f(x) = \sqrt[3]{-x^2 + 4x + 48}$.

5. Sapendo che f e' continua nel punto $x_0 = 5$ e $f(5) = 2$, e' possibile dedurre il segno di $f(4.9)$ o il segno di $f(5.01)$?

   Per la seguente funzione

   $$f(x) = \begin{cases} \log x & \text{se } x \leq 3 \\ x + 1 & \text{se } x > 3 \end{cases}$$

si determini il dominio e si dica se sono soddisfatte le ipotesi del teorema della permanenza del segno nei seguenti punti:

$$x_0 = -1,\ x_1 = 1,\ x_2 = 3,\ x_3 = 4$$

Soluzioni

1. La funzione f_1 e' continua in ogni punto $x_0 < 0$, dato che (i) per ogni $x_0 < 0$ esiste un intorno $B(x_0, \delta)$ di x_0 (con $\delta > 0$ piccolo) tale che $f_1(x) = 3\cos(x) - 7^{\sin(x)}$ per ogni $x \in B(x_0, \delta)$; (ii) in base all'osservazione (i) deduciamo che $\lim_{x\to x_0} f_1(x) = \lim_{x\to x_0} \left(3\cos(x) - 7^{\sin(x)}\right)$ e $\lim_{x\to x_0} \left(3\cos(x) - 7^{\sin(x)}\right) = 3\cos(x_0) - 7^{\sin(x_0)} = f_1(x_0)$ dato che le funzioni coseno, seno ed esponenziale sono continue in ogni $x_0 < 0$ (per la precisione, lo sono in ogni punto della retta reale), e ogni funzione che e' composizione/somma/prodotto/quoziente di funzioni continue e' continua dove e' definita. Ragionando in maniera analoga si conclude che f_1 e' continua in ogni punto $x_0 > 0$, ovvero f_1 e' continua in $\mathbb{R} - \{0\}$. L'unico possibile punto di discontinuita' per f_1 e' dunque $x_0 = 0$. Calcoliamo allora

$$\begin{aligned}
\lim_{x\to 0^-} f_1(x) &= 3 - 1 = 2 \\
\lim_{x\to 0^+} f_1(x) &= \lim_{x\to 0^+} \frac{(\sqrt{2+x} - \sqrt{2})(\sqrt{2+x} + \sqrt{2})}{x(\sqrt{2+x} + \sqrt{2})} = \lim_{x\to 0^+} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{4}\sqrt{2}
\end{aligned}$$

Poiche' $2 \neq \frac{1}{4}\sqrt{2}$, si deduce che f_1 non e' continua in $x_0 = 0$.

Per f_2 si ragiona in maniera analoga e si deduce che l'unico possibile punto di discontinuita' e' $x_0 = 1$. Calcoliamo allora

$$\begin{aligned}
\lim_{x\to 1^-} f_2(x) &= \lim_{x\to 1^-} \frac{2\,(x-1)\,(x+8)}{(2x-5)\,(x-1)} = -6 \\
\lim_{x\to 1^+} f_2(x) &= \lim_{x\to 1^+} \left(-4 + 5\cos(\frac{\pi}{2}x) - 2\sin(\frac{\pi}{2}x)\right) = -6
\end{aligned}$$

Questo rivela che $\lim_{x\to 1} f_2(x) = -6$, ma poiche' $f_2(0) = 2$ che e' diverso da -6, si deduce che f_2 non e' continua in $x_0 = 1$.

Per f_3 si ragiona in maniera analoga e si deduce che gli unici possibili punti di discontinuita' per f_3 sono i punti dell'insieme $\{0, 3, 4\}$. Risulta che f_3 e' continua in $x_0 = 0$ dato che

$$\begin{aligned}
\lim_{x\to 0^-} f_3(x) &= 0 + 1 + 3 = 4 = f_3(0) \\
\lim_{x\to 0^+} f_3(x) &= \lim_{x\to 0^+} \frac{\sqrt{1+8x} - 1}{x + x^2} = \lim_{x\to 0^+} \frac{8x}{(x+x^2)(\sqrt{1+8x} + 1)} = 4
\end{aligned}$$

Inoltre, f_3 e' continua anche in $x_1 = 3$ dato che

$$\begin{aligned}
\lim_{x\to 3^-} f_3(x) &= \lim_{x\to 3^-} \frac{\sqrt{1+8x} - 1}{x + x^2} = \frac{1}{3} \\
\lim_{x\to 3^+} f_3(x) &= \lim_{x\to 3^+} \frac{2\sqrt{x-2}}{12 + x - x^2} = \frac{1}{3} = f_3(3)
\end{aligned}$$

Tuttavia, f_3 non e' continua in $x_2 = 4$, dato che $\lim_{x\to 4^-} f_3(x) = +\infty$, $\lim_{x\to 4^+} f_3(x) = -\infty$.

2. (a) Ragionando come nella soluzione dell'esercizio 1 si deduce che f e' continua in $\mathbb{R} - \{1\}$. Dunque f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 1$, e questa proprieta' e' soddisfatta se e solo se $\lim_{x\to 1^-} f(x) = 5 - a = f(1)$ e' uguale a $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x+1} = \frac{1}{2}$. L'uguaglianza $5 - a = \frac{1}{2}$ vale se e solo se $a = \frac{9}{2}$.

   (b) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 0$, e questa proprieta' e' soddisfatta se e solo se $\lim_{x\to 0^-} f(x) = 1 = f(0)$ e' uguale a $\lim_{x\to 0^+} f(x) = -a^2$. L'uguaglianza $1 = -a^2$ e' violata per ogni a reale.

   (c) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 0$, e questa proprieta' e' soddisfatta se e solo se $\lim_{x\to 0^-} f(x) = 0 = f(0)$ e' uguale a $\lim_{x\to 0^+} f(x) = 0$. L'uguaglianza $0 = 0$ e' soddisfatta per ogni a reale.

   (d) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 1$ e in $x_1 = 2$. La continuita' in $x_0 = 1$ e' soddisfatta se e solo se $\lim_{x\to 1^-} f(x) = -2 + b = f(1)$ e' uguale a $\lim_{x\to 1^+} f(x) = 2 + a$. La continuita' in $x_1 = 2$ e' soddisfatta se e solo se $\lim_{x\to 2^-} f(x) = 5 + 2a$ e' uguale a

$\lim_{x\to 2^+} f(x) = 8-b = f(2)$. Le uguaglianze $-2+b = 2+a$ e $5+2a = 8-b$ sono soddisfatte se e solo se $a = -\frac{1}{3}$, e $b = \frac{11}{3}$.

(e) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 0$, e questa proprieta' e' soddisfatta se e solo se $\lim_{x\to 0^-} f(x) = 2-a = f(0)$ e' uguale a $\lim_{x\to 0^+} f(x) = a^2$. L'uguaglianza $2-a = a^2$ e' soddisfatta se e solo se $a = 1$ o $a = -2$.

(f) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 0$ e in $x_1 = 1$. La continuita' in $x_0 = 0$ e' soddisfatta se e solo se $\lim_{x\to 0^-} f(x) = 1+2a$ e' uguale a $\lim_{x\to 0^+} f(x) = -1 = f(0)$. La continuita' in $x_1 = 1$ e' soddisfatta se e solo se $\lim_{x\to 1^-} f(x) = a^2 - 1 = f(1)$ e' uguale a $\lim_{x\to 1^+} f(x) = 0$. Le uguaglianze $1+2a = -1$ e $a^2 - 1 = 0$ sono soddisfatte se e solo se $a = -1$.

(g) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 0$, e questa proprieta' e' soddisfatta se e solo se $\lim_{x\to 0^-} f(x) = 1 = f(0)$ e' uguale a $\lim_{x\to 0^+} f(x) = 0$. L'uguaglianza $1 = 0$ e' violata per ogni a reale.

(h) f e' continua in $\mathbb{R}$ se e solo se e' continua in $x_0 = 0$, e questa proprieta' e' soddisfatta se e solo se $\lim_{x\to 0^-} f(x) = 5$ e' uguale a $\lim_{x\to 0^+} f(x) = a+1 = f(0)$. L'uguaglianza $5 = 1+a$ vale se e solo se $a = 4$.

3. Sia $f(x) = \frac{1}{8}x^3 + 2x - 3$ e si noti che x_0 e' soluzione dell'equazione se e solo se $f(x_0) = 0$. Risulta che $f(0) = -3 < 0$ e $f(2) = 2 > 0$. Poiche' f e' continua in $[0,2]$ (f e' la somma di funzioni continue) si deduce che esiste un $x_0 \in (0,2)$ tale che $f(x_0) = 0$, ovvero esiste in $(0,2)$ una soluzione dell'equazione. Inoltre tale soluzione e' unica (in $(0,2)$) poiche' f e' monotona strettamente crescente in $(0,2)$ (f e' la somma di funzioni monotone strettamente crescenti e di una costante).

   Per applicare l'algoritmo di bisezione si calcola f in $x = \frac{1}{2}(0+2) = 1$; poiche' $f(1) = -\frac{7}{8} < 0$ si deduce che la soluzione si trova in $(1,2)$. Il passo successivo consiste nel calcolare f in $x = \frac{1}{2}(1+2) = \frac{3}{2}$; poiche' $f(\frac{3}{2}) = \frac{27}{64} > 0$ si deduce che la soluzione si trova in $(1, \frac{3}{2})$. Il passo successivo consiste nel calcolare f in $x = \frac{1}{2}(1+\frac{3}{2}) = \frac{5}{4}$; poiche' $f(\frac{5}{4}) = -\frac{131}{512} < 0$ si deduce che la soluzione si trova in $(\frac{5}{4}, \frac{3}{2})$. Il passo successivo consiste nel calcolare f in $x = \frac{1}{2}(\frac{5}{4}+\frac{3}{2}) = \frac{11}{8}$; poiche' $f(\frac{11}{8}) = \frac{307}{4096} > 0$ si deduce che la soluzione si trova in $(\frac{5}{4}, \frac{11}{8})$. Pertanto $x = \frac{1}{2}(\frac{5}{4}+\frac{11}{8}) = \frac{21}{16} = 1.312\,5$ dista meno di $\frac{1}{16}$ dalla soluzione.

   (a) Sia $f(x) = -2x^3 + 13x^2 - 26x + 15$ e si noti che x_0 e' soluzione dell'equazione se e solo se $f(x_0) = 0$. Risulta che $f(0) = 15 > 0$ e $f(4) = -9 < 0$. Poiche' f e' continua in $[0,4]$, si deduce che esiste un $x_0 \in (0,4)$ tale che $f(x_0) = 0$, ovvero esiste in $(0,4)$ una soluzione dell'equazione. Si noti tuttavia che f non e' la somma di funzioni tutte monotone strettamente crescenti o tutte monotone strettamente decrescenti, e pertanto e' impossibile, con i metodi presentati finora, stabilire se f e' strettamente monotona o no. Dunque non possiamo garantire che l'intervallo $(0,4)$ contenga un'unica soluzione (in realta' ne contiene tre: $x_0 = 1$, $x_1 = \frac{5}{2}$, $x_2 = 3$).

   (b) Sia $f(x) = -x^3 + 3 - \log_2 x$ e si noti che x_0 e' soluzione dell'equazione se e solo se $f(x_0) = 0$. Risulta che $f(1) = 2 > 0$ e $f(2) = -6 < 0$. Poiche' f e' continua in $[1,2]$, si deduce che esiste un $x_0 \in (1,2)$ tale che $f(x_0) = 0$, ovvero esiste in $(1,2)$ una soluzione dell'equazione. Poiche' f e' monotona strettamente decrescente in $[1,2]$ (e' la somma di funzioni monotone strettamente decrescenti e di una costante), si deduce che nell'intervallo $(1,2)$ si trova un'unica soluzione dell'equazione.

   (c) Sia $f(x) = 3 + 6\sqrt{x} - 2^x$ e si noti che x_0 e' soluzione dell'equazione se e solo se $f(x_0) = 0$. Risulta che $f(1) = 7 > 0$ e $f(3) = 6\sqrt{3} - 5 > 0$. Dunque non si puo' applicare il teorema di esistenza degli zeri, e in effetti $(1,3)$ non contiene alcuna soluzione all'equazione, dato che $6\sqrt{x} - 2^x > 0$ per ogni $x \in [1,3]$, e dunque $f(x) > 0$ per ogni $x \in [1,3]$.

   (d) Risulta che $f(0) = 5 > 0$ e $f(2) = -2 < 0$. Poiche' f e' continua in $[0,2]$ (dato che $\lim_{x\to 1^-} f(x) = 3 = f(1) = \lim_{x\to 1^+} f(x)$), si deduce che esiste almeno una soluzione in $(0,2)$. In effetti, dato che f e' decrescente in $(0,1)$ e $f(1) = 3 > 0$, l'intervallo $(0,1)$ non contiene alcuna soluzione e quindi almeno una soluzione si trova in $(1,2)$.

   (e) Sia $f(x) = x^3 + x^4 - 1$ e si noti che x_0 e' soluzione dell'equazione se e solo se $f(x_0) = 0$. Risulta che $f(-2) = 7 > 0$ e $f(1) = 1 > 0$. Dunque non si puo' applicare il teorema di esistenza degli zeri per concludere che esiste una soluzione in $(-2,1)$, ma in realta' risulta

che $f(0) = -1 < 0$ e dunque il teorema di esistenza degli zeri stabilisce l'esistenza di una soluzione in $(-2, 0)$ e di un'altra soluzione in $(0, 1)$ (f e' monotona strettamente crescente in $(0, 1)$, pertanto la soluzione e' unica in $(0, 1)$).

(f) Risulta che $f(0) = -4 < 0$ e $f(3) = \frac{65}{6} > 0$. Tuttavia, non e' possibile applicare il teorema di esistenza degli zeri perche' f non e' continua nel punto $x_0 = 2$: $\lim_{x\to 2^-} f(x) = -2$ e $\lim_{x\to 2^+} f(x) = \frac{3}{2}$. In effetti, non esiste alcuna soluzione in $(0, 3)$ all'equazione $f(x) = 0$ poiche' $f(x) < 0$ per ogni $x \in (0, 2)$ e $f(x) > 0$ per ogni $x \in [2, 3)$.

4. Ciascuna delle funzioni in questo esercizio e' continua, quindi e' possibile applicare il teorema dei valori intermedi calcolando $\sup f$ (o $\max f$) e $\inf f$ (o $\min f$).

   (a) f e' monotona strettamente crescente (e' la somma di funzioni monotone strettamente crescenti) e $\inf f = \lim_{x\to 0^+} f(x) = -\infty$, $\sup f = \max f = f(4) = 639$. Dunque $\mathrm{Im}(f) = (-\infty, 639]$.

   (b) f e' monotona strettamente decrescente (e' la somma di funzioni monotone strettamente decrescenti) e $\min f = f(1) = -\frac{7}{2}$, $\max f = f(0) = 1$. Dunque $\mathrm{Im}(f) = [-\frac{7}{2}, 1]$.

   (c) $\lim_{x\to+\infty} f(x) = +\infty$ e $\min f = f(1) = \log_3 7$. Dunque $\mathrm{Im}(f) = [\log_3 7, +\infty)$.

   (d) $\lim_{x\to+\infty} f(x) = -\infty$ e $\max f = f(2) = \sqrt[3]{52}$. Dunque $\mathrm{Im}(f) = (-\infty, 52]$.

5. In x_0, x_1, x_2 le ipotesi del teorema della permanenza del segno non sono soddisfatte perche': (i) f non e' definita in x_0; (ii) $f(x_1) = 0$; (iii) f non e' continua in x_2. In x_3, le ipotesi del teorema della permanenza del segno sono soddisfatte perche' $f(x_3) > 0$ e f e' continua in x_3; pertanto esiste un intorno di x_3 tale che $f(x) > 0$ per ogni x nell'intorno.

Chapter 5

Elementi di Calcolo Differenziale

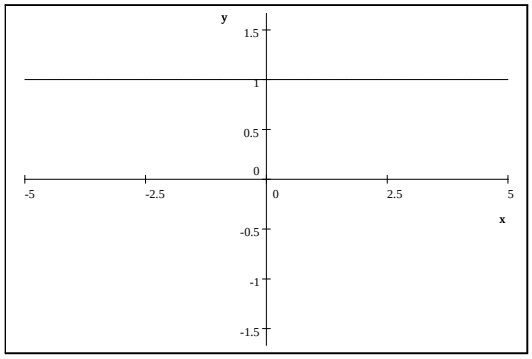
5.1 Introduzione

Il concetto fondamentale del calcolo differenziale e' quello di derivata, che formalizza il concetto di "pendenza" del grafico di una funzione in un punto. Conoscere la pendenza in ogni punto ci da' molte informazioni sul grafico.

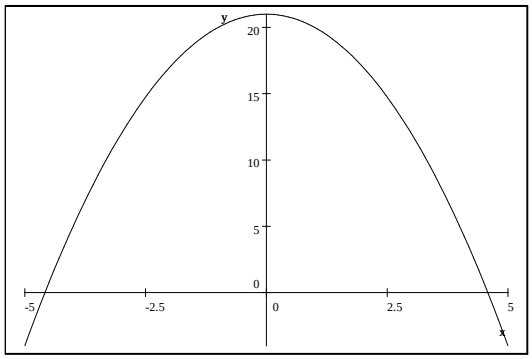
Pendenza sui segnali stradali:

la pendenza e' zero:

1



la pendenza e' prima positiva e poi negativa:



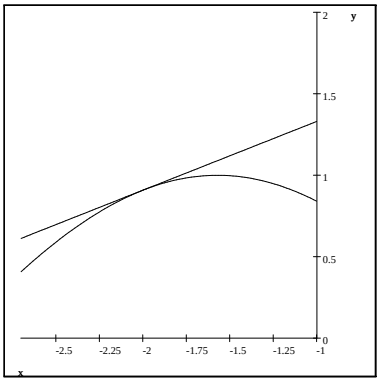
Si ricordi che la pendenza di una retta e' il coefficiente angolare della retta. Data la equazione della retta non verticale

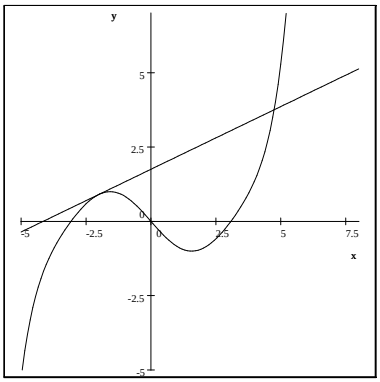
$$y = mx + n$$

il coefficiente angolare di tale retta e' m.

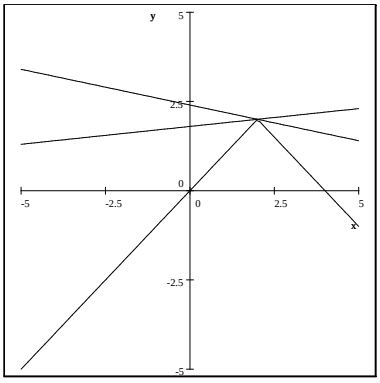
Definiremo la pendenza di una curva in un punto come il coefficiente angolare della retta tangente alla curva in quel punto. Ma cos'e' la retta tangente?

Le informazioni che vogliamo dalla retta tangente sono di natura locale: riguardano cosa succede vicino al punto. Trascuriamo cosa succede lontano e ci concentriamo su cosa succede in un parte del grafico vicino al punto. Vogliamo trovare una retta che approssima localmente il grafico della funzione.





Si osservi anche che in alcuni casi non e' chiaro quale sia la retta tangente.



Sulla base di queste osservazioni possiamo ora dare la definizione di derivata: essa verra' usata per definire in maniera rigorosa e "utile" il concetto di retta tangente al grafico di una funzione.

5.2 Il concetto di derivata

Definition 128 *Sia datai la funzione $f : S \subseteq \mathbb{R} \to \mathbb{R}$ e $x_0 \in Int\ S$. Si dice che f e' derivabile in x_0 se esiste ed e' finito il seguente limite*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{5.1}$$

In tal caso, il limite si dice derivata di f in x_0 e si indica con uno dei seguenti simboli

$$f'(x_0) \qquad o \qquad \frac{df(x)}{dx}_{|x=x_0} \qquad o \qquad \frac{df(x_0)}{dx}.$$

$\frac{f(x)-f(x_0)}{x-x_0}$ *si dice rapporto incrementale di f in x_0.*

Remark 129 *Si osservi che x_0 e' un punto interno di S e dunque di accumulazione per S e perciò ha senso calcolare il limite (5.1).*

Remark 130 *Si ponga $h := x - x_0$. Dunque $x = x_0 + h$ e se $x \to x_0$ allora $h \to 0$. Dunque il limite in* (5.1) *puo' essere riscritta come segue*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Definition 131 *Si consideri la funzione $f : S \subseteq \mathbb{R} \to \mathbb{R}$ e $x_0 \in Int\ S$. Si dice derivata sinistra di f in x_0, e si indica con $f'_-(x_0)$, il seguente limite*

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \tag{5.2}$$

se esiste ed e' finito; analogamente, si dice derivata destra di f in x_0, e si indica con $f'_+(x_0)$, iil seguente limite

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \tag{5.3}$$

se esiste ed e' finito.

Remark 132 *Altri modi per scrivere il limite in (5.2).*

a. Si ponga $h := x - x_0 < 0$; si ha dunque $x = x_0 + h$ e

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

b. Si ponga $k := x_0 - x > 0$; si ha dunque $x = x_0 - k$ e

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{k \to 0} \frac{f(x_0 - k) - f(x_0)}{-k}$$

Scritture analoghe valgono per il limite in (5.3.)

Remark 133 *La funzione f e' derivabile in x_0 se*

1. *esiste ed e' finito il limite in (5.2);*
2. *esiste ed e' finito il limite in (5.1);*
3. *tali limiti sono uguali.*

Definition 134 *$f : S \subseteq \mathbb{R} \to \mathbb{R}$ e' derivabile in $A \subseteq Int\ S$ se e' derivabile in ogni $x \in A$.*
f e' derivabile se e' derivabile in S (e dunque $S = Int\ S$).

Definition 135 *Se $f : S \subseteq \mathbb{R} \to \mathbb{R}$ e' derivabile in $A \subseteq Int\ S$, allora*

$$f' : A \to \mathbb{R}, \quad f' : x \mapsto f'(x)$$

si dice funzione derivata prima - o semplicemente funzione derivata - di f in A.

Definition 136 *Sia $n \in \mathbb{N}\setminus\{0,1\}$. La (funzione) derivata n-esima di f si indica con $f^{(n)}$ ed e' la (funzione) derivata prima della (funzione) derivata $(n-1)$-esima di f.*

Per comodita' di notazione si pone anche $f^{(0)} := f$ e $f^{(1)} := f'$. L'indice (n) puo' essere sostituito da un numero romano.

Calcoliamo le derivata di varie funzioni.

Example 137 *$f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto ax + b$, con $a, b \in \mathbb{R}$.*

$$\lim_{x\to x_0} \frac{ax+b-(ax_0+b)}{x-x_0} = \lim_{x\to x_0} \frac{a(x-x_0)}{x-x_0} = \lim_{x\to x_0} a = a$$

Possiamo dunque concludere che f e' derivabile su tutto $\mathbb{R}$ e dunque su ogni sottoinsieme aperto di $\mathbb{R}$.

Example 138 *$f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto ax^2 + bx + c$, con $a \neq 0$.*

$$\lim_{x\to x_0} \frac{ax^2+bx+c-\left(ax_0^2+bx_0+c\right)}{x-x_0} = \lim_{x\to x_0} \frac{a\left(x^2-x_0^2\right)+b(x-x_0)}{x-x_0} =$$

$$= \lim_{x\to x_0} \frac{a(x-x_0)(x+x_0)}{x-x_0} + \lim_{x\to x_0} \frac{b(x-x_0)}{x-x_0} = 2ax_0 + b$$

Example 139

$$f : \mathbb{R} \to \mathbb{R}, f : x \mapsto |x| := \begin{cases} -x & se \quad x < 0 \\ 0 & se \quad x = 0 \\ x & se \quad x > 0 \end{cases}$$

Dall'Esempio137 segue che

$$f' : \mathbb{R}\setminus\{0\} \to \mathbb{R}, \quad f' : x \mapsto := \begin{cases} -1 & se \quad x < 0 \\ 1 & se \quad x > 0 \end{cases}$$

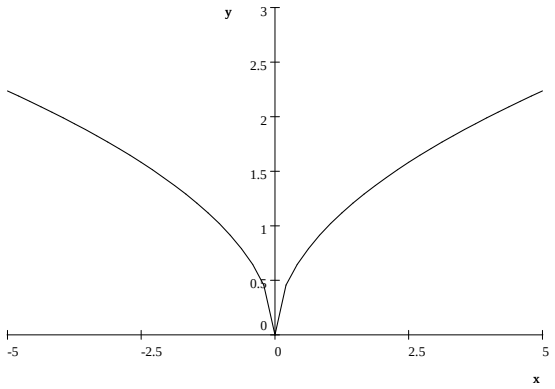
Si osservi che la funzione non e' derivabile in 0:

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{-x-0}{x} = \lim_{x\to 0^-} -1 = -1 = f'_-(0)$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{x-0}{x} = \lim_{x\to 0^+} 1 = 1 = f'_+(0)$$

Example 140 *Si dica se la seguente funzione e' derivabile in zero.*

$$f : \mathbb{R} \to \mathbb{R}, \quad f : x \mapsto \sqrt{|x|} = \begin{cases} \sqrt{-x} & se \quad x < 0 \\ 0 & se \quad x = 0 \\ \sqrt{x} & se \quad x > 0 \end{cases}$$



$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\sqrt{-x}}{x} = \lim_{x\to 0^-} \frac{\sqrt{-x}\cdot\sqrt{-x}}{x\sqrt{-x}} = \lim_{x\to 0^-} \frac{-x}{x\sqrt{-x}} = \lim_{x\to 0^-} \frac{-1}{\sqrt{-x}} = -\infty$.

Dunque, la funzione non e' derivabile in zero.

5.3 Il significato geometrico di derivata

Si consideri il seguente grafico

`inserire figura`

L'equazione della retta passante per i punti A di coordinate $(x_0, f(x_0))$ e $B(h)$ di coordinate $(x_0+h, f(x_0+h))$ e'

$$\frac{y-f(x_0)}{f(x_0+h)-f(x_0)} = \frac{x-x_0}{(x_0+h)-x_0}$$

ovvero

$$y = f(x_0) + \frac{f(x_0+h)-f(x_0)}{h}(x-x_0)$$

Si osservi che $\frac{f(x_0+h)-f(x_0)}{h}$, ovvero il rapporto incrementale di f in x_0, e' la pendenza della retta passante per i punti A e $B(h)$ ed e' anche $\tan\alpha(h)$, dove $\alpha(h)$ e' la misura dell'angolo che la retta forma con l'asse delle ascisse in senso orario.

Si osservi anche che se $h \to 0$, $B(h)$ si muove lungo il grafico di f verso A. Possiamo perciò dire che $f'(x_0) := \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ e' la pendenza della retta tangente nel punto $A = B(0)$ del grafico di f. Diamo dunque la seguente definizione.

Definition 141 *Sia $f : S \subseteq \mathbb{R} \to \mathbb{R}$ derivabile in x_0. L'equazione della retta tangente al grafico nel punto di coordinate $(x_0, f(x_0))$ e'*

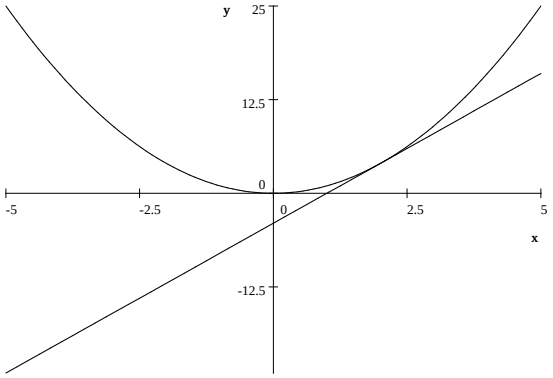
$$y = f(x_0) + f'(x_0)\cdot(x-x_0)$$

Remark 142 *E' possibile dare la definizione vista solo nel caso in cui f sia derivabile in x_0.*

Example 143 *Se esiste, si trovi la retta tangente al grafico di $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^2$ nel punto di ascissa 2.*

Abbiamo visto che $f'(x) = 2x$. Dunque la equazione cercata e' $y = 4 + 4\cdot(x-2)$ ovvero

$$y = 4x - 4$$



5.4 Derivabilita' e continuita'

Nell'Esempio 139 abbiamo il caso di una funzione continua in zero e non derivabile in zero. Vale invece il seguente Teorema.

Theorem 144 *Sia data $f : S \subseteq \mathbb{R} \to \mathbb{R}$, $f : x \mapsto f(x)$. Se f e' derivabile in x_0, allora f e' continua in x_0.*

Proof. Vogliamo dimostrare che $\lim_{x\to x_0} f(x) = f(x_0)$. Si osservi che

$$\begin{aligned}
&(\lim_{x\to x_0} f(x)) - f(x_0) \overset{(1)}{=} \\
&= (\lim_{x\to x_0} f(x)) - (\lim_{x\to x_0} f(x_0)) \overset{(2)}{=} \\
&= \lim_{x\to x_0} (f(x) - f(x_0)) = \\
&= \lim_{x\to x_0} \tfrac{f(x)-f(x_0)}{x-x_0}(x-x_0) \overset{(3)}{=} \\
&= \left(\lim_{x\to x_0} \tfrac{f(x)-f(x_0)}{x-x_0}\right)(\lim_{x\to x_0}(x-x_0)) \overset{(4)}{=} \\
&= 0
\end{aligned}$$

dove la spiegazione delle uguaglianze indicate con (1), ..., (4) e' presentata di seguito.

(1) : Il limite di una costante e' la costante stessa;

(2) : Poiche' uno dei due limiti esiste finito, il limite della differenza e' la differenza dei limiti;

(3) : Il limite del prodotto e' il prodotto dei limiti;

(4) : Poiche' f e' derivabile in x_0, $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ esiste ed e' finito. ■

In sintesi abbiamo dunque

$$\langle f \text{ derivabile in } x_0\rangle \overset{\Rightarrow}{\nLeftarrow} \langle f \text{ continua in } x_0\rangle$$

Example 145 *Varie figure su funzioni derivabili e non.*

Remark 146 *In maniera intuitiva, per dire se una funzione definita su un intervallo aperto e' derivabile, si procede come segue: a. Si cerca di capire se la funzione e' continua; se non lo e', allora non e' derivabile; se e' continua, si va al passo successivo. b. se il grafico della funzione non presenta "spigoli", e' liscio, allora la funzione e' derivabile.*

5.5 Calcolo di derivate

In quanto segue, calcoleremo le derivate delle funzioni elementari e di funzioni ottenute a partire da altre attraverso le operazioni di somma algebrica, prodotto, quoziente, potenza, radice,composizione, inversione.

In quanto segue assumiamo che S e Y sono sottoinsiemi di $\mathbb{R}$.

Proposition 147 *Se $f : S \to Y$, $f : x \to k$, dove $k \in \mathbb{R}$, allora $\forall x \in Int\ S$, $f'(x) = 0$.*

Proof.

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{k-k}{x-x_0} = \lim_{x\to x_0} 0 = 0.$$

■

Proposition 148 *Se $f : S \to Y$, e $g : S \to Y$ sono derivabili in x_0, allora $f + g$ e' derivabile in x_0, e*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

Proof.

$$(f+g)'(x_0) := \lim_{x\to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} := \lim_{x\to x_0} \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0} =$$

$$= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0).$$

■

Proposition 149 *Se $f : S \to Y$, e $g : S \to Y$ sono derivabili in x_0, allora $f \cdot g$ e' derivabile in x_0, e*

$$(f\cdot g)'(x_0) = f'(x_0)\cdot g(x_0) + f(x_0)\cdot g'(x_0)$$

Proof. (Idea: Aggiungi e sottrai $f(x)\cdot g(x_0)$.)

$$(f\cdot g)'(x_0) := \lim_{x\to x_0} \frac{(f\cdot g)(x) - (f\cdot g)(x_0)}{x - x_0} :=$$

$$:= \lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0) + f(x)\cdot g(x_0) - f(x)\cdot g(x_0)}{x - x_0} =$$

$$= \lim_{x\to x_0} \frac{[f(x)\,g(x) - f(x)\cdot g(x_0)] + [-f(x_0)\,g(x_0) + f(x)\cdot g(x_0)]}{x - x_0} =$$

$$= \lim_{x\to x_0} \frac{f(x)\,[g(x) - g(x_0)] + g(x_0)\,[f(x) - f(x_0)]}{x - x_0} =$$

$$= \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} + \lim_{x\to x_0} g(x_0) \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} =$$
$$= f'(x_0)\cdot g(x_0) + f(x_0)\cdot g'(x_0)$$

Si osservi che nell'ultimo passaggio si e' usato il Teorema 144. ■

Proposition 150 *Se $f : S \to Y$ e' derivabile in x_0, e $k \in \mathbb{R}$, allora $g := kf$ e' derivabile in x_0, e $g'(x_0) = k \cdot f'(x_0)$.*

Proof.
$$g'(x_0) := \lim_{x\to x_0} \frac{k\cdot f(x) - k\cdot(x_0)}{x-x_0} = \lim_{x\to x_0} k\cdot\frac{f(x)-(x_0)}{x-x_0} = kf'(x_0)$$

■

Proposition 151 *Se $n \in \mathbb{N}\setminus\{0\}$ e $f : S \to Y$, $f : x \mapsto x^n$, allora $\forall x \in S$,*

$$f'(x) = nx^{n-1}$$

Proof. La dimostrazione procede per induzione.
L'affermazione e' vera per $n = 1$:
Si veda l'esempio 137.
Assumendo l'affermazione per n, si dimostra l'affermazione per $n+1$.
Definita $g(x) := x^{n+1}$, vogliamo dimostrare che $g'(x) = (n+1)x^n$.
Si osservi che $g(x) = x^n \cdot x$. Dunque, applicando la regola di derivazione del prodotto, si ha $g'(x) = nx^{n-1}\cdot x + x^n \cdot 1 = nx^n + x^n = (n+1)x^n$. ■

Proposition 152 *Se $g : S \to Y$ e' derivabile in x_0 e $g(x_0) \neq 0$, allora $\frac{1}{g}$ e' derivabile in x_0, e*

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{[g(x_0)]^2}$$

Proof. (Idea: Usa la definizione di derivata)

$$\left(\frac{1}{g}\right)'(x_0) := \lim_{x\to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x-x_0} = \lim_{x\to x_0} -\frac{g(x)-g(x_0)}{(x-x_0)\,g(x)\,g(x_0)} =$$
$$= -\lim_{x\to x_0} \frac{1}{g(x)\,g(x_0)} \lim_{x\to x_0} \frac{g(x)-g(x_0)}{(x-x_0)} = -\frac{g'(x_0)}{[g(x_0)]^2}$$

Si osservi che abbiamo usato la continuita' di g in x_0. ■

Proposition 153 *Se $f : S \to Y$, e $g : S \to Y$ sono derivabili in x_0, e $g(x_0) \neq 0$, allora $\frac{f}{g}$ e' derivabile in x_0 e*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{[g(x_0)]^2}$$

Proof. Segue dal fatto che $\frac{f}{g} = f\cdot\frac{1}{g}$, dalla Proposizione precedente e dalla regola di derivazione del prodotto. ■

Proposition 154 *Se $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto \sin x$, allora $\forall x \in \mathbb{R}$, $f'(x) = \cos x$.*

Proof. (Idea: formula di prostaferesi e $\lim_{x\to 0}\frac{\sin x}{x} = 1$).
La dimostrazione usa la seguente formula di prostaferesi:

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$$

e la definizione di derivata.

$$f'(x_0) := \lim_{x\to x_0} \frac{\sin x - \sin x_0}{x-x_0} = \lim_{x\to x_0} \frac{2\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2}}{x-x_0} =$$

$$= \left(\lim_{x\to x_0} 2\frac{1}{x-x_0} \sin\frac{x-x_0}{2} \right) \lim_{x\to x_0} \cos\frac{x+x_0}{2} = \left(\lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} \right) \cos x_0$$

Posto $y := \frac{x-x_0}{2}$, si osserva che se $x \to x_0$, allora $y \to 1$ e dunque

$$\lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

Anche in questo caso abbiamo usato la continuita' della funzione f. ■

Proposition 155 *Se $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto \cos x$, allora $\forall x \in \mathbb{R}$, $f'(x) = -\sin x$.*

Proof. La dimostrazione usa la seguente formula di prostaferesi:

$$\cos a - \cos b = -2 \sin\frac{a-b}{2} \sin\frac{a+b}{2}$$

e la definizione di derivata.

$$f'(x_0) := \lim_{x\to x_0} \frac{\cos x - \cos x_0}{x - x_0} = \lim_{x\to x_0} \frac{-2\sin\frac{x-x_0}{2}\sin\frac{x+x_0}{2}}{x-x_0} =$$

$$-\left(\lim_{x\to x_0} 2\frac{1}{x-x_0} \sin\frac{x-x_0}{2} \right) \lim_{x\to x_0} \sin\frac{x+x_0}{2} =$$

$$-\left(\lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} \right) \sin x_0 = -\sin x_0$$

■

Proposition 156 *Given $a \in \mathbb{R}_{++} \setminus \{1\}$, se $f : \mathbb{R}_{++} \to \mathbb{R}$, $f : x \mapsto \log_a x$, allora $\forall x \in \mathbb{R}_{++}$,*

$$f'(x) = \frac{1}{x} \log_a e.$$

Proof. Idea: far comparire $y = \frac{x}{x_0} - 1 > -1$.
Si osservi che

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\log_a x - \log_a x_0}{x - x_0} = \frac{\log_a \frac{x}{x_0}}{x_0\left(\frac{x}{x_0} - 1\right)} = \frac{1}{x_0} \frac{\log_a\left(1 + \frac{x}{x_0} - 1\right)}{\frac{x}{x_0} - 1}$$

Si ponga $y := \frac{x}{x_0} - 1 > -1$ poiche' $x, x_0 \in \mathbb{R}_{++}$ e percio' $\log_a(1+y)$ e' ben definito. Dunque $x = x_0(1+y)$ e se $x \to x_0$, allora $y \to 0$.

$$f'(x_0) = \lim_{x\to x_0} \frac{1}{x_0} \frac{\log_a\left(1 + \frac{x}{x_0} - 1\right)}{\frac{x}{x_0} - 1} = \frac{1}{x_0} \lim_{y\to 0} \frac{\log_a(1+y)}{y} = \frac{1}{x_0} \log_a e.$$

Si osservi che abbiamo usato la continuita' della funzione logaritmo. ■

Remark 157 *Si osservi che*

$$\frac{d \ln x}{dx} = \log_e e \frac{1}{x} = \frac{1}{x}.$$

Proposition 158 *(Derivata di una funzione composta) Siano date le funzioni*

$$f : S \to Y, f : x \mapsto f(x)$$

$$g : Z \to W, g : y \mapsto g(y)$$

con $S, Y, Z, W \subseteq \mathbb{R}$, $\operatorname{Im} f \subseteq Z$. Se f e' derivabile in x_0 e g e' derivabile in $f(x_0) := y_0$, allora $h := g \circ f$ e' derivabile in x_0, e

$$h'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

ovvero

$$\frac{dh(x)}{dx}_{|x=x_0} = \frac{dg(y)}{dy}_{|y=f(x_0)} \cdot \frac{df(x)}{dx}_{|x=x_0}$$

Proof. Omessa. ■

Example 159 *Si calcoli la derivata di* $h(x) = \sin\left(x^2\right)$.
Nella notazione del Teorema precedente, si ha $y = f(x) = x^2$ *e* $g(y) = \sin y$. *Dunque*

$$h'(x) = \cos y_{|y=x^2} \cdot 2x = 2x \cos x^2$$

Example 160 *Si calcoli la derivata di* $h(x) = \sin^2 x$.
Nella notazione del Teorema precedente, si ha $y = f(x) = \sin x$ *e* $g(y) = y^2$. *Dunque*

$$h'(x) = 2y_{|y=\sin x} \cdot \cos x = 2 \sin x \cos x$$

Exercise 161 *Si calcolino le derivate di* $\ln^2 x$, $\ln^n x$, $f(g(x) \cdot h(x)) + g(f(x) \cdot h(x))$.

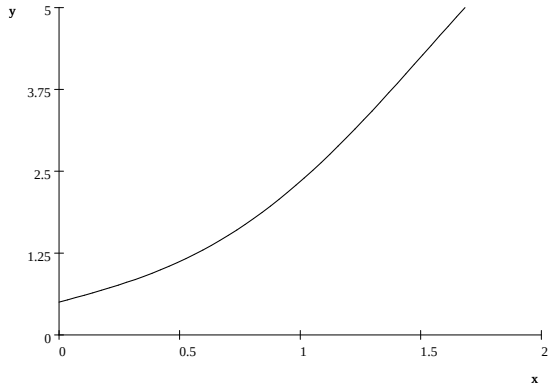
Proposition 162 *(Derivata della funzione inversa) Se*
$f : [a,b] \to \mathbb{R}$ *e' strettamente monotona e continua,*
derivabile in $x_0 \in (a,b)$,
$f'(x_0) \neq 0$,
allora la funzione $g : \operatorname{Im} f \to [a,b]$, $g : y \mapsto f^{-1}(y)$ *e' derivabile in* $y_0 := f(x_0)$ *e*

$$g'(y_0) = \frac{1}{f'(x_0)}$$

ovvero

$$\frac{dg(y)}{dy}_{|y=y_0} = \frac{1}{\frac{df(x)}{dx}_{|x=g(y_0)}}$$

Proof. Strategia: 1. Si disegni una figura con x, x_0, y, y_0.
2. Si scriva la relazione che lega gli incrementi di g a quelli di f;
3. Si calcoli il limite.
`figura sotto da completare`



Vogliamo dimostrare che

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

Si osservi che

$$g(y) - g(y_0) = x - x_0 \qquad \text{e} \qquad y - y_0 = f(x) - f(x_0).$$

Poiche' f e' continua, anche g e' continua e

$$0 = \lim_{y \to y_0} g(y) - g(y_0) = \lim_{y \to y_0} x - x_0$$

ovvero se $y \to y_0$, allora $x \to x_0$. Dunque,

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}.$$

■

Proposition 163 *Se* $g : \mathbb{R} \to \mathbb{R}$, $g : y \mapsto a^y$, *allora* $\forall y \in \mathbb{R}$, $g'(y) = a^y \log_e a$.

Proof. La dimostrazione usa la precedente proposizione e la regola di derivazione della funzione logaritmo. Sia $y = f(x) = \log_a x$ e dunque

$$\frac{dg(y)}{dy} = \frac{1}{\frac{d\log_a x}{dx}_{|x=g(y)}} = \frac{1}{\frac{1}{\log_e a \cdot x}_{|x=g(y)}} = \log_e a \cdot a^y$$

■

Remark 164

$$\frac{de^x}{dx} = \log_e e \cdot e^x = e^x$$

Example 165 *Si consideri la funzione invertibile*

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x + ex$$

si calcoli la derivata della funzione inversa nel punto $\overline{y} = 1$. *Suggerimento: la soluzione dell'equazione* $e^x + ex = 1$ *appartiene all'insieme* $\{-1, 0, +1\}$.

La derivata della funzione inversa $g : \mathbb{R} \to \mathbb{R}$, $g : y \to g(y)$ *nel punto* $\overline{y}$ *e'*

$$\frac{dg(y)}{dy}_{|y=\overline{y}} = \frac{1}{\frac{df(x)}{dx}_{|\overline{x}=g(\overline{y})}}$$

Per trovare x *tale che* $\overline{x} = g(\overline{y})$ *ovvero tale che* $f(\overline{x}) = \overline{y}$ *occorre risolvere l'equazione*

$$e^x + ex = 1$$

che ha unica soluzione per $\overline{x} = 0$ *:* $e^0 + e \cdot 0 = 1$ *Dunque,*

$$\frac{dg(y)}{dy}_{|y=\overline{y}} = \frac{1}{e^x + e_{|\overline{x}=0}} = \frac{1}{1+e}$$

Definition 166 arcsin *è la funzione inversa di* $\sin_{|\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$*;*
arccos *è la funzione inversa di* $\cos_{|[0,\pi]}$ *;*
arctan *è la funzione inversa di* $\tan_{|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$.

Proposition 167 *La funzione* $\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *e' derivabile in* $(-1, 1)$ *e* $\forall x \in (-1, 1)$,

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Proof. Usando la notazione del Teorema di derivazione della funzione inversa, poniamo

$$g(y) = \arcsin y$$

e

$$f(x) = \sin x = y$$

Si ha dunque

$$g'(y) = \frac{1}{f'(x)_{|x=g(y)}} = \frac{1}{\cos x_{|x=\arcsin y}} = \frac{1}{\cos \arcsin y}$$

`figura`
Dalla definizione di cos e sin e dal Teorema di Pitagora segue che

$$\frac{1}{\cos \arcsin y} = \frac{1}{\sqrt{1 - y^2}}$$

■

Exercise 168 *1. La funzione* $\arccos : [-1,1] \to [0,\pi]$ *e' derivabile in* $(-1,1)$ *e* $\forall x \in (-1,1)$,

$$\frac{d \arccos x}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

2. La funzione $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *e' derivabile e* $\forall x \in \mathbb{R}$,

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

Example 169 *Se* $f : \mathbb{R}_+ \to \mathbb{R}$, $f : x \mapsto x^{\frac{1}{n}}$, *allora*

$$\forall x \in dom\ f, f'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$$

Si osservi che la funzione inversa di f *e' la potenza n-esima di* x.

Dalla regola di derivazione delle funzioni inverse, identificando $g(y)$ *con* $y^{\frac{1}{n}}$ *e* $f(x)$ *con* x^n *, si ha*

$$\frac{dy^{\frac{1}{n}}}{dy} = \frac{1}{\frac{dx^n}{dx}_{|x=y^{\frac{1}{n}}}} = \frac{1}{nx^{n-1}_{|x=y^{\frac{1}{n}}}} = \frac{1}{ny^{\frac{n-1}{n}}} = \frac{1}{n} y^{\frac{1}{n}-1}$$

Example 170 *Se* $f : \mathbb{R}_{++} \to \mathbb{R}$, $f : x \mapsto x^b$, *con* $\underline{b \in \mathbb{R}}$, $\frac{dx^b}{dx} = bx^{b-1}$.

Usando la notazione del Teorema di derivazione della funzione composta si ha:

$$g[f(x)] = h(x) = x^b = e^{b \ln x}; \qquad f(x) = b \ln x = y \qquad ; \ g(y) = e^y$$

Dunque

$$h'(x) = g'[f(x)] \cdot g'(x) = (e^y)_{|y=b \ln x} \cdot b\frac{1}{x} = b\frac{x^b}{x}$$

Example 171

$$\frac{df(x)^{g(x)}}{dx} = \frac{de^{g(x) \ln f(x)}}{dx} = e^{g(x) \ln f(x)} \left[g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right] =$$

$$= f(x)^{g(x)} \left[g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right]$$

The following table summarizes the derivatives we computed.

$f(x)$	$f'(x)$	
k	0	$k \in \mathbb{R}$
x^a	ax^{a-1}	$x \in \mathbb{R}_{++}, a \in \mathbb{R} \setminus \{0\}$
a^x	$a^x \log a$	$x \in \mathbb{R},\ a \in \mathbb{R}_{++} \setminus \{1\}$
e^x	e^x	$x \in \mathbb{R}$
$\log_a x$	$\frac{1}{x} \log_a e$	$x \in \mathbb{R}_{++}, a \in \mathbb{R}_{++} \setminus \{1\}$
$\log x$	$\frac{1}{x}$	$x \in \mathbb{R}_{++}$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \in \mathbb{R} \setminus \left\{x' \in \mathbb{R} : x' = \frac{\pi}{2} + k\pi, \text{ with } \pi \in \mathbb{Z}\right\}$,
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in \{x' \in \mathbb{R} : \lvert x' \rvert < 1\}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$x \in \{x' \in \mathbb{R} : \lvert x' \rvert < 1\}$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$

Main rules of differentiations ar listed below. We assume all the involved functions are differentiable and all expressions are well defined.

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

$$\frac{d(f(g(x)))}{dx} = \frac{d(f(y))}{dy}_{|y=g(x)} \cdot \frac{dg(x)}{dx}$$

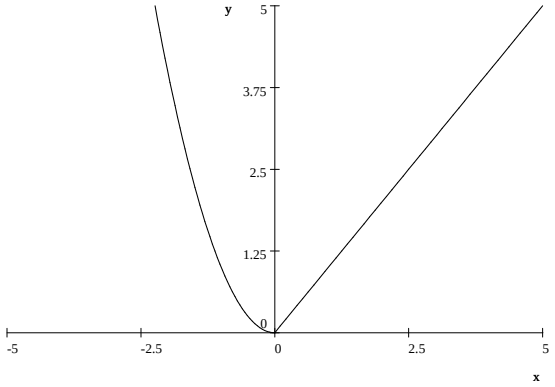
$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}_{|x=g(y)}}$$

Alcuni esercizi con svolgimento.

1. Si calcoli la derivata della funzione

$$f : \mathbb{R} \to \mathbb{R}, \quad f : x \mapsto \begin{cases} x^2 & \text{if} \quad x \leq 0 \\ x & if \quad x > 0 \end{cases}$$

Soluzione.



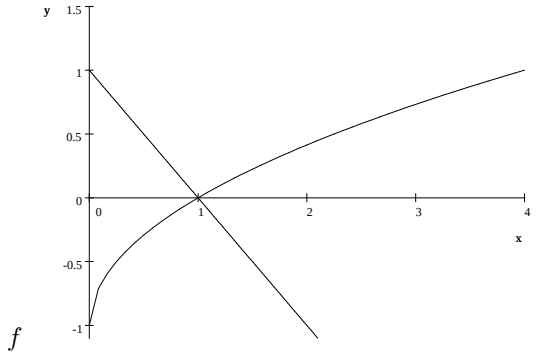
$$f'(x) = \begin{cases} 2x & \text{if} \quad x < 0 \\ 1 & if \quad x > 0 \end{cases}$$

$f'_-(0) := \lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{x^2}{x} = \lim_{x \to 0^-} x = 0;$
$f'_+(0) := \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1;$
Dunque la funzione non e' derivabile in zero.

2. Se possibile, si calcoli la derivata della funzione

$$f : \mathbb{R}_+ \to \mathbb{R}, \quad f : x \mapsto \max\left\{\sqrt{x} - 1, 1 - x\right\}$$

Soluzione.



f

Disegnando il grafico di $\sqrt{x}-1$ e di $1-x$, e' facile vedere che

$$f(x)=\begin{cases} 1-x & \text{if} \quad x\in[0,1] \\ \sqrt{x}-1 & if \quad x\in(1,+\infty) \end{cases}$$

$f'_+(1):=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{(\sqrt{x}-1)-0}{x-1}=\lim_{x\to1^+}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}=\lim_{x\to1^+}\frac{1}{(\sqrt{x}+1)}=\frac{1}{2}$.

$f'_-(1):=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{1-x}{x-1}=-1$.

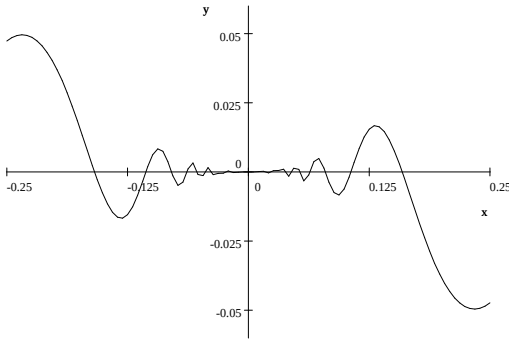
e dunque la funzione non e' derivabile in 1.

3. Se possibile, si calcoli la derivata della funzione

$$f:\mathbb{R}\to\mathbb{R},\quad f:x\mapsto\begin{cases} x^2\sin\frac{1}{x} & \text{if} \quad x\neq0 \\ 0 & if \quad x=0 \end{cases}$$

in zero.

Soluzione.

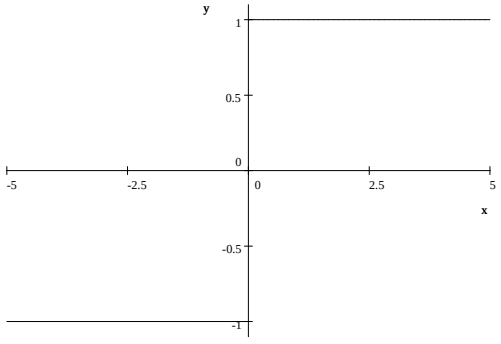


$\lim_{x\to0}\frac{x^2\sin\frac{1}{x}-0}{x-0}\lim_{x\to0}x\sin\frac{1}{x}=0$, e dunque $f'(0)=0$.

4. Se possibile, si calcoli la derivata della funzione

$$f:\mathbb{R}\to\mathbb{R},\quad f:x\mapsto\begin{cases} -1 & \text{if} \quad x<0 \\ 0 & if \quad x=0 \\ +1 & if \quad x>0 \end{cases}$$

Soluzione.



$f'_-(0):=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{-1-0}{x}=+\infty$

$f'_+(0):=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{1-0}{x}=+\infty$, e

$$f':\mathbb{R}\backslash\{0\}\to\mathbb{R},\quad f:x\mapsto\begin{cases} 0 & \text{if} \quad x<0 \\ 0 & if \quad x>0 \end{cases}$$

5. Assumendo che f,g e h siano derivabili, si calcolino le derivate di

a. $\ln^n x$;

b. $f(g(x)\cdot h(x))+g(f(x)\cdot h(x))$.

Soluzione.

a. $n \ln^{n-1} x \frac{1}{x}$;
b. $f'(g(x) h(x)) [g'(x) h(x) + g(x) h'(x)] + g'(f(x) h(x)) [f'(x) h(x) + f(x) h'(x)]$.
6. Si calcoli la derivata di $\arccos x$ (nell'appropriato sottoinsieme di $\mathbb{R}$).
Soluzione.
Usando la notazione del Teorema di derivazione della funzione inversa, poniamo

$$g(y) = \arccos y$$

e

$$f(x) = \cos x = y$$

Si ha dunque

$$g'(y) = \frac{1}{f'(x)_{|x=g(y)}} = \frac{1}{\sin x_{|x=\arcsin y}} = \frac{1}{\cos \arcsin y}$$

`figura`
Dalla definizione di sin e dal Teorema di Pitagora segue che

$$g'(y) = \frac{1}{-\sqrt{1-y^2}}$$

7. Si calcoli la derivata di $\arctan x$ (nell'appropriato sottoinsieme di $\mathbb{R}$).
Soluzione.
Si ponga $y = f(x) = \tan x$ e dunque $x = g(y) = \arctan y$.

$$\frac{d \arctan y}{dy} = \frac{1}{\frac{d \tan x}{dx}_{|x=\arctan y}} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

5.6 Esercizi

Nannicini, Verdi and Vessella (2008): Sezione 8.7. Esercizi 11,12b,13,14,16,17,18,21,22,23,24.

Barozzi e Corradi (1989). Capitolo 7. Paragrafo su "Derivata, differenziale e elasticità": Esercizi 1, 2, 3 e 4. Paragrafo su "Regole di derivazione": Esercizi da 9.1 a 9.16.

Chapter 6

Applicazioni del Calcolo Differenziale

6.1 Derivate e Grafici di Funzioni - I

In questo Capitolo vogliamo verificare tra l'altro che

"conoscendo poco di f', riusciamo a dire molto di f".

Si ricordino le definizioni di (punto di) massimo e di minimo locale (o relativo) e globale (o assoluto).

Definition 172 *x_0 e' un punto di massimo locale per $f : S \subseteq \mathbb{R} \to \mathbb{R}$ se $x_0 \in S$ and*

$$\exists \delta > 0 \text{ tale che } \forall x \in (x_0 - \delta, x_0 + \delta) \cap S \text{ risulta che } f(x_0) \geq f(x)$$

x_0 e' un punto di massimo globale per f se

$$\forall x \in S \text{ risulta che } f(x_0) \geq f(x)$$

In questo paragrafo assumiamo che Se Y sono sottoinsiemi di $\mathbb{R}$ e $f : S \to Y, x \mapsto f(x)$.

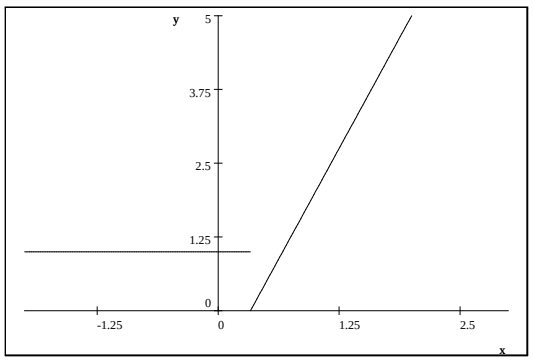
Definition 173 *$x_0 \in S$ e' un punto di estremo (locale) per f se x_0 e' un punto di massimo o di minimo (locale) per f.*

Example 174 ***1.*** *Si consideri la funzione*

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1 - 3x & se \quad x \geq \frac{1}{3} \\ 1 & se \quad x < \frac{1}{3} \end{cases}$$

Si determino eventuali punti di massimo locale e globale e punti di minimo locale e globale di $|f|$.

Soluzione



$x = \frac{1}{3}$ e' un punto di minimo locale e globale;

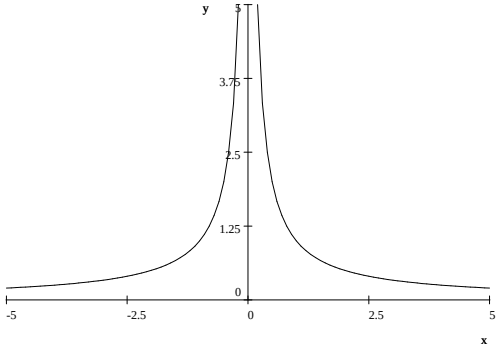
i punti in $\left(-\infty, \frac{1}{3}\right)$ sono punti di massimo e minimo locale;

la funzione non ha punti di massimo globale.

2. *Si enuncino le definizioni di massimo globale (o assoluto) e minimo globale (o assoluto) e si dica se la seguente funzione ammette massimi o minimi globali:*

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} 0 & se \quad x \in \mathbb{Q} \\ \left|\frac{1}{x}\right| & se \quad x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Soluzione.



L'insieme dei minimi globali e' $\mathbb{Q}$. La funzione non ammette massimo globale.

Per la dimostrazione della seguente Proposizione e' necessario il seguente Lemma.

Lemma 175 *Si consideri $f : S \subseteq \mathbb{R} \to \mathbb{R}$. Se $x_0 \in D(S)$, esiste $\gamma > 0$ tale che $\forall x \in (x_0, x_0 + \gamma) \cap S$ risulta $f(x) \geq 0$, $\lim_{x \to x_0^+} f(x)$ esiste, allora $\lim_{x \to x_0^+} f(x) \geq 0$.*

Proof. Strategia: si proceda per assurdo e si prenda $\varepsilon = -\frac{\alpha}{2} > 0$.

Si supponga che la conclusione del teorema e' falsa, ovvero

$$\lim_{x \to x_0^+} f(x) := \alpha < 0$$

Dalla definizione di limite, si ha che

$\forall \varepsilon > 0 \ \ \exists \delta > 0$ tale che $\forall x \in (x_0, x_0 + \delta) \cap S$ risulta $|f(x) - \alpha| < \varepsilon$ e dunque $f(x) < \alpha + \varepsilon$.

Si prenda $\varepsilon = -\frac{\alpha}{2} > 0$. Si ha dunque che $\forall x \in (x_0, x_0 + \delta) \cap (x_0, x_0 + \gamma) \cap S$ risulta

$$f(x) < \alpha + \left(-\frac{\alpha}{2}\right) = \frac{\alpha}{2} < 0$$

,il che contraddice l'ipotesi che $\forall x \in (x_0, x_0 + \gamma) \cap S$, $f(x) \geq 0$ ■

Remark 176 *Risultati analoghi valgono per funzioni non positive o non negative in intorni sinistri o completi.*

Proposition 177 *Se*

1. x_0 e' un punto di estremo per f, e

2. f e' derivabile in x_0

allora $f'(x_0) = 0$.

Proof. (Passi principali:1. Fai un disegno; 2.Calcola derivata sinistra e destra.)

Si consideri il caso in cui x_0 e' un punto di massimo locale. La dimostrazione del caso in cui x_0 e' un punto di minimo locale e' analoga.

Si osservi che poiche' f e' derivabile, x_0 e' un punto interno di S e dunque $\exists \delta > 0$ tale che $(x_0 - \delta, x_0 + \delta) \subseteq S$ and $\forall x \in (x_0 - \delta, x_0 + \delta)$ risulta che $f(x_0) \geq f(x)$.

Si prenda $h \in (0, \delta)$. Dunque $x_0 + h \in (x_0, x_0 + \delta)$. Dalla definizione di massimo segue dunque che

$$f(x_0) \geq f(x_0 + h)$$

e

$$\frac{f(x_0+h)-f(x_0)}{h} \leq 0 \tag{6.1}$$

In maniera analoga, si prenda $k \in (-\delta, 0)$. Then $x_0 + k \in (x_0 - \delta, x_0)$. Dalla definizione di massimo segue dunque che

$$f(x_0) \geq f(x_0+k)$$

e

$$\frac{f(x_0+k)-f(x_0)}{k} \geq 0 \tag{6.2}$$

Prendendo il limite per h che tende a zero del membro di sinistra di (6.1), e il limite per k che tende a zero del membro di sinistra di (6.2), e usando il Lemma 175, otteniamo

$$f'_+(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$$

e

$$f'_-(x_0) = \lim_{k\to 0} \frac{f(x_0+k)-f(x_0)}{k} \geq 0$$

Poiche' f e' derivabile in x_0 si ha dunque

$$0 \leq f'_-(x_0) = f'(x_0) = f'_+(x_0) \leq 0$$

e quindi

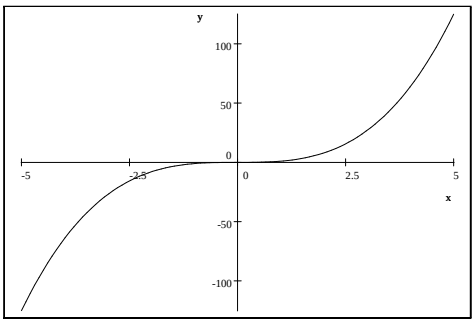
$$f'(x_0) = 0$$

■

Remark 178 *L'implicazione opposta di quella contenuta nella Proposizione 177 e' falsa. Piu' precisamente, si puo' avere $f'(x_0) = 0$ e x_0 puo' essere un punto non di estremo. Il classico (contro)esempio usato per illustrare questa affermazione e' il seguente. Si consideri*

$$f : \mathbb{R} \to \mathbb{R}, \quad f : x \longmapsto x^3$$

Si ha dunque $f'(0) = 0$. D'altra parte, $\forall \delta > 0$, $\forall x \in (-\delta, 0)$, $f(x) < 0$ and $\forall x \in (0, \delta)$, $f(x) > 0$: dunque zero non e' punto ne' di massimo ne' di minimo.



Definition 179 *Si consideri la funzione derivabile $f : S \subseteq \mathbb{R} \to \mathbb{R}$. $x \in S$ e' un punto critico per f se $f'(x) = 0$. $y \in \mathbb{R}$ e' un valore critico per f se esiste $x \in S$ tale che $y = f(x)$.*

Sulla base della precedente Definizione possiamo sintetizzare la Proposizione 177 e la Osservazione 178 come segue:

Se f e' derivabile,

$$\left\langle \; x \text{ e' un punto di estremo per } f \; \right\rangle \underset{\not\Leftarrow}{\Rightarrow} \left\langle \; x \text{ e' un punto critico per } f \; \right\rangle$$

o, in termini di insiemi,

punti critici | punti di estremo

Per continuare la nostra analisi delle relazioni esistenti tra f e f', abbiamo bisogno di alcuni importanti teoremi.

6.2 Teoremi di Rolle, Lagrange e Cauchy

Theorem 180 ***di Rolle.*** *Si consideri* $f : [a,b] \to \mathbb{R}$.

$$\left\langle \begin{array}{l} 1.\ f \text{ e' continua su } [a,b], \\ 2.\ f \text{ e' derivabile su } (a,b) \\ 3.\ f(a) = f(b) \end{array} \right\rangle \Rightarrow \left\langle \ \exists\, x_0 \in (a,b) \text{ tale che } f'(x_0) = 0 \ \right\rangle$$

Proof. (Idea di fondo: Si usi il Teorema del valore estremo e si distinguano due casi).

Grazie all'ipotesi 1 e al Teorema del valore estremo, f assume valore massimo M e valore minimo m, ovvero,esistono $x_M, x_m \in [a,b]$ tali che

$$\forall x \in [a,b], \qquad f(x_m) \leq f(x) \leq f(x_M) \tag{6.3}$$

I suguenti due casi sono gli unici possibili:

1. $x_m \in (a,b) \vee x_M \in (a,b)$;
2. $x_m \in \{a,b\} \wedge\ x_M \in \{a,b\}$.

Si osservi che $\neg\left[x_m \in (a,b) \vee\ x_M \in (a,b)\right] = \left[x_m \notin (a,b) \wedge\ x_M \notin (a,b)\right] = \left[x_m \in \{a,b\} \wedge\ x_M \in \{a,b\}\right]$.

Caso 1.

Dalla Proposizione 177, si ha che $f'(x_m) = 0 \vee f'(x_M) = 0$.

Caso 2.

Dall'ipotesi 3 si ha che

$$f(x_m) = f(a) = f(b) = f(x_M)$$

Ma allora da (6.3), si ha che

$$\forall x \in [a,b], \qquad f(a) \leq f(x) \leq f(a) \tag{6.4}$$

ovvero la funzione e' costante e dunque $\forall x \in (a,b)$ si ha che $f'(x) = 0$. ■

Remark 181 *La eliminazione di anche una sola delle ipotesi del teorema rende falsa la conclusione.*
tre figure

Usando il Teorema di Rolle, possiamo dimostrare il seguente Teorema.

Theorem 182 ***di Lagrange.*** *Si consideri* $f : [a,b] \to \mathbb{R}$.

$$\left\langle \begin{array}{l} 1.\ f \text{ e' continua su } [a,b], \\ 2.\ f \text{ e' derivabile su } (a,b) \end{array} \right\rangle \Rightarrow \left\langle \ \exists\, x_0 \in (a,b) \text{ tale che } f'(x_0) = \frac{f(b)-f(a)}{b-a} \ \right\rangle$$

Proof. La dimostrazione del Teorema di Lagrange consiste nella applicazione del Teorema di Rolle a una ben scelta funzione.

Passo 1. Si consideri la funzione

$$h : [a,b] \to \mathbb{R}, \qquad h : x \longmapsto f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$$

figura

Si osservi che

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

e' l'equazione della retta passante per i punti di coordinate $(a, f(a))$ e $(b, f(b))$.

Passo 2. La funzione h verifica le ipotesi del teorema di Rolle:

a. h e' continua su $[a,b]$ perche' f e' continua su $[a,b]$;

b. h e' derivabile su (a,b) perche' f e' derivabile su (a,b);

c.

$$h(a) = 0 = h(b)$$

Passo 3. Si applichi a h la conclusione del Teorema di Rolle:

$$\exists\, x_0 \in (a,b) \text{ tale che } h'(x_0) = 0$$

Poiche'

$$h'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a}$$

il risultato voluto segue.

(Si verifichi che si ottiene la dimostrazione anche con $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$ ■

Remark 183 *Talvolta, puo' essere utile scrivere la conclusione del Teorema di Lagrange nel seguente (equivalente) modo:*

$$\exists \theta \in (0,1) \text{ tale che } f'(a+\theta(b-a)) = \tfrac{f(b)-f(a)}{b-a}$$

Usando il Teorema di Rolle, possiamo anche dimostrare il seguente Teorema.

Theorem 184 ***di Lagrange generalizzato.*** *Si consideri* $f, g : [a,b] \to \mathbb{R}$.

$$\left\langle \begin{array}{l} 1.\ f,g \text{ sono continue su } [a,b], \\ 2.\ f,g \text{ sono derivabili su } (a,b) \end{array} \right\rangle \Rightarrow \left\langle \begin{array}{l} \exists\ x_0 \in (a,b) \text{ tale che} \\ f'(x_0)[g(b)-g(a)] = g'(x_0)[f(b)-f(a)] \end{array} \right\rangle$$

Proof. Si applichi il Teorema di Rolle alla funzione

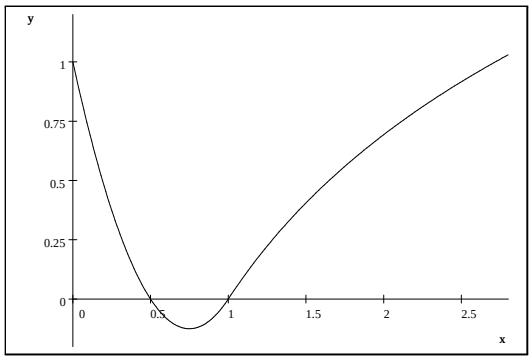
$$h : [a,b] \to \mathbb{R}, \qquad h : x \longmapsto f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)]$$

■

Exercise 185 *Si dica se la funzione*

$$f : [0,e] \to \mathbb{R}, \qquad f : x \longmapsto \begin{cases} 2x^2-3x+1 & se \quad x < 1 \\ \log x & se \quad x \geq 1 \end{cases}$$

soddisfa le ipotesi del Teorema di Rolle.
Soluzione.



f *e' chiaramente derivabile e continua in* $[0,e] \setminus \{1\}$.
$\lim_{x\to 1}\left(2x^2-3x+1\right) = 0$;
$\lim_{x\to 1} \ln x = 0$; $f(0) = 0$.
Dunque la funzione e' continua in 1.
$\lim_{h\to 0} \frac{2(1+h)^2-3(1+h)+1-0}{h} = \lim_{h\to 0} \frac{2+4h+2h^2-3-3h+1}{h} = \lim_{h\to 0} \frac{2h^2+h}{h} = 1$;
$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1$, *e dunque la funzione e' derivabile in* 1.
Infine, $f(0) = f(e) = 1$.

The following proposition provides an easier way of computing derivatives in situations as the one presented in the above exercise.

Proposition 186 *If*
1. $f : S \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable in* $[a,b] \setminus \{x_0\} \subseteq S$,
2. it iscontinuous in x_0, *and*
3. $\lim_{x\to x_0} f'(x) = l \in \mathbb{R}$, *i.e., the limit exists and it is finite,*
then f *is differentiable in* x_0 *and* $f'(x_0) = l$.

Proof. See, for example, Marcellini e Sbordone (2002), page 408. ■

Useremo il Teorema di Lagrange generalizzato per dimostrare due importanti Teoremi: il primo Teorema di de l'Hopital (nel paragrafo successivo) e il Teorema di Taylor (nel paragrafo 6.4).

6.3 Teoremi di de l'Hopital

Theorem 187 ***(Primo Teorema di de l'Hopital)***
Si considerino due funzioni $f : S \subseteq \mathbb{R} \to \mathbb{R}$, $f : x \longmapsto f(x)$ *and* $g : T \subseteq \mathbb{R} \to \mathbb{R}$, $g : x \longmapsto g(x)$ *tali che* $\exists \delta > 0$ *tale che*

1. f *e* g *sono derivabili in* $B(x_0, \delta) \setminus \{x_0\}$ *e continue in* $B(x_0, \delta)$;
2. $f(x_0) = g(x_0) = 0$;
3. $\forall x \in B(x_0, \delta) \setminus \{x_0\}$, $g(x) \neq 0$ *e* $g'(x) \neq 0$;
4. $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ *esiste (finito o infinito).*

Allora

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Proof. Strategia: Si applichi il Teorema di Lagrange generalizzato a f, g su $[x_0, \overline{x}] \subseteq B(x_0, \delta)$ per dimostrare che $\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = \lim_{x \to x_0^+} \frac{f'(x)}{g'(x)}$. La dimostrazione per il limite sinistro e' strettamente analoga.

Si consideri $[x_0, \overline{x}] \subseteq B(x_0, \delta)$ e si osservi che f e g rispettano le ipotesi del Teorema di Lagrange generalizzato. Dunque

$$\exists\, \xi \in (x_0, \overline{x}) \text{ tale che } f'(\xi)\,[g(\overline{x}) - g(x_0)] = g'(\xi)\,[f(\overline{x}) - f(x_0)]$$

ovvero, poiche' per ipotesi $f(x_0) = g(x_0) = 0$,

$$\exists\, \xi \in (x_0, \overline{x}) \text{ tale che } f'(\xi)\, g(\overline{x}) = g'(\xi)\, f(\overline{x})$$

and

$$\exists\, \xi \in (x_0, \overline{x}) \text{ tale che } \frac{f(\overline{x})}{g(\overline{x})} = \frac{f'(\xi)}{g'(\xi)}$$

Si prenda il limite per $\overline{x} \to x_0$ di entrambi i lati della uguaglianza:

$$\lim_{\overline{x} \to x_0} \frac{f(\overline{x})}{g(\overline{x})} = \lim_{\overline{x} \to x_0} \frac{f'(\xi)}{g'(\xi)} \tag{6.5}$$

Poiche' $\xi \in (x_0, \overline{x})$,

$$\lim_{\overline{x} \to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)} \tag{6.6}$$

Da (6.5) e (6.6) il risultato voluto segue. ■

Remark 188 *Il teorema vale anche nel caso in cui limite e' calcolato per* $x \to \pm\infty$.

Example 189 *(da Spivak) Si calcoli*

$$\lim_{x \to 0} \frac{x - \tan x}{x - \sin x} = \text{“}\, \frac{0}{0} \,\text{”}$$

Definite $f(x) := x - \tan x$ *e* $g(x) := x - \sin x$, *si ha*

$$\frac{f'(x)}{g'(x)} = \frac{1 - \frac{1}{\cos^2 x}}{1 - \cos x} = \frac{\cos^2 x - 1}{(1 - \cos x)\cos^2 x} = -\frac{(1 - \cos x)(1 + \cos x)}{(1 - \cos x)\cos^2 x} = -\frac{(1 + \cos x)}{\cos^2 x}$$

e

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} -\frac{(1 + \cos x)}{\cos^2 x} = -2$$

In conclusione

$$\lim_{x \to 0} \frac{x - \tan x}{x - \sin x} = -2$$

Remark 190 *Dopo aver calcolato il rapporto* $\frac{f'(x)}{g'(x)}$ *e' opportuno semplificare la espressione ottenuta, se possibile.*

Example 191 *(da Spivak) Dato $a \in \mathbb{R}$, si calcoli*

$$\lim_{x\to 1} \frac{x^a - ax + a - 1}{\left(x^2-1\right)^2}$$

Definite $f(x) := x^a - ax + a - 1$ e $g(x) := \left(x^2-1\right)^2$, si ha

$$\frac{f'(x)}{g'(x)} = \frac{ax^{a-1} - a}{4x\left(x^2-1\right)}$$

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \text{“}\frac{0}{0}\text{”}$$

$$\frac{f''(x)}{g''(x)} = \frac{a(a-1)x^{a-2}}{4\left(3x^2-1\right)}$$

$$\lim_{x\to 1} \frac{f''(x)}{g''(x)} = \frac{a(a-1)}{8}$$

Remark 192 *Il Primo Teorema di de l'Hopital vale anche nel caso in cui $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} f(x) = \pm\infty$, con le oppurtune modifiche. Questo risultato e' in genere noto come Secondo Teorema di de l'Hopital.*

Theorem 193 ***(Secondo Teorema di de l'Hopital)***
Si considerino due funzioni $f : S \subseteq \mathbb{R} \to \mathbb{R}$, $f : x \longmapsto f(x)$ and $g : T \subseteq \mathbb{R} \to \mathbb{R}$, $g : x \longmapsto g(x)$ tali che $\exists \delta > 0$ tale che

1. *f e g sono derivabili in un intorno $B(x_0, \delta) \setminus \{x_0\}$;*
2. *$\lim_{x\to x_0} f(x) = \pm\infty$ e $\lim_{x\to x_0} g(x) = \pm\infty$;*
3. *$\forall x \in B(x_0, \delta) \setminus \{x_0\}$, $g(x) \neq 0$ e $g'(x) \neq 0$;*
4. *$\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ esiste (finito o infinito).*

Allora

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

Proof. Omessa. ■

Remark 194 *Il teorema vale anche nel caso in cui limite e' calcolato per $x \to \pm\infty$.*

Example 195 *(da Spivak)Si calcoli*

$$L := \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \text{“}\frac{0}{0}\text{”}$$

Definite $f(x) := e^{-\frac{1}{x}}$ e $g(x) := x$, si ha

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{x^2}e^{-\frac{1}{x}}}{1} = \frac{e^{-\frac{1}{x}}}{x^2}$$

che e' ancora una forma indeterminata. Un ulteriore tentativo di applicazione del (Primo) Teorema di de l'Hopital non e' conclusiva. In effetti,

$$\frac{f''(x)}{g''(x)} = \frac{e^{-\frac{1}{x}}}{2x^3}$$

Occorre dunque procedere in modo diverso. Si ponga $y := \frac{1}{x}$, e si osservi che se $x \to 0^+$, allora $y \to +\infty$. Dunque

$$L = \lim_{y\to+\infty} \frac{e^{-y}}{\frac{1}{y}} = \lim_{y\to+\infty} \frac{y}{e^y} = \lim_{y\to+\infty} \frac{1}{e^y} = 0 \tag{6.7}$$

Remark 196 *Nella espressione (6.7) e in quelle che seguono, conveniamo che l'esistenza di ogni limite implica l'esistenza di quello precedente e anche la loro uguaglianza.*

Due errori che possono essere commessi applicando i Teoremi di de l'Hopital in modo improprio sono illustrati nei seguenti esempi.

Example 197 *(da Spivak) Il seguente calcolo e' **errato***

$$\lim_{x\to 1}\frac{3x^2-2x-1}{x^2-x}=\lim_{x\to 1}\frac{6x-2}{2x-1}=\lim_{x\to 1}\frac{6}{2}=3$$

Il primo passaggio e' corretto, ma il secondo non lo e'. $\frac{6x-2}{2x-1}$ ***non*** *e' una forma indeterminata per* $x\to 1$*. Il risultato corretto ,* 4*, si ottiene semplicemente sostituendo* 1 *al posto di* x *nella espressione* $\frac{6x-2}{2x-1}$*.*

Example 198 *(da Spivak) Il seguente calcolo e' **errato***

$$\lim_{x\to+\infty}\frac{x+\sin x}{x}=\lim_{x\to+\infty}\frac{1+\cos x}{1}\ \textit{che non esiste}$$

*La ipotesi 4 del Secondo Teorema di de l'Hopital (cosi' come del Primo Teorema) consentono di scrivere l'uguaglianza contenuta nella conclusione del Teorema stesso solo se il limite del rapporto tra derivate **esiste**. Il corretto calcolo del limite in questione e' il seguente.*

$$\lim_{x\to+\infty}\frac{x+\sin x}{x}=\lim_{x\to+\infty}1+\frac{\sin x}{x}=1$$

(Abbiamo usato il fatto che $-\frac{1}{x}\leq\frac{\sin x}{x}\leq+\frac{1}{x}$*).*

Oltre alle forme "$\frac{0}{0}$" e "$\frac{\infty}{\infty}$", i Teoremi di de l'Hopital sono utili per calcolare limiti anche nelle forme indeterminate del tipo "$0\cdot\infty$", "0^0", "∞^0", "1^∞". Il trucco da usare consiste semplicemente nel ricondurre questo secondo insieme di forme indeterminate a una di quello studiate nei due teoremi visti.

Si considerino funzioni tali che

$$\begin{array}{c}\lim_{x\to x_0}f(x)=0\\ \lim_{x\to x_0}g(x)=\infty\\ \lim_{x\to x_0}h(x)=0\\ \lim_{x\to x_0}l(x)=1\end{array}$$

Dunque

1.

$$\lim_{x\to x_0}f(x)\,g(x)=\text{“ }0\cdot\infty\text{ ”}$$

e

$$\lim_{x\to x_0}f(x)\,g(x)=\lim_{x\to x_0}\frac{f(x)}{\frac{1}{g(x)}}=\text{“ }\frac{0}{0}\text{ ”}$$

Si osservi che in generale

$$f(x)^{\phi(x)}=e^{\log f(x)^{\phi(x)}}=e^{\phi(x)\log f(x)}$$

2.

$$\lim_{x\to x_0}f(x)^{h(x)}=\text{“ }0^0\text{ ”}$$

e

$$\lim_{x\to x_0}f(x)^{h(x)}=\lim_{x\to x_0}e^{h(x)\log f(x)}=e^{\text{“ }0\cdot\infty\text{ ”}}$$

3.

$$\lim_{x\to x_0}g(x)^{f(x)}=\text{“}\infty^0\text{”}$$

e

$$\lim_{x\to x_0} g(x)^{f(x)} = \lim_{x\to x_0} e^{f(x)\log g(x)} = e^{\text{“ } 0\cdot\infty \text{ ”}}$$

3.

$$\lim_{x\to x_0} l(x)^{g(x)} = \text{“}1^{\infty}\text{”}$$

e

$$\lim_{x\to x_0} l(x)^{g(x)} = \lim_{x\to x_0} e^{g(x)\log l(x)} = e^{\text{“ } \infty\cdot 0 \text{ ”}}$$

Example 199 *Si calcoli*

$$L := \lim_{x\to 0^+} x^x$$

$L = \lim_{x\to 0^+} e^{x\log x}$.
$L_1 := \lim_{x\to 0^+} x\log x = \text{“ } 0\cdot(+\infty) \text{ ”}$.
$L_1 = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$.
In conclusione

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

Alcuni esercizi con svolgimento
Si calcolino i seguenti limiti:
1. $\lim_{x\to 0} \frac{\sin x^2+1-\cos x}{x^2}$;
2. $\lim_{x\to 0} \frac{2\cos x+e^{x^2}-3}{x^2+x}$;
3. $\lim_{x\to 2} \frac{\log\left(x^2-4x+5\right)}{\sin\frac{\pi}{2}x}$;
4. $\lim_{x\to 0} \frac{x^2\sin\frac{1}{x}}{\sin x}$.

Soluzioni.
Cerchiamo di applicare i Teoremi di de l'Hopital.
1.

$$\lim_{x\to 0} \frac{\sin x^2+1-\cos x}{x^2} = \text{“ } \frac{0}{0} \text{ ”}$$

$$\lim_{x\to 0} \frac{2x\cos x^2+\sin x}{2x} = \text{“ } \frac{0}{0} \text{ ”}$$

$$\lim_{x\to 0} \frac{2\cos x^2-4x^2\sin x^2+\cos x}{2} = \frac{3}{2}$$

2.

$$\lim_{x\to 0} \frac{2\cos x+e^{x^2}-3}{x^2+x} = \text{“ } \frac{2+1-3}{0} \text{ ”}$$

$$\lim_{x\to 0} \frac{-2\sin x+2xe^{x^2}}{2x+1} = 0$$

3.

$$\lim_{x\to 2} \frac{\log\left(x^2-4x+5\right)}{\sin\frac{\pi}{2}x} = \text{“ } \frac{0}{0} \text{ ”}$$

$$\lim_{x\to 2} \frac{2x-4}{x^2-4x+5}\frac{1}{\frac{\pi}{2}\cos\frac{\pi}{2}x} = 0$$

4.

$$\lim_{x\to 0} \frac{x^2\sin\frac{1}{x}}{\sin x} = \text{“ } \frac{0}{0} \text{ ”}$$

$$\lim_{x\to 0} \frac{2x^2\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x} \text{ non esiste}$$

Dunque non possiamo applicare il teorema. In effetti,

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim_{x\to 0}\frac{x\sin\frac{1}{x}}{\frac{\sin x}{x}}=0$$

6.4 Teoremi di Taylor

Premessa sul concetto di infinitesimo.

Definition 200 *La funzione $f:S\subseteq\mathbb{R}\to\mathbb{R}$ e' un infinitesimo per x che tende a $x_0\in D(S)$ (rispettivamente, a $\pm\infty$, assumendo S superiormente o inferiormente illimitato) se*

$$\lim_{x\to x_0}f(x)=0$$

(rispettivamente, $\lim_{x\to\pm\infty}f(x)=0$).

Definition 201 *Si considerino due funzioni f,g infinitesime per $x\to x_0$; sia inoltre $g(x)\neq 0$ in un intorno di x_0 escluso x_0. Si possono distinguere i seguenti casi.*

1. *se $\lim_{x\to x_0}\frac{f(x)}{g(x)}=0$, allora si dice che f e' un infinitesimo di ordine superiore rispetto a g (per $x\to x_0$);*
2. *se $\lim_{x\to x_0}\frac{f(x)}{g(x)}=k\neq 0$, allora si dice che f e' un infinitesimo dello stesso ordine di g;*
3. *se $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\pm\infty$, allora si dice che f e' un infinitesimo di ordine inferiore rispetto a g;*
4. *se $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ non esiste, allora si dice che f e g sono infinitesimi non confrontabile.*

Definition 202 *Se f e' un infinitesimo di ordine superiore rispetto a g, si dice anche che f e' un "o piccolo"di g e si scrive*

$$f(x)=o(g(x))\qquad\qquad(\text{per } x\to x_0)$$

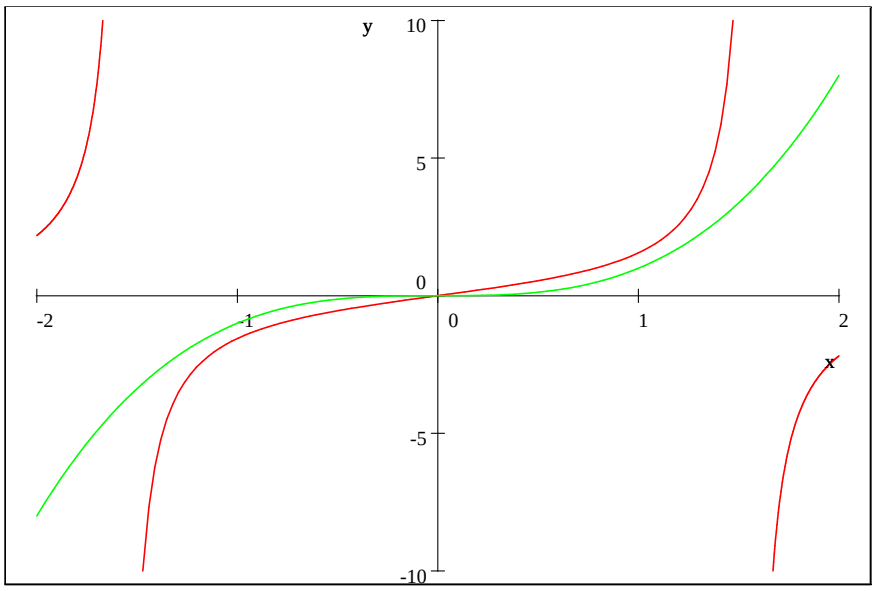
Example 203 *Si prenda $x_0=0$.*

1. *$f(x)=x^2$ e $g(x)=x$;*
2. *$f(x)=2\sin x$ e $g(x)=x$;*
3. *$f(x)=x^2$ e $g(x)=\sin x$;*
4. *$f(x)=x\sin\frac{1}{x}$ e $g(x)=x$.*

Remark 204 *Se f e' un infinitesimo di ordine superiore a g (per $x\to x_0$) possiamo dire che quando x si avvicina a x_0, f si avvicina a zero piu' velocemente di g.*

Example 205

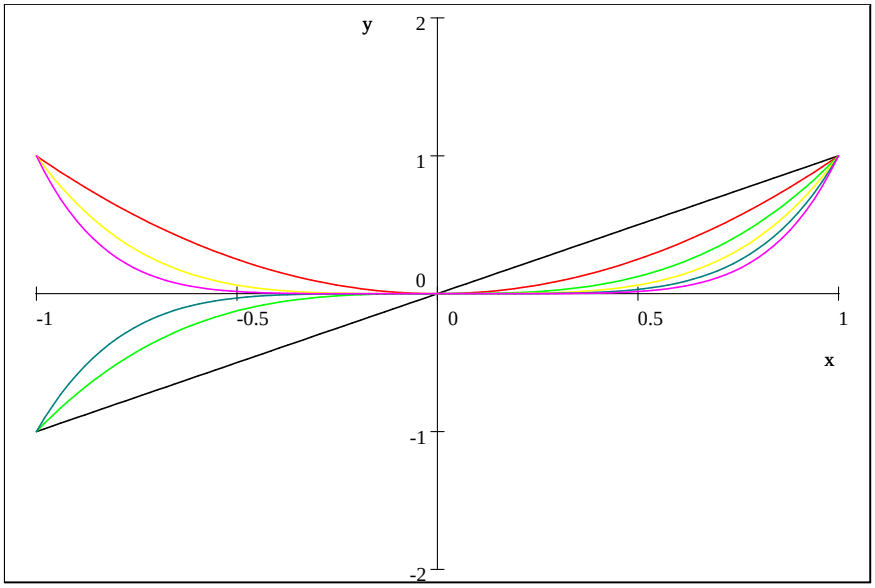
$$\lim_{x\to 0}\frac{x^3}{\tan x}=\lim_{x\to 0}\frac{x^2\cos x}{\frac{\sin x}{x}}=0$$



Example 206 *Per qualsiasi* $n \in \mathbb{N}\setminus\{0\}$,

$$\lim_{x\to 0} \frac{(x-x_0)^n}{(x-x_0)^{n-1}} = 0$$

Dunque tanto piu' n e' grande, tanto piu' $(x-x_0)^n$ *si avvicina velocemente a* x_0.



Fine della Premessa sul concetto di infinitesimo.

Come abbiamo visto, se f e' derivabile in x_0, allora f puo' essere approssimata dalla retta tangente al grafico nel punto $(x_0, f(x_0))$, in un intorno di x_0 sufficientemente piccolo. In altre parole, "vicino" a x_0

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

In maniera rigorosa, se f e' derivabile in x_0, allora

$$\lim_{x\to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x-x_0)]}{x-x_0} = 0$$

ovvero

$$f(x) - [f(x_0) + f'(x_0)(x-x_0)]$$

e' un infinitesimo di ordine superiore a $(x-x_0)^n$ con $n = 1$.

E' possibile ottenere una "migliore approssimazione" di f in un intorno di x_0?

In altre parole, e' possibile trovare una espressione $T_{(f,x_0,n)}(x)$tale che

$$\lim_{x\to x_0} \frac{f(x) - T_{(f,x_0,n)}(x)}{(x-x_0)^n} = 0 \quad ?$$

I Teoremi di Taylor presentano condizioni sotto le quali e' possibile rispondere affermativamente a questa domanda.

Un corollario di quei teoremi permettera' anche di trovare un metodo relativamente generale per la determinazione di punti di estremo locale.

Definition 207 $f^{(0)} := f$.

Theorem 208 ***(di Taylor con resto nella forma di Peano)***
Given $f : S \subseteq \mathbb{R} \to \mathbb{R}$, if $\exists \delta > 0$ such that f is differentiable at least n times in $(x_0 - \delta, x_0 + \delta)$, then $\forall x \in (x_0 - \delta, x_0 + \delta)$

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!} + R_n(x) \tag{6.8}$$

where

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = 0,$$

i.e., $R_n(x)$ e' un infinitesimo di ordine superiore a $(x - x_0)^n$, o anche $R_n(x) = o((x - x_0)^n)$.

Definition 209 *$\sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x-x_0)^k}{k!}$ si indica con $T(x, f, x_0, n)$ o, piu' semplicemente, con $T_n(x)$, e si dice Polinomio di Taylor di grado n in x_0.*

La dimostrazione del Teorema e' preceduta da due Lemmi.

Lemma 210 *Se $G(x) \equiv (x - x_0)^n$, allora per $\forall i \in \{1, ..., n\}$*

$$G^{(i)}(x) = \frac{n!}{(n-i)!} (x - x_0)^{n-i}$$

Proof. Si definisca

$$g(x, i) \equiv \frac{n!}{(n-i)!} (x - x_0)^{n-i}$$

La dimostrazione procede per induzione su i.
1. $G'(x) = n(x - x_0)^{n-1}$;
$g(x, 1) = \frac{n!}{(n-1)!}(x - x_0)^{n-1} = n(x - x_0)^{n-1}$.
2. $G^{(i+1)} = \left(G^{(i)}\right)'(x) = \frac{d}{dx}\left(\frac{n!}{(n-i)!}(x - x_0)^{n-i}\right) = \frac{n!(n-i)}{(n-i)!}(x - x_0)^{n-i-1} = \frac{n!}{(n-(i+1))!}(x - x_0)^{n-(i+1)}$.
$g(x, i+1) \equiv \frac{n!}{(n-(i+1))!}(x - x_0)^{n-(i+1)}$. ■

Lemma 211 *Se $F(x) \equiv \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x-x_0)^k}{k!}$, allora per $\forall i \in \{1, ..., n\}$*

$$F^{(i)}(x) = \sum_{k=i}^{n} f^{(k)}(x_0) \frac{(x - x_0)^{k-i}}{(k-i)!}$$

Proof. Si definisca

$$f(x, i) \equiv \sum_{k=i}^{n} f^{(k)}(x_0) \frac{(x - x_0)^{k-i}}{(k-i)!}$$

Per definizione

$$F(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2} + ... + f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

La dimostrazione procede per induzione su i.
1. $F'(x) = f'(x_0) + f''(x_0)(x - x_0) + ... + f^{(n)}(x_0) \frac{(x-x_0)^{n-1}}{(n-1)!}$.
$f(x, 1) = \sum_{k=1}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k-1}}{(k-1)!}$.
2. $F^{(i+1)} = \left(F^{(i)}\right)'(x) = \frac{d}{dx}\left(\sum_{k=i}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k-i}}{(k-i)!}\right) =$
$= \frac{d}{dx}\left(f^{(i)}(x_0) \frac{(x-x_0)^{i-i}}{(i-i)!} + f^{(i+1)}(x_0) \frac{(x-x_0)^{(i+1)-i}}{((i+1)-i)!} + ... + f^{(n)}(x_0) \frac{(x-x_0)^{n-i}}{(n-i)!}\right) =$
$= \frac{d}{dx}\left(f^{(i)}(x_0) + f^{(i+1)}(x_0)(x - x_0) + ... + f^{(n)}(x_0) \frac{(x-x_0)^{n-i}}{(n-i)!}\right) =$
$= f^{(i+1)}(x_0) + ... + f^{(n)}(x_0)(n - i) \frac{(x-x_0)^{n-i-1}}{(n-i)!} =$
$= f^{(i+1)}(x_0) + ... + f^{(n)}(x_0) \frac{(x-x_0)^{n-(i+1)}}{(n-(i+1))!}$.

$f(x, i+1) \equiv \sum_{k=i+1}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k-i}}{(k-i)!}$. ■

Proof. del Teorema di Taylor.

Strategia: si applichi $n-1$ volte il Teorema di de l'Hopital.

Vogliamo dimostrare che

$$\lim_{x \to x_0} \frac{f(x) - \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x-x_0)^k}{k!}}{(x-x_0)^n} = 0$$

Possiamo applicare $(n-1)$ volte il teorema di de l'Hopital. Usando i Lemmi precedenti possiamo calcolare la derivata $(n-1)$-esima di $R(x)$ e di $(x-x_0)^n$. Si ha dunque

$$\lim_{x \to x_0} \frac{R(x)}{(x-x_0)^n} = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - F^{(n-1)}(x)}{G^{(n-1)}(x)} =$$

$$= \lim_{x \to x_0} \frac{f^{(n-1)}(x) - \sum_{k=n-1}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k-(n-1)}}{(k-(n-1))!}}{\frac{n!}{(n-(n-1))!}(x-x_0)^{n-(n-1)}} =$$

$$= \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) \frac{(x-x_0)^{(n-1)-(n-1)}}{((n-1)-(n-1))!} - f^{(n)}(x_0) \frac{(x-x_0)^{n-(n-1)}}{(n-(n-1))!}}{n!(x-x_0)} =$$

$$= \frac{1}{n!} \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{(x-x_0)} =$$

$$= \frac{1}{n!} \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{(x-x_0)} =$$

$$= \frac{1}{n!} \left[\lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{(x-x_0)} - f^{(n)}(x_0) \right] = 0$$

■

Remark 212 *La espressione* (6.8) *nell'enunciato del teorema di Taylor con resto nella forma di Peano, prende il nome di formula di Taylor (con resto nella forma di Peano). Se* $x_0 = 0$*, allora la formula prende il nome di formula di McLaurin.*

E' possibile verificare la validita' delle seguenti uguaglianze.

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o\left(x^{2n+1}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+1}\right)$$

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2n)!} + o\left(x^{2n+1}\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o\left(x^8\right)$$

$$\arctan x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o\left(x^{2n+1}\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + (-1)^n \frac{x^{2n+1}}{2n+1} + o\left(x^{2n+1}\right)$$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + o(x^n)$$

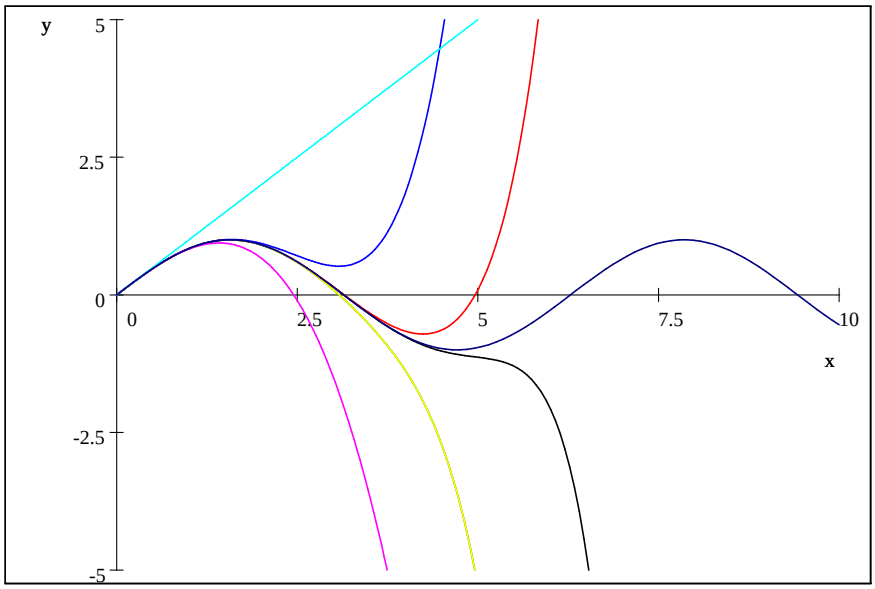
$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + o(x^n) = 1 + x + x^2 + x^3 + ... + x^n + o(x^n)$$

$$\log(1+x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + o(x^n)$$

Le formule di Taylor e McLaurin esprimono una funzione come somma di un polinomio di grado n (il polinomio di Taylor) e un termine (detto resto) che puo' "ritenersi trascurabile", nel senso che e' un infinitesimo di ordine superiore a $(x-x_0)^n$, e quindi tende a zero "piu rapidamente" di

$(x-x_0)^n$. Le formule di Taylor e McLaurin propongono dunque una formula di approssimazione locale tanto migliore quanto piu' alto e' il grado n del polinomio "approssimante".

Nella figura, sono proposti il grafico della funzione sin e di alcuni polinomino approssimanti.



Theorem 213 ***(di Taylor con resto nella forma di Lagrange)***

Se f e' derivabile almeno $n+1$ volte in un intorno $(x_0-\delta, x_0+\delta)$ di x_0 con derivata $f^{(n+1)}$ continua,

allora $\forall x \in (x_0-\delta, x_0+\delta)$, $\exists \xi \in (x_0, x)$ o $\xi \in (x, x_0)$ tae che

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x-x_0)^k}{k!} + f^{(n+1)}(\xi) \frac{(x-x_0)^{n+1}}{(n+1)!} \tag{6.9}$$

Proof. Omessa. ■

Remark 214 *La espressione $f^{(n+1)}(\xi) \frac{(x-x_0)^{n+1}}{(n+1)!}$ e' detta resto nella forma di Lagrange. Si osservi che*

$$\lim_{x\to x_0} \frac{f^{(n+1)}(\xi) \frac{(x-x_0)^{n+1}}{(n+1)!}}{(x-x_0)^n} = \lim_{x\to x_0} \left(\frac{f^{(n+1)}(\xi)}{(n+1)!} \frac{(x-x_0)^{n+1}}{(x-x_0)^n} \right) = 0$$

ovvero il resto e' un infinitesimo di ordine superiore a $(x-x_0)^n$.

Example 215 *(Come usare il Teorema di Taylor con resto nella forma di Lagrange per approssimazioni numeriche)*

Si calcoli e con una errore in valore assoluto minore di 10^{-3}.

1. *La funzione da usare e' $f : \mathbb{R} \to \mathbb{R}$, $f : x \longmapsto e^x$.*
2. *Si identifichi x con 1 e x_0 con 0 e si usi la formula di McLaurin. Si ottiene dunque*

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^{\xi}}{(n+1)!}$$

Vogliare ora determinare n in modo che $\forall \xi \in (0,1)$

$$\left| \frac{e^{\xi}}{(n+1)!} \right| = \frac{e^{\xi}}{(n+1)!} < \frac{1}{1000}$$

3. *Si osservi che $\xi \in (0,1)$ e che, poiche' la funzione esponenziale e' strettamente crescente*

$$1 = e^0 < e^{\xi} < e^1 < 3$$

Quindi

$$\frac{1}{(n+1)!} < \frac{e^{\xi}}{(n+1)!} < \frac{3}{(n+1)!}$$

E se

$$\frac{3}{(n+1)!} < \frac{1}{1000}$$

ovvero $(n+1)! > 3000$*, a maggior ragione, l'errore e' minore di* 10^{-3}*. Occorre dunque trovare* n *in modo che la equazione sia risolta. E' sufficiente consultare la seguente tabella.*

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40 320	362 880	3628 800

Dunque occorre scegliere $(n+1) = 7$ *e il valore cercato e'*

$$\sum_{k=0}^{7-1} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} = \frac{1957}{720} = 2.718\,1$$

Remark 216 *Un passo fondamentale nella procedura illustrata sopra consiste nella cosiddetta maggiorazione dell'errore - vedi passo 3. In generale, l'idea e' trovare* $a > 0$ *tale che* $\forall \xi \in (x_0, x)$ *risulti*

$$\left| f^{(n+1)}(\xi) \right| < a$$

ovvero

$$-a \leq f^{(n+1)}(\xi) \leq a$$

da cui segue, essendo $x > x_0$*,*

$$-a\frac{(x-x_0)^{n+1}}{(n+1)!} \leq f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!} \leq a\frac{(x-x_0)^{n+1}}{(n+1)!}$$

dove il termine centrale e' proprio il resto nella forma di Lagrange.

Exercise 217 *Si calcoli* $\sin 2$ *con un errore in valore assoluto minore di un decimo.*

Soluzione.
Usando il Teorema di Tayor si ha

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!}$$

con $\xi \in (x, x_0)$ o $\xi \in (x_0, x)$.
Si ponga $f = \sin$, $x = 2$, $x_0 = 0$.
Si osservi anche che

$$f(x) = \sin x \quad f'(x) = \cos x \quad f''(x) = -\sin x$$

$$f'''(x) = -\cos x \quad f^{iv}(x) = \sin x \quad f^{v}(x) = \cos x$$

Quindi per qualsiasi $\forall k$, $\left| f^{(k)}(\xi) \right| \leq 1$. Dunque

$$\left| f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!} \right| = \left| f^{(n+1)}(\xi) \right| \left| \frac{2^{n+1}}{(n+1)!} \right| \leq \frac{2^{n+1}}{(n+1)!}$$

Dunque si deve trovare $n \in \mathbb{N}$ tale che

$$\frac{2^{n+1}}{(n+1)!} < \frac{1}{10}$$

Si osservi che

n		1	2	3	4	5
$\frac{2^{n+1}}{(n+1)!}$		$\frac{4}{2}$	$\frac{8}{6}$	$\frac{16}{24}$	$\frac{32}{120}$	$\frac{64}{720}$

Dunque $n = 5$.
Si osservi anche che

$$\begin{array}{lll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x \\ f(0) = 0 & f'(0) = 1 & f''(0) = 0 \end{array}$$

$$\begin{array}{lll} f'''(x) = -\cos x & f^{iv}(x) = \sin x & f^{v}(x) = \cos x \\ f'''(0) = -1 & f^{iv}(0) = 0 & f^{v}(0) = 1 \end{array}$$

In conclusione,

$$\sin 2 = 0 + 1 \cdot 2 + 0 + (-1)\frac{8}{6} + 0 + 1\frac{32}{120} = 2 - \frac{8}{6} + \frac{32}{120} = \frac{240 - 160 + 32}{120} = \frac{112}{120} = \frac{14}{15}$$

6.5 Derivate e Grafici di Funzioni - II

Theorem 218 *Sia $f : [a,b] \to \mathbb{R}$ continua su $[a,b]$ e derivabile su (a,b).*
a. $\forall x \in (a,b)$, $f'(x) > 0 \Rightarrow$ f e' strettamente crescente su $[a,b]$;
b. $\forall x \in (a,b)$, $f'(x) < 0 \Rightarrow$ f e' strettamente decrescente su $[a,b]$;
c. $\forall x \in (a,b)$, $f'(x) = 0 \Rightarrow$ f e' costante su $[a,b]$.

Proof. Strategia: si applichi il teorema di Lagrange a f su $[x,y] \subseteq [a,b]$.
Si applichi il teorema di Lagrange a f su $[x,y] \subseteq [a,b]$. Si ha dunque che esiste $x_0 \in (x,y)$ tale che

$$f'(x_0) = \frac{f(y) - f(x)}{y - x}$$

ovvero

$$f(y) - f(x) = f'(x_0)(y - x) \tag{6.10}$$

Da (6.10), le conclusioni desiderate seguono immediatamente. ■

Remark 219 *Le implicazioni inverse contenute nelle affermazioni a. e b. del Teorema 218 sono false.*
a. Si consideri $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^3$. f e' strettamente crescente, ma $f'(0) = 0$.
b. Si consideri $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto -x^3$. f e' strettamente decrescente, ma $f'(0) = 0$.
La implicazione inversa contenuta nella affermazione c. del Teorema 218 e' vera. Nella Proposizione 147, abbiamo visto che se una funzione e' costante, allora ha derivata nulla.

Una versione indebolita del Teorema 218 consente di avere entrambe le implicazioni.

Theorem 220 *Sia $f : [a,b] \to \mathbb{R}$ continua su $[a,b]$ e derivabile su (a,b).*
a. $\forall x \in (a,b)$, $f'(x) \geq 0 \Leftrightarrow$ f e' crescente su $[a,b]$;
b. $\forall x \in (a,b)$, $f'(x) \leq 0 \Leftrightarrow$ f e' decrescente su $[a,b]$.

Proof. Per brevita' presentiamo la dimostrazione solo della affermazione a.
[$\Rightarrow$] La dimostrazione e' strettamente analoga alla dimostrazione del Teorema 218.
[$\Leftarrow$] Strategia: osserva che se $x \geq x_0$, allora $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$.
Si prenda un arbitrario $x_0 \in (a,b)$. Si consideri $x \in (a,b)$ e $x > x_0$. Poiche' f e' crescente

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Prendendo $\lim_{x \to x_0^+}$ del membro di sinistra, usando il fatto che f e'derivabile e il Lemma 175 si ha che

$$f'(x_0) \geq 0$$

■

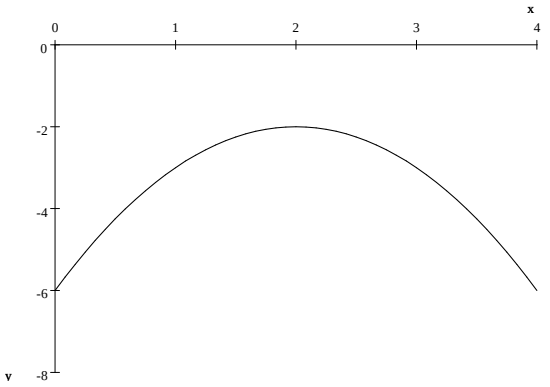
Riassumendo, in maniera schematica si ha quanto segue. Data una funzione f derivabile,

$$\langle f' > 0 \rangle \overset{\Rightarrow}{\underset{\not\Leftarrow}{}} \langle f \text{ strettamente crescente} \rangle \overset{\Rightarrow}{\underset{\not\Leftarrow}{}} \langle f \text{ crescente} \rangle \overset{\Rightarrow}{\underset{\Leftarrow}{}} \langle f' \geq 0 \rangle$$

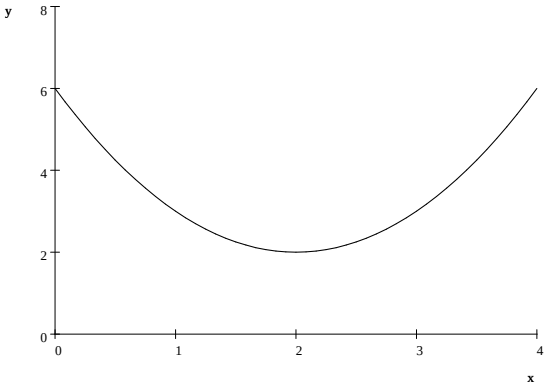
Dai risultati visti si ha il seguente teorema.

Theorem 221 *Si consideri $f : S \subseteq \mathbb{R} \to \mathbb{R}$. Si assuma che $\exists \delta > 0$ tale che $[x_0 - \delta, x_0 + \delta] \subseteq S$ e f e' continua su $[x_0 - \delta, x_0 + \delta]$ e derivabile su $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.*

a. Se $\forall x \in (x_0 - \delta, x_0)$ risulta $f'(x) \geq 0$ e $\forall x \in (x_0, x_0 + \delta)$ risulta $f'(x) \leq 0$, allora x_0 e' un punto di massimo locale per f;



b. Se $\forall x \in (x_0 - \delta, x_0)$ risulta $f'(x) \leq 0$ e $\forall x \in (x_0, x_0 + \delta)$ risulta $f'(x) \geq 0$, allora x_0 e' un punto di minimo locale per f.



Proof. Presentiamo la dimostrazione del solo caso a.

La funzione $f_{|[x_0-\delta,x_0]}$ soddisfa le ipotesi del Teorema 220, parte a.; dunque e' crescente su $[x_0 - \delta, x_0]$ e quindi

$$\forall x \in [x_0 - \delta, x_0], \quad x \leq x_0 \text{ and therefore } f(x_0) \geq f(x)$$

La funzione $f_{|[x_0,x_0+\delta]}$ soddisfa le ipotesi del Teorema 220, parte b.; dunque e' decrescente su $[x_0, x_0 + \delta]$ e quindi

$$\forall x \in [x_0, x_0 + \delta], \quad x \geq x_0 \text{ and therefore } \quad f(x_0) \geq f(x)$$

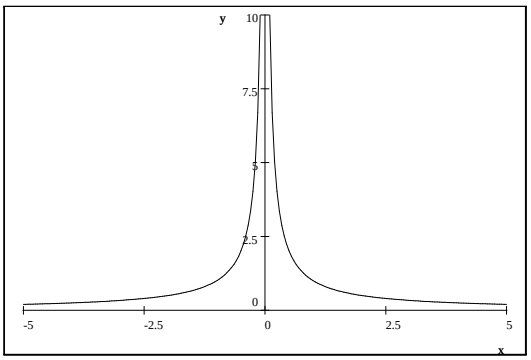
■

Example 222 *1. $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^2$ ha un punto di minimo in zero.*

2. $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto |x|$ ha un punto di minimo in zero.

I seguenti esempi sottolineano l'importanza della ipotesi di continuita' del precedente teorema.

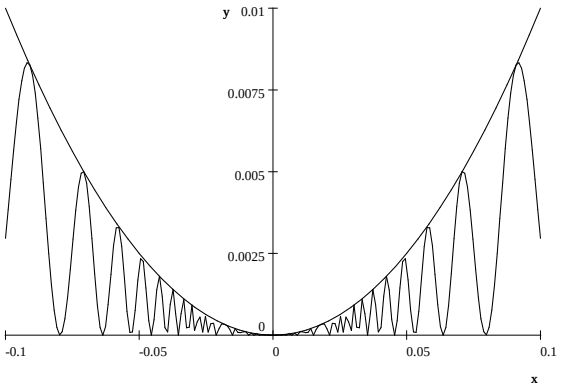
3 . $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f : x \mapsto \left|\frac{1}{x}\right|$ non ha un punto di minimo in zero



4 . $f:\mathbb{R}\to\mathbb{R},\ f:x\mapsto\begin{cases}|x| & se \quad x\neq 0\\ 1 & se \quad x=0\end{cases}$ *non ha un punto di minimo in zero.*

Remark 223 *Nel precedente teorema non valgono le implicazioni inverse delle affermazioni a. o b. La seguente funzione e' il controesempio cercato relativo alla implicazione inversa della affermazione b. .*

$$f:\mathbb{R}\to\mathbb{R},\ f:x\mapsto\begin{cases}x^2\sin^2\frac{1}{x} & se \quad x\neq 0\\ \\ 0 & se \quad x=0\end{cases}$$



Poiche' per qualsiasi $x\in\mathbb{R}$, $f(x)\geq 0$, *e* $f(0)=0$, *si ha che* 0 *e' un punto di minimo globale e dunque locale. D'altra parte, per* $x\neq 0$, $f'(x)=\frac{dx^2\sin^2\frac{1}{x}}{dx}=2x\sin^2\frac{1}{x}+x^2\left(-2\left(\sin\frac{1}{x}\right)\frac{\cos\frac{1}{x}}{x^2}\right)=2x\sin^2\frac{1}{x}-2\sin\frac{1}{x}\cos\frac{1}{x}$ *assume sia valori positivi che negativi in ogni intorno di* 0.

Dal Teorema precedente si ottiene la seguente regola.

Per determinare i punti di estremo locale appartenenti all'Interno del dominio di una funzione derivabile, occorre studiare la disequazione

$$f'(x)\geq 0$$

I punti per cui in un intorno sinistro $f'(x)\geq 0$ e in un intorno destro $f'(x)\leq 0$ sono punti di massimo locale;

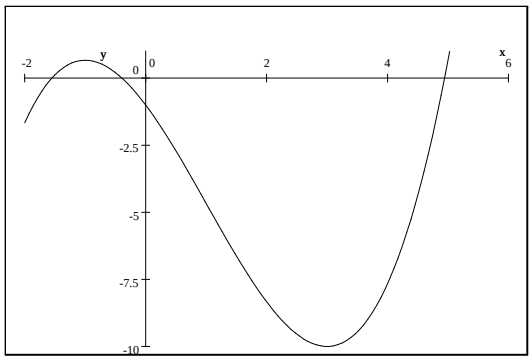
i punti per cui in un intorno sinistro $f'(x)\leq 0$ e in un intorno destro $f'(x)\geq 0$ sono punti di minimo locale.

Example 224 *Compute points of minimum and maximum for* $f:\mathbb{R}\to\mathbb{R},\ f(x)=\frac{1}{3}x^3-x^2-3x-1$.

Since $f'(x)=x^2-2x-3$, *and solutions of* $x^2-2x-3=0$ *are* $-1,3$,

f *is strictly increasing in* $(-\infty,-1]\cup[3,+\infty)$ *and strictly decreasing in* $[-1,3]$;

-1 *is a maximum point and* 3 *is a minimum point.*



Theorem 225 *Si consideri $f : S \subseteq \mathbb{R} \to \mathbb{R}$ e $x_0 \in \mathrm{Int}(S)$. Si assuma che esiste $\delta > 0$ tale che f sia derivabile almeno $n+1$ volte in $(x_0 - \delta, x_0 + \delta)$.*

Se $f'(x_0) = ... = f^{(n)}(x_0) = 0$ e $f^{(n+1)}(x_0) \neq 0$ con $n \geq 1$, allora si ha quanto segue.

1. se $n+1$ e' pari e $f^{(n+1)}(x_0) > 0$, allora x_0 e' un punto di minimo locale stretto;

2. se $n+1$ e' pari e $f^{(n+1)}(x_0) < 0$, allora x_0 e' un punto di massimo locale stretto;

3. se $n+1$ e' dispari, allora x_0 non e' ne' un punto di minimo locale, ne' un punto di massimo locale.

Proof. (Strategia: applica il Teorema della permanenza del segno per i limiti e il Teorema di Taylor con resto nella forma di Peano).

Dal Teorema di Taylor con resto nella forma di Peano si ha che $\forall x \in (x_0 - \delta, x_0 + \delta)$

$$f(x) = \sum_{k=0}^{n+1} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!} + R_{n+1}(x), \tag{6.11}$$

dove

$$\lim_{x \to x_0} \frac{R_{n+1}(x)}{(x - x_0)^{n+1}} = 0, \tag{6.12}$$

Per ipotesi $f'(x_0) = ... = f^{(n)}(x_0) = 0$; dunque da (6.11) si ottiene

$$f(x) - f(x_0) = f^{(n+1)}(x_0) \frac{(x - x_0)^{n+1}}{(n+1)!} + R_{n+1}(x) \tag{6.13}$$

e dunque

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^{n+1}} \overset{(6.12)}{=} \lim_{x \to x_0} \frac{f^{(n+1)}(x_0)}{(n+1)!} + \lim_{x \to x_0} \frac{R_{n+1}(x)}{(x - x_0)^{n+1}} \overset{(6.13)}{=}$$

$$\lim_{x \to x_0} \frac{f^{(n+1)}(x_0)}{(n+1)!} = \frac{f^{(n+1)}(x_0)}{(n+1)!}.$$

Dal Teorema della permanenza del segno per i limiti - vedi Proposizione 60 - ■

esiste $\rho > 0$ tale che $\forall x \in (x_0 - \rho, x_0 + \rho) \setminus \{x_0\}$, we have that

$$\text{segno } \frac{f(x) - f(x_0)}{(x - x_0)^{n+1}} = \text{segno } f^{(n+1)}(x_0). \tag{6.14}$$

Consideriamo ora le tre affermazioni contenute nell'enunciato del teorema.

Proof. 1. Poiche' $n+1$ e' pari, $(x - x_0)^{n+1} > 0$ e da (6.14)si ha che $\forall x \in (x_0 - \rho, x_0 + \rho) \setminus \{x_0\}$, $f(x) > f(x_0)$, ovvero, x_0 e' un punto di minimo locale (stretto).

2. ...

3. Per fissare le idee, si assuma $f^{(n+1)}(x_0) > 0$ - la dimostrazione del caso $f^{(n+1)}(x_0) < 0$ e' strettamente analoga. Da (6.14) si ha che

$\forall x \in (x_0, x_0 + \rho)$, $f(x) - f(x_0) > 0$ ovvero $f(x) > f(x_0)$, e

$\forall x \in (x_0 - \rho, x_0)$, $f(x) - f(x_0) < 0$ ovvero $f(x) < f(x_0)$.

`disegno` ■

Using the above theorem we can also get a necessary condition for the existence of local extrema.

Corollary 226 *Let $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathrm{Int}(S)$ be given. Assume there exists $\delta > 0$ such that f is at least twice differentiable on $(x_0 - \delta, x_0 + \delta)$. Then,*

1. x_0 is a local maximum point for $f \Rightarrow f''(x_0) \leq 0$, and

2. x_0 is a local minimum point for $f \Rightarrow f''(x_0) \geq 0$.

Proof. 1.

Since x_0 is a local maximum point from Proposition 177, we have $f'(x_0) = 0$. Then we can apply Theorem 225 for $n = 2$ to get that

$$f^{(n+1)}(x_0) > 0 \Rightarrow \ x_0 \text{ e' un punto di minimo locale;}$$

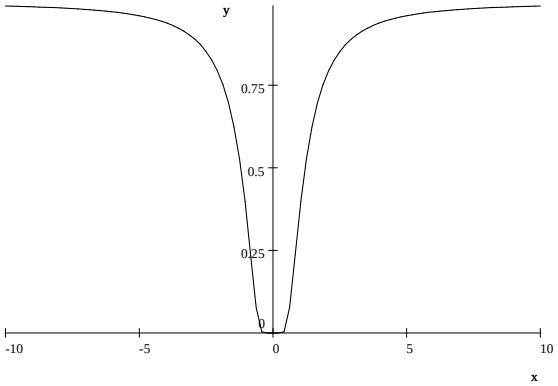
taking the contropositive of the above statement, we get thte desired result.

2.

The proof is quite similar to the proof of statement 1 above. ■

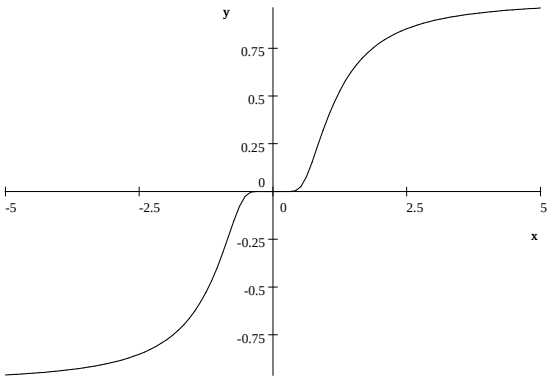
Il Teorema 225 non consente di risolvere il problema della ricerca dei punti di massimo o minimo locale per qualsiasi funzione, semplicemente perche' $f^{(n+1)}(x_0)$ puo' essere uguale a zero per qualsiasi $n + 1$. Si pensi alla funzione costante o alle seguenti funzione.

$$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto \begin{cases} e^{-\frac{1}{x^2}} & se \quad x \neq 0 \\ \\ 0 & se \quad x = 0 \end{cases}$$



Per tale funzione, $\forall n \in \mathbb{N},\ f^{(n)}(0) = 0$ e la funzione ha un minimo in zero. Si osservi che tutte le derivate si annullano in zero anche per le seguenti funzioni: $-f$ e

$$g : \mathbb{R} \to \mathbb{R},\ g : x \mapsto \begin{cases} -e^{-\frac{1}{x^2}} & se \quad x < 0 \\ 0 & se \quad x = 0 \\ e^{-\frac{1}{x^2}} & se \quad x > 0 \end{cases}$$



$-f$ ha un punto massimo locale in zero, e g non ha ne' punto di massimo ne' di minimo in zero.

Dal Teorema precedente si ottiene la seguente regola per il calcolo di punti di massimo o di minimo per funzioni derivabili un numero sufficiente grande di volte.

1. Si risolva la equazione

$$f'(x) = 0$$

2. Sia x_0 una eventuale soluzione della equazione. Si calcolino in x_0 le derivate successive alla prima fino a trovarne una, se esiste, diversa da zero. Sia essa quella di ordine $n+1$. Si ha dunque

$$\begin{array}{lcl} n+1 \text{ pari, } f^{(n+1)}(x_0) < 0 & \Rightarrow & x_0 \text{ e' un punto di massimo locale} \\ n+1 \text{ pari, } f^{(n+1)}(x_0) > 0 & \Rightarrow & x_0 \text{ e' un punto di minimo locale} \\ n+1 \text{ dispari} & \Rightarrow & x_0 \text{ non e' un punto di estremo} \end{array}$$

Example 227 *(Applicazione dei due metodi visti)*
Si calcolino eventuali punti di massimo e minimo locale della funzione

$$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto \frac{x^3}{3} + \frac{3}{2}x^2 + 2x + 2$$

$$f'(x) = x^2 + 3x + 2.$$

Le soluzioni di $f'(x) = 0$ *sono* $x = \frac{-3 \pm \sqrt{9-8}}{2} = \left\langle \begin{array}{c} -2 \\ -1 \end{array} \right.$.

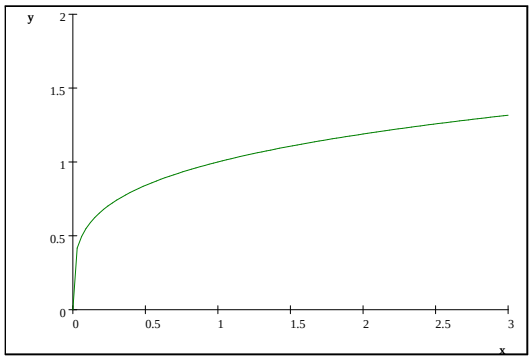
Primo metodo: poiche' $f'(x) > 0$ *per* $x \in (-\infty, -2) \cup (-1, +\infty)$*, si ha che* -2 *e' un punto di massimo locale e* -1 *e' un punto di minimo locale.*

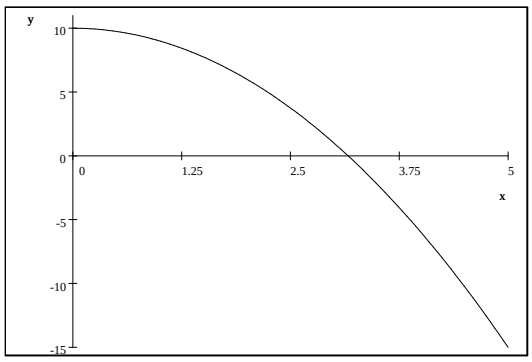
Secondo metodo. $f''(x) = 2x + 3$*.* $f''(-2) = -1 < 0$ *e* $f''(-1) = 1 > 0$ *e dunque* -2 *e' un punto di massimo locale e* -1 *e' un punto di minimo locale.*

6.6 Funzioni Concave e Convesse

Prima di tutto presentiamo una descrizione geometrica e intuitiva delle definizioni di funzioni concave e convesse e di alcuni risultati.

Una funzione e' concava (rispettivamente, convessa) se ogni segmento che unisce due arbitrari punti del grafico sta non sopra (rispettivamente, non sotto) il grafico stesso.





E' immediato ricavare dai grafici presentati - e verra' dimostrato nel seguito - che una funzione derivabile e' concava (rispettivamente, convessa) se ogni retta tangente sta non sotto (rispettivamente, non sopra) il grafico stesso.

Cerchiamo ora di formalizzare e dimostrare quanto detto.

Definition 228 *Dati $x', x'' \in \mathbb{R}$ e $\lambda \in [0,1]$, il numero reale*

$$x(\lambda) := (1-\lambda) x' + \lambda x''$$

si dice combinazione convessa di x' e x'' (per mezzo di λ).

Remark 229 *Si osservi che $x(0) = x'$ e $x(1) = x''$. Inoltre se, senza perdita di generalita', e' $x'' \geq x'$, si ha che $x' \leq x(\lambda) \leq x''$, come verificato di seguito.*

$$(1-\lambda) x' + \lambda x'' = x' + \lambda (x'' - x') \geq x'$$

$$(1-\lambda) x' + \lambda x'' = (1-\lambda) x' + \lambda x'' - x'' + x'' = x'' - (1-\lambda)(x'' - x') \leq x''$$

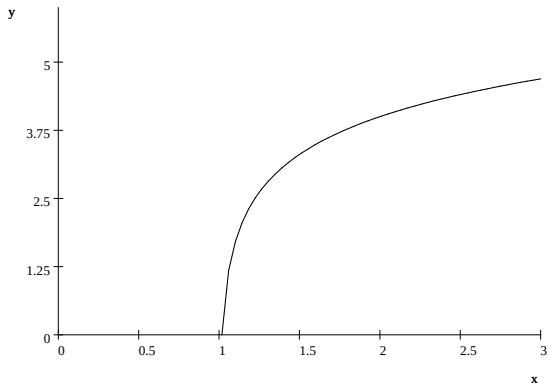
Inoltre, se $x'' > x'$, allora

$$\lambda = \frac{x(\lambda) - x'}{x'' - x'}$$

Dunque $x(\lambda)$ e' uguale a x' piu' una proporzione λ della distanza tra x' e x''.

Dalla definizione informale che abbiamo presentato di funzione concava e dall'osservazione vista sopra si ha che f e' concava se $\forall \lambda \in [0,1]$, $f(x(\lambda)) \geq AB(\lambda)$:

`figura da completare`



Vogliamo verificare che $AB(\lambda) = (1-\lambda) f(x') + \lambda f(x'')$. Si consideri il caso in cui la funzione f non sia constante. Procediamo come segue.

a. Troviamo la equazione della retta passante per i punti $(x', f(x'))$ e $(x'', f(x''))$. Sia $y = l(x)$ la equazione di tale retta.

b. Verifichiamo che $l(x(\lambda)) = (1-\lambda) f(x') + \lambda f(x'')$.

a.

$$\frac{y - f(x')}{f(x'') - f(x')} = \frac{x - x'}{x'' - x'}$$

$$y = f(x') + \frac{f(x'') - f(x')}{x'' - x'} (x - x')$$

b.

$$l(x(\lambda)) = f(x') + \frac{f(x'') - f(x')}{x'' - x'} ((1-\lambda) x' + \lambda x'' - x') \quad =$$
$$= f(x') + \frac{f(x'') - f(x')}{x'' - x'} \lambda (x'' - x') = (1-\lambda) f(x') + \lambda f(x'')$$

Possiamo dunque dare la seguente definizione.

Definition 230 *Dato un intervallo I e una funzione $f : I \to \mathbb{R}$, f e' concava se $\forall x', x'' \in I$ e $\forall \lambda \in [0,1]$ si ha*

$$f((1-\lambda) x' + \lambda x'') \geq (1-\lambda) f(x') + \lambda f(x'')$$

Il fatto che la corda stia "strettamente" sotto il grafico, si formalizza come segue.

Definition 231 *Dato un intervallo I e una funzione $f : I \to \mathbb{R}$, f e' strettamente concava se $\forall x', x'' \in I$ tali che $x' \neq x''$ e $\forall \lambda \in (0,1)$ si ha*

$$f\left((1-\lambda)\, x' + \lambda x''\right) > (1-\lambda)\, f\left(x'\right) + \lambda f\left(x''\right)$$

Remark 232 *Se una funzione e' strettamente concava, allora e' concava. Esistono funzioni concave che non sono strettamente concave, come $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x$.*

Il fatto che la corda stia non sotto il grafico, si formalizza come segue.

Definition 233 *Dato un intervallo I e una funzione $f : I \to \mathbb{R}$, f e' convessa se $\forall x', x'' \in I$ e $\forall \lambda \in [0,1]$ si ha*

$$f\left((1-\lambda)\, x' + \lambda x''\right) \leq (1-\lambda)\, f\left(x'\right) + \lambda f\left(x''\right)$$

Il fatto che la corda stia "strettamente" sopra il grafico, si formalizza come segue.

Definition 234 *Dato un intervallo I e una funzione $f : I \to \mathbb{R}$, f e' strettamente convessa se $\forall x', x'' \in I$ tali che $x' \neq x''$ e $\forall \lambda \in (0,1)$ si ha*

$$f\left((1-\lambda)\, x' + \lambda x''\right) < (1-\lambda)\, f\left(x'\right) + \lambda f\left(x''\right)$$

Le figure **??** e **??** suggeriscono che un'altra interpretazione geometrica della concavita' di una funzione *derivabile* e' che la retta tangente al grafico stia non al di sotto del grafico stesso. Il seguente teorema dimostra la validita' di tale interpretazione.

Theorem 235 *Dato un intervallo aperto I e una funzione derivabile $f : I \to \mathbb{R}$,*
f e' concava $\Leftrightarrow \forall x', x'' \in I,\ \ f(x'') \leq f(x') + f'(x')(x'' - x')$.

Proof.
[$\Rightarrow$]
Se $x' = x''$, si ha che $f(x'') \leq f(x') + f'(x')(x'' - x')$ diventa $f(x) \leq f(x') + f'(x')(x' - x')$, il che e' ovviamente vero. Si consideri dunque il caso in cui $x' \neq x''$.

$$f(x' + \lambda\,(x'' - x')) - f\left(x'\right) \geq \lambda\,(f(x'') - \lambda f(x')$$

From the definition of concavity, we have that for $\lambda \in (0,1)$

$$(1-\lambda)\, f(x') + \lambda f(x'') \leq f(x' + \lambda\,(x'' - x'))$$
$$f(x' + \lambda\,(x'' - x')) - f\left(x'\right) \geq \lambda\,(f(x'') - f(x'))$$

$$\frac{f(x' + \lambda\,(x'' - x')) - f\left(x'\right)}{\lambda} \geq f(x'') - f(x')$$

e, moltiplicando numeratore e denominatore del lato di sinistra per $(x'' - x')$,

$$\frac{f(x' + \lambda\,(x'' - x')) - f\left(x'\right)}{\lambda\,(x'' - x')}\,(x'' - x') \geq f(x'') - f(x')$$

Si prenda ora il limite di entrambi i lati per $\lambda \to 0$,

$$(x'' - x') \lim_{\lambda \to 0} \frac{f(x' + \lambda\,(x'' - x')) - f\left(x'\right)}{\lambda\,(x'' - x')} \geq \lim_{\lambda \to 0} \left(f(x'') - f(x')\right)$$

ovvero

$$(x'' - x') \cdot f'\left(x'\right) \geq f(x'') - f(x')$$

come desiderato.
[$\Leftarrow$]
Il risultato desiderato e' chiaramente vero se $\lambda \in \{0,1\}$. Consideriamo il caso $\lambda \in (0,1)$. Per ipotesi

$$f(x'') - f(x\,(\lambda)) \leq f'(x\,(\lambda))(x'' - x\,(\lambda)) \tag{6.15}$$

e

$$f(x') - f(x(\lambda)) \leq f'(x(\lambda))(x' - x(\lambda)) \tag{6.16}$$

Moltiplicando (6.15)per λ, e (6.16)per $(1-\lambda)$ e sommando membro a membro otteniamo

$$\lambda(f(x'') - f(x(\lambda))) + (1-\lambda)(f(x') - f(x(\lambda))) \leq f'(x(\lambda))(\lambda(x'' - x(\lambda)) + (1-\lambda)(x' - x(\lambda)))$$

Poiche'

$$\lambda(x'' - x(\lambda)) + (1-\lambda)(x' - x(\lambda)) = x(\lambda) - x(\lambda) = 0$$

otteniamo

$$\lambda f(x'') + (1-\lambda) f(x') \leq f(x(\lambda))$$

come desiderato.

■

Remark 236 *Analoghi risultati e dimostrazioni valgono per le altre defnizioni viste.*

Theorem 237 *Sia $f : (a,b) \to \mathbb{R}$ derivabile almeno due volte.*

$$\langle \forall x \in (a,b),\ f''(x) \leq 0 \rangle \overset{(1)}{\underset{(2)}{\Rightarrow \atop \Leftarrow}} \langle f \text{ e' concava} \rangle$$

$$\langle \forall x \in (a,b),\ f''(x) < 0 \rangle \overset{(3)}{\underset{(4)}{\Rightarrow \atop \not\Leftarrow}} \langle f \text{ e' strettamente concava} \rangle$$

Proof. $\left[\overset{(1)}{\Rightarrow}\right]$

Strategia: Teorema di Lagrange, Teorema precedente, $f'' \leq 0 \Rightarrow f'$ decrescente.

Vogliamo dimostrare che $\forall x, x_0 \in (a,b)$, $f(x) - f(x_0) - f'(x_0)(x - x_0) \leq 0$. Senza perdita di generalita' sia $x > x_0$ e si consideri $f_{|[x_0,x]}$. Dal Teorema di Lagrange,

$$\exists \xi \in (x_0, x) \text{ tale che } f(x) - f(x_0) = f'(\xi)(x - x_0)$$

Dunque

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = [f'(\xi) - f'(x_0)](x - x_0) \leq 0$$

dove la disuguaglianza segue dal fatto che $x > x_0$ e dal fatto che $f'' \leq 0$ e $\xi > x$ implica che $f'(\xi) \leq f'(x_0)$. Si ricordi che $f'' \geq 0 \Rightarrow f'$ decrescente.

$\left[\overset{(2)}{\Leftarrow}\right]$

Strategia: per assurdo, Teorema di Taylor e della permanenza del segno.

Si supponga per assurdo che $\exists x_0 \in (a,b)$ tale che $f''(x_0) > 0$. Dal Primo Teorema di Taylor, si ha che preso $x \in (a,b)$ risulta

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = \frac{1}{2} f''(x_0)(x - x_0)^2 + R_{2,x_0}(x)$$

con

$$\lim_{x \to x_0} \frac{R_{2,x_0}(x)}{(x - x_0)^2} = 0$$

Dunque

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} = \frac{1}{2} \lim_{x \to x_0} \frac{f''(x_0)(x - x_0)^2}{(x - x_0)^2} + \lim_{x \to x_0} \frac{R_{2,x_0}(x)}{(x - x_0)^2} = \frac{1}{2} f''(x_0) > 0$$

Poiche' f e' derivabile due volte, f' e' continua e e' continua anche la funzione definita da

$$\frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2}$$

in un intorno di x_0 escluso x_0. Dunque in tale insieme, per il teorema della permanenza del segno per le funzioni continue, si ha che

$$f(x) - f(x_0) - f'(x_0)(x - x_0) > 0$$

contraddicendo la concavita' di f.

$\left[\overset{(3)}{\Rightarrow}\right]$

Dimostrazione analoga alla (1): e' sufficiente sostituire " $\leq$ " con " $<$ ".

$\left[\overset{(4)}{\nLeftarrow}\right]$

$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto -x^4$ e' strettamente concava, ma $f''(0) = 12x^2_{|x=0} = 0$. ■

Remark 238 *Risultati analoghi valgono per le funzioni convesse e strettamente convesse.*

Funzioni concave e massimi globali.

Proposition 239 *Consider an open interval I and a concave function $f : I \to \mathbb{R}$. If x_0 is a local maximum point, then it is a global maximum point.*

Proof.

By definition of local maximum point, we know that $\exists \delta > 0$ such that $\forall x \in B(x_0, \delta) := (x_0 - \delta, x_0 + \delta)$, $f(x_0) \geq f(x)$. Take $y \in I$; we want to show that $f\left(x^0\right) \geq f(y)$.

Since S is convex, $\forall\, \lambda \in [0,1]$, $(1-\lambda)\, x^0 + \lambda y \in S$. Take $\lambda^0 > 0$ and sufficiently small to have $\left(1-\lambda^0\right) x^0 + \lambda^0 y \in B(x^0, \delta)$. (To find such λ^0, just solve the inequality $\left|\left(1-\lambda^0\right) x^0 + \lambda^0 y - x^0\right| = \left|\lambda^0 \left(y - x^0\right)\right| = \left|\lambda^0\right| \bullet \left|y - x^0\right| < \delta$, where, without loss of generality, $y \neq x^0$).

Then,

$$f\left(x^0\right) \geq f\left(\left(1-\lambda^0\right) x^0 + \lambda^0 y\right) \overset{f \text{ concave}}{\geq} \left(1-\lambda^0\right) f(x^0) + \lambda^0 f(y)$$

or $\lambda^0 f(x^0) \geq \lambda^0 f(y)$. Dividing both sides of the inequality by $\lambda^0 > 0$, we get $f(x^0) \geq f(y)$.

■

Proposition 240 *Consider a differentiable and concave function f. If $f'(x^0) = 0$, then x^0 is a global maximum point.*

Proof.

From Proposition 235, if $f'(x^0) = 0$, we get that $\forall\, x \in I,\ f\left(x^0\right) \geq f(x)$, the desired result.

■

6.7 Asymptotes

Definition 241 *(Vertical asymptotes) Given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in D(\{x \in S : x > x_0\})$, we say that the line with equation*

$$x = x_0$$

is a right vertical asymptote for f if

$$\lim_{x \to x_0^+} f(x) = \pm\infty.$$

Similarly, given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in D(\{x \in S : x < x_0\})$, we say that the line with equation

$$x = x_0$$

is a left vertical asymptote for f if

$$\lim_{x \to x_0^-} f(x) = \pm\infty.$$

Example 242 *The function $f : \mathbb{R} \setminus \{0\}$, $f(x) = \frac{1}{x}$ has $x = 0$ as both left and right vertical asymptote. Observe that the "horyzhontal distance" between the graph of f and the graph of the vertical line becomes smaller when x gets closer to 0. Observe also that f is not continuous in 0.*

Definition 243 *Given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded above, we say that the line with with equation*

$$y = px + q,$$

with $p, q \in \mathbb{R}$, is an asymptote for f at $+\infty$ if

$$\lim_{x \to +\infty} [f(x) - (px + q)] = 0.$$

Similarly, given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded below, we say that the line with with equation

$$y = px + q,$$

with $p, q \in \mathbb{R}$, is an asymptote for f at $-\infty$ if

$$\lim_{x \to -\infty} [f(x) - (px + q)] = 0.$$

Example 244 *The function $f : \mathbb{R} \setminus \{0\}$, $f(x) = \frac{1}{x}$ has $y = 0$ as an asymptote at $\pm\infty$:*

$$\lim_{x \to \pm\infty} \left(\frac{1}{x} - 0 \right) = 0.$$

Observe that the "vertical distance" between the graph of f and the graph of the line becomes smaller when x gets closer and closer to $+\infty$ or $-\infty$.

Definition 245 *Given $f : S \subseteq \mathbb{R} \to \mathbb{R}$ with S unbounded above,*

$$\langle \lim_{x \to +\infty} [f(x) - (px + q)] = 0 \rangle \quad \Leftrightarrow \quad \left\langle \begin{array}{l} 1. \lim_{x \to +\infty} \frac{f(x)}{x} = p \in \mathbb{R}, \\ \\ 2. \lim_{x \to +\infty} [f(x) - px] = q \in \mathbb{R}. \end{array} \right\rangle .$$

Proof. [⇒]
Observe that the Assumption implies that

$$0 = \lim_{x \to +\infty} \frac{[f(x) - (px + q)]}{x} = \lim_{x \to +\infty} \left(\frac{f(x)}{x} - \frac{px}{x} - \frac{q}{x} \right) = \lim_{x \to +\infty} \left(\frac{f(x)}{x} - p \right),$$

or

$$\lim_{x \to +\infty} \frac{f(x)}{x} = p.$$

Again from the assumption,

$$\lim_{x \to +\infty} [f(x) - px] = q.$$

[⇐]
Assumption 2. implies the desired conclusion. ■
A particular case of Definition 243 is obtained if $p = 0$.

Definition 246 *Let the function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ be given. Assume that S is unbounded above. Then we say that*

$$y = k \in \mathbb{R}$$

is a horizontal asymptote for f at $+\infty$ if

$$\lim_{x \to +\infty} f(x) = k.$$

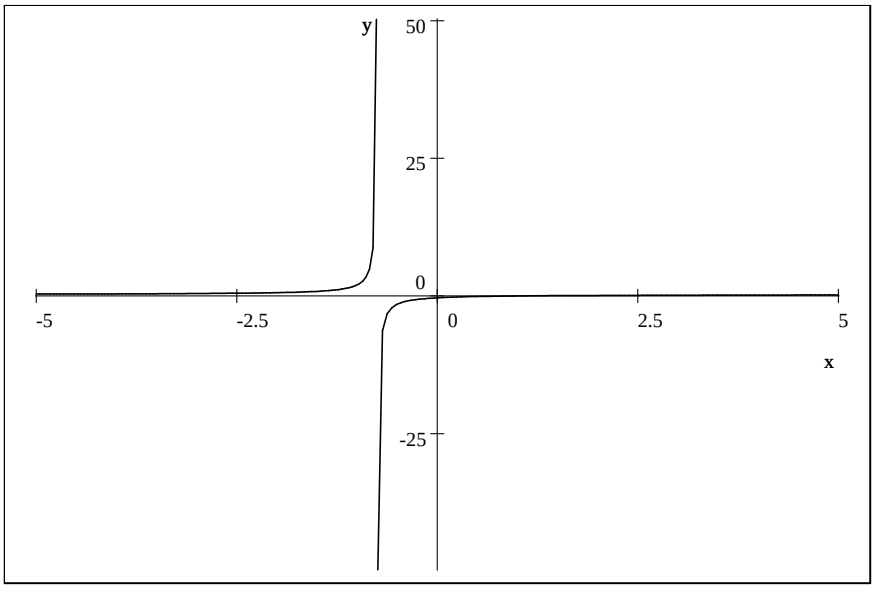
Similarly if S is unbounded below, we say that

$$y = k \in \mathbb{R}$$

is a horizontal asymptote for f at $-\infty$ if

$$\lim_{x \to -\infty} f(x) = k.$$

Example 247 *Let* $f(x) = \frac{x-1}{4x+3}$



Then f *has horizontal asymptote* $y = \frac{1}{4}$ *at* $\pm\infty$ *and a vertical asymptote in* $x = -\frac{3}{4}$*; indeed*

$$\lim_{x\to\pm\infty} \frac{x-1}{4x+3} = \lim_{x\to\pm\infty} \frac{1-\frac{1}{x}}{4+\frac{3}{4}} = \frac{1}{4},$$

and

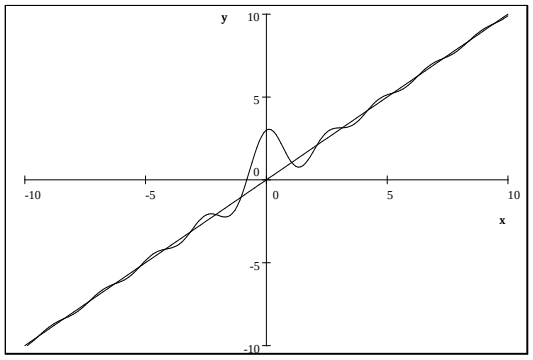
$$\lim_{x\to-\frac{3}{4}^-} \frac{x-1}{4x+3} = "\frac{-\frac{1}{4}}{0^-}" = +\infty,$$

$$\lim_{x\to-\frac{3}{4}^+} \frac{x-1}{4x+3} = "\frac{-\frac{1}{4}}{0^+}" = -\infty,$$

Example 248 *Compute the aymptotes at* $\pm\infty$ *for*

$$f(x) = x + \frac{\sin x}{x}.$$

$p = \lim_{x\to\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(\frac{x+\frac{\sin x}{x}}{x}\right) = \lim_{x\to\pm\infty} \left(1+\frac{\sin x}{x^2}\right) = 1 := p.$
$q = \lim_{x\to+\infty} [f(x) - px] = \lim_{x\to+\infty} \left[\left(x+\frac{\sin x}{x}\right) - x\right] = 0.$
Therefore, the equation of the asymptote is $y = x$.



6.8 Studio di funzioni

Di seguito proponiamo un elenco di passi da seguire che sono spesso utili nel disegnare il grafico di funzioni.

1. Determinazione del dominio di una funzione.

Cio' e' richiesto se il dominio della funzione non e' dato contestualmente alla "legge" analitica con la quale essa e' definita.

2. Studio del segno della funzione e delle eventuali intercette con gli assi.

Si tratta di studiare la equazione $f(x) = 0$ e la disequazione $f(x) \geq 0$.

3. Studio della continuita della funzione e individuazione degli eventuali punti di discountinuita' (e asintoti verticali).

4. Studio del comportamento a $+\infty$ e $-\infty$ e determinazione degli eventuali asintoti non verticali.

Si ricordi che se un asintoto non verticale a $+\infty$ esiste ed ha equazione $y = px + q$, si ha che $p = \lim_{x\to+\infty} \frac{f(x)}{x}$ e $q = \lim_{x\to+\infty} [f(x) - px]$.

5. Studio del segno della derivata prima (nel caso la funzione sia derivabile).

Tale studio permette di determinare gli intervalli in cui la funzione e' crescente ($f' \geq 0$) o decrescente ($f' \leq 0$) e punti di massimo e di minimo locale.

6. Studio del segno della derivata seconda (nel caso la funzione sia derivabile due volte).

Tale studio permette di determinare gli intervalli in cui la funzione e' concava ($f' \geq 0$) o convessa ($f' \leq 0$).

Si tenga presente che lo studio del grafico puo' essere di molto semplificato usando le conoscenze acquisite nella prima parte del corso su funzioni elementari e le loro trasformazioni, funzioni pari, funzioni dispari e funzioni periodiche.

Example 249 *Si studi il grafico di*

$$f(x) = \frac{e^x}{|x^2 - 1|}$$

Si tralasci lo studio della derivata seconda.

Soluzione

1. Dominio.

$dom\ f = \mathbb{R} \setminus \{-1, +1\}$.

$f(x) = \frac{-e^x}{|x^2-1|} = \begin{cases} \frac{e^x}{x^2-1} := f_1(x) & se \quad x \notin [-1,1] \\ \frac{e^x}{-x^2+1} := f_2(x) & se \quad x \in (-1,1) \end{cases}$

2. Segno.

Il segno di $f(x)$ e' sempre positivo.

3. Punti di discontinuita'.

$\lim_{x\to-1^{\pm}} \frac{e^x}{|x^2-1|} = "\frac{1/e}{0^+}" = \infty$;

$\lim_{x\to1^{\pm}} \frac{e^x}{|x^2-1|} = "\frac{e}{0^+}" = \infty$;

4. Comportamento a $\pm\infty$ e eventuali asintoti non verticali.

$\lim_{x\to+\infty} \frac{e^x}{x^2-1} = +\infty$;

$\lim_{x\to+\infty} \frac{e^x}{x^2-1}\frac{1}{x} = +\infty$ *e dunque non esiste un asintoto obliquo a* $+\infty$.

$\lim_{x\to-\infty} \frac{e^x}{-x^2+1} = 0$

5. Derivata prima.

Per $x \notin [-1,1]$,

$f_1'(x) = \frac{e^x(x^2-1)-2xe^x}{(x^2-1)^2} = \frac{e^x(x^2-1-2x)}{(x^2-1)^2} \geq 0 \Leftrightarrow$

$x \in \left(-\infty, 1-\sqrt{2}\right] \cup \left[1+\sqrt{2}, +\infty\right)$, *e dunque*

			-1		$1-\sqrt{2}$		$+1$		$1+\sqrt{2}$	
f_1'		$+$		*non definita*		*non definita*		$-$		$+$
f_1		*crescente*						*decrescente*		*crescente*

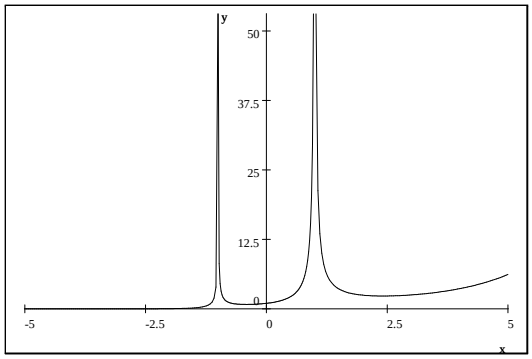
e $1+\sqrt{2}$ *e' un punto di minimo,* $f\left(1+\sqrt{2}\right) = \frac{e^{1+\sqrt{2}}}{\left(1+\sqrt{2}\right)^2-1} = 2.3157$.

Per $x \in (-1,1)$,

$f_2'(x) = \frac{e^x\left(-x^2+1\right)+2xe^x}{(x^2-1)^2} = \frac{e^x\left(-x^2+1+2x\right)}{(x^2-1)^2} \geq 0 \Leftrightarrow$
$x \in \left[1-\sqrt{2}, 1+\sqrt{2}\right]$, *e dunque*

			-1		$1-\sqrt{2}$		$+1$		$1+\sqrt{2}$	
f'		$\underset{definita}{non}$		$-$		$+$		$\underset{definita}{non}$		$\underset{definita}{non}$
f				*decrescente*		*crescente*				

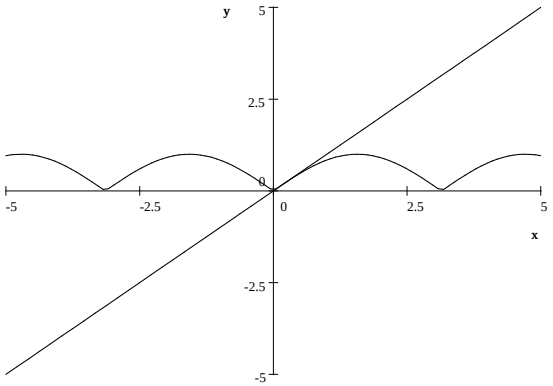
e $1-\sqrt{2}$ *e' un punto di minimo;* $f\left(1-\sqrt{2}\right) = \frac{e^{1-\sqrt{2}}}{\left(1-\sqrt{2}\right)^2-1}; = .79773.$



Example 250 *Si determino eventuali punti di massimo locale e globale e punti di minimo locale e globale di* f.

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} |\sin x| & se \quad x \in \mathbb{R}\backslash\mathbb{Q} \\ x & se \quad x \in \mathbb{Q} \end{cases}$$

Soluzione.



La funzione e' illimitata e dunque non ammette massimi o minimi globali.

Poiche' $\sin x = 0$ *se* $x = k\pi$ *con* $k \in \mathbb{Z}$, *allora*

$$\{x \in \mathbb{R} : x > 0 \text{ } e \text{ } \exists k \in \mathbb{Z} \text{ } tale \text{ } che \text{ } x = k\pi\} \subseteq \mathbb{R}\backslash\mathbb{Q}$$

e' l'insieme dei punti di minimo locale di f.

Poiche' $|\sin x| = 1$ *se* $x = k\frac{\pi}{2}$ *con* $k \in \mathbb{Z}$, *allora*

$$\left\{x \in \mathbb{R} : x < 0 \text{ } e \text{ } \exists k \in \mathbb{Z} \text{ } tale \text{ } che \text{ } x = k\frac{\pi}{2}\right\} \subseteq \mathbb{R}\backslash\mathbb{Q}$$

e' l'insieme dei punti di massimo locale di f.

6.9 Esercizi

Nannicini, Verdi and Vessella (2008):
Paragrafo 9.12.1; Esercizi: 6,7,8,9,17,21,24.
Paragrafo 9.12.2; Esercizi: da1 a 10 compresi.
Paragrafo 9.12.3; Esercizi: 1,2,3,5..
Paragrafo 10.2; Esercizi: tutti
Barozzi e Corradi (1979).
Capitolo 7. Paragrafo su "I teoremi di de l'Hopital": Esercizi da 25 a 29 compresi.
Capitolo 7 (Paragrafo: La formula di Taylor) Esercizi 31, 32 e 33a, b.
Capitolo 7. Paragrafo"Massimi e Minimi": Esercizi 38, 39, 40.
Capitolo 7. Paragrafo "Funzioni convesse": Esercizi 42, 43, 46 i., ii., iii.
Capitolo 7. Paragrafo "Applicazioni I. Studio del grafico di una funzione": Esercizio 53.

References.

Barozzi G. C., Corradi C., (1998), *Corso di Matematica per le Scienze Economiche: Esercizi*, Il Mulino, Bologna.

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