Problem set 2

(1) Find the total derivatives of the following functions.

1. $f(x_1,x_2) = x_1^3 - 4x_1^2 + x_2 + 5x_2^2$ at $x_1 = 1$ $x_2 = 1$.
2. $f(x_1,x_2) = 2x_1 - x_2 / x_1 + x_2$ " "
3. $f(x_1,x_2) = x_1^2 + 3x_1x_2 - x_2^2 + e^{x_1x_2}$ " "
4. $f(x_1,x_2x_3) = x_2 \log x_1 + x_3$.
5. $f(x,y,z) = x^{yz}$ $(x y z) = (e,1,1)$.

(2) Let $f(x_1 x_2 x_3) = \frac{1}{3}\log x_1 + \frac{1}{6}\log x_2 + \frac{1}{2}\log x_3$.

$x > 0$ $\bar{x} = (1,1,2)$ $h = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Find: (i) $\nabla f(\bar{x})$

(ii) the directional derivative at $\bar{x}$ in direction h.

(iii) the total derivative at $\bar{x}$.

(2b) Let $f(x_1x_2x_3) = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 6x_2x_3$

$\bar{x} = (1,0,-1)$ $h = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

find (i),(ii) (iii) as in ex. 2.

NO (3) for the two functions in 2, 2b, find the tangent plane at $\bar{x}$. and the Hessian matrix.

(4) find the directional derivatives of the following functions at the given points in the given direction

(a) $f(x,y) = x + 5y - 1$ at $(2,1)$ in direction $(1,1)$
(b) $g(x,y,z) = x e^{y/x} - z^2$ at $(0,1,1)$ " $(1,1,1)$
(c) $h(x,y) = x^y$ at $(e,1)$ " $(1,2)$
(d) $f(x,y) = x e^{xy}$ $(0,0)$ $(2,3)$
(e) $f(x,y) = x^2 - 2yx - y^2$ $(3,1)$ $(1,1)$
(f) $f(x,y) = 2x + y - 1$. $(-1,1)$ $(+2,-3)$

NO (5) Find the tangent plane for

$x^2+y^2 = f(x,y) = z$ at $(z_0, x_0, y_0) = (5,4,1)$

$f(x,y) = (y-x^2)(y-2x^2) = z$ at $(z_0, x_0, y_0) = (2,3,1)$

NO (6) Find a second order polynomial approximation in (x,y) around $(0,0)$.

$f(x,y) = (1+x)^{1+y}$

$f(x,y) = \ln 1+x^2+y^2$

$f(x,y) = \exp(x\cdot e^y)$.

(7) Prove that the following equations define y as a function of x in a neighbourhood of x_0. compute $\partial y/\partial x_0$ at $x=x_0$

$y^3+y-x^3=0 \qquad x_0=0$

$x^y - y^x = 0 \qquad x_0=1.$

(8) find $\partial z/\partial t$ and $\partial z/\partial s$ in the following ~~faces~~ cases.

(i) $z = xy^2 \qquad x = t+s^2$ and $y = t^2 s$

(ii) $z = (x-y)/(x+y) \qquad x = e^{t+s} \qquad y = e^{ts}$

(9) Let $f.(x,y)$ be an homogeneous function of degree k

ie $\forall \lambda > 0 \quad f(\lambda x, \lambda y) = \lambda^k f(x,y)$.

Show

(i) Thm (Euler's thm)

Let $f: \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$. f is homogeneous of degree k.

if and only if, $\forall x \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} \bar{x}_i f_{x_i}(\bar{x}) = k f(\bar{x}).$$

(ii) under the maintained assumption $F_x(\cdot)$ is homogeneous of degree $k-1$.

(10) Let $f(x,y)$ be defined on an open $S \subset \mathbb{R}^2$. Assume f_x, f_y exist everywhere and are bounded everywhere on S.

Show that f is a continuous function on S.

(11) Let $f(x,y,z) = x^2 + yz^2 + y^2x + 1 = 0$. be the equation relating the variables (x,y,z). Find $\partial y/\partial z$ and $\partial y/\partial x$ at $x=-1$, $z=1$.

(12) Let $f \in C^1$, $f: \mathbb{R}^2 \to \mathbb{R}$, $f(1,1) = 1$ $f_{x^1}(1,1) = a$
$f_{x^2}(1,1) = b$.
Let $\Phi(x) = f(x, f(x,x))$.
Find $\Phi(1)$ and $\Phi'(1)$.

(13) Given n functions $f_i : \mathbb{R}^n \to \mathbb{R}$, each differentiable on $S \subset \mathbb{R}^n$; let $x \in S$. Show (i) that the function $f(x) = \sum_{i=1}^n f_i(x)$ is differentiable at x $\forall x \in S$.
(ii) Show
$$f'(x)(u) = \sum_{i=1}^n f_i'(x_i)\, u_i \qquad u = (u_1 \dots u_n)$$

(14) Show the implicit function theorem for $f: \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x,y)=0$, $f_x(x,y) \neq 0$

NO (15) , Find the values of x_i (depending on the parameters p) such that $u(x)$ is a maximum

(1) $U(x) = \prod_{i=1}^n x^{i\alpha}$ s.t. $\sum_{i=1}^n p^i x^i \leq y$ $\alpha > 0$. (Cobb Douglas)

(2) $U(x) = \min\{x^1, 2x^2\}$ $p^1x^1 + p^2x^2 = y$. (Leontief)

(3) $U(x) = A\left(\delta x_1^{-e} + (1-\delta)\, x_2^{-e}\right)^{-\frac{1}{e}}$ $e \neq 0$ $e \geq -1$, $\delta \in [0,1]$ $A > 0$. (CES).

(4) $\max\ (y+z-3)^2$ st. (1) $x^2 + y + z = 1$
(2) $x + y^2 + 2z = 2$.

(3) $U(x) = \sum_{i=1}^n x_i^\alpha$ $\alpha > 0$, $\alpha \neq 1$ $\sum p^i x^i = y$

(4) $U(x) = \sum_{i=1}^n x_i$ $\sum p^i x^i = y$.

Answers to problem set 2.

Ex (i) In all the following exercises you must check that the function is differentiable at the given point.
In each case we will compute $\nabla f(x)$, if the partial derivatives are continuous (at the given point) then, $\nabla f(x)$ is the representation of the total derivative.

(1) $\nabla f(x) = \left[3x_1^2 - 8x_1,\ 1 + 10x_2\right]$. Clearly f_{x^i} are continuous and $\nabla f(1,1) = [-5, 11]$.

(2) $\nabla f(x) = \left[3x_2/(x_1+x_2)^2,\ -3x_1/(x_1+x_2)^2\right] = \left[3/4 \quad -3/4\right]$ Clearly f_{x^i} are continuous at $(1,1)$ (of course they are not continuous, and hence $f(\cdot)$ is not necessarily differentiable, at say $x_1 = k$, $x_2 = -k$ $\forall k \in \mathbb{R}$.

(3) $\nabla f(x) = \left[2x_1 + 3x_2 + x_2 e^{x_1x_2};\ 3x_1 - 2x_2 + x_1 e^{x_1x_2}\right]$ $f \in C^1$.
$\Rightarrow$ $\nabla f(1,1) = [5+e,\ 1+e]$.

(4) $\nabla f(x) = \left[\frac{x_2}{x_1},\ \log x_1,\ 1\right] = [1, 0, 1]$ at $x = (1,1,1)$.
Again, $f \in C^1$ on $\mathbb{R}^2_{++}$.

(5) $\nabla f(x) = \left[(yz)(x)^{yz-1},\ z(x)^{yz}\log x,\ y(x)^{yz}\log x\right]$. f is C^1 on $\mathbb{R}^3_{++}$ and, at $(e,1,1)$ $\nabla f(e,1,1) = [1, e, e]$.

Ex (2). $\nabla f(x) = \left[\frac{1}{3x_1}, \frac{1}{6x_2}, \frac{1}{2x_3}\right]$. Hence f is C^1 at $\bar{x}$, $\nabla f(\bar{x}) = \left[\frac{1}{3}, \frac{1}{6}, \frac{1}{4}\right]$
Hence the vector $\nabla f(\bar{x})$ is also the representation of the total derivative at $\bar{x}$. While $\frac{\sqrt{3}}{4} = f'(x,h)$.

Ex (2b) $\nabla f(x) = \left[2x_1 + 2x_2,\ 4x_2 - 2x_1 - 6x_3,\ -2x_3 - 6x_2\right]$. Clearly $f \in C^1$.
$\nabla f(\bar{x}) = [2, +4, 2]$, $f'(\bar{x}; h) = 0$.

Ex 3. (i) Hessian of ex 2a is
$$H = \begin{bmatrix} -\frac{1}{3x_1^2} & 0 & 0 \\ 0 & -\frac{1}{6x_2^2} & 0 \\ 0 & 0 & -\frac{1}{2x_3^2} \end{bmatrix}$$

at $\bar{x}$
$$H(1,1,2) = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{8} \end{bmatrix}$$

the tg. plane is given by the solution to

$$y = f_{x^1}(\bar{x})(x_1-1) + f_{x^2}(\bar{x})(x_2-1) + f_{x^3}(\bar{x})(x^3-2) + f(\bar{x})$$

$$y = \frac{1}{3}x_1 - \frac{1}{3} + \frac{1}{6}x_2 - \frac{1}{6} + \frac{1}{4}x_3 - \frac{1}{2} + \frac{1}{2}\log 2.$$

$$y = \frac{1}{3}x_1 + \frac{1}{6}x_2 + \frac{1}{4}x_3 - 1 + \frac{1}{2}\log 2.$$

(ii). Hessian of ex 2b is.
$$H = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -6 \\ 0 & -6 & -2 \end{bmatrix}.$$

tg plane is given by
$$y = 2(x_1-1) + 4(x_2-0) + 2(x_3+1) + f(\bar{x})$$
$$\Rightarrow y = 2x_1 + 4x_2 + 2x_3 + 0.$$

Ex 4. (a) $\nabla f(x) = [1, 5]$ $\quad f'(\bar{x}, v) = 6/\sqrt{2}$ (clearly C^1).

(b) $\nabla f(x) = [e^{yx} + xye^{yx},\ x^2e^{yx}, -2z]$ Clearly C^1.

$\nabla f(\bar{x}) = [1+0,\ 0, -2]$. $\Rightarrow f'(\bar{x}, v) = -2/\sqrt{3}$

(3) $\nabla f(x) = [y x^{y-1},\ x^y \ln x]$. clearly C^1 at $\bar{x}$.

$\nabla f(\bar{x}) = [1e^0,\ e^1 \ln e] = [1, e]$. $\quad f'(\bar{x}, v) = (1+2e)/\sqrt{5}$

(4) $\nabla f(x) = [e^{xy} + xye^{xy},\ x^2e^{xy}]$ - $(f \in C^1)$ $\Rightarrow$ $\nabla f(\bar{x}) = (1, 0) \Rightarrow f'(\bar{x}, v) = 2/\sqrt{13}$

(5) $\nabla f(x) = [2x-2y, -2x-2y]$ $(f \in C^1)$ $\Rightarrow \nabla f(\bar{x}) = [4, -8]$ $\quad f'(\bar{x}, u) = -4/\sqrt{2}$

(6) $\nabla f(x) = [2, 1]$ $\quad f \in C^1$, $\quad f'(\bar{x}, u) = 1/\sqrt{2}$

Ex (5) (i) The tg plane is clearly the solution to at $(5,2,1)$.

$$f_x(\bar{x})(x-x_0) + f_y(\cdot)(y-y_0) - (z-z_0) = 0.$$

ie $$4(x-2) + 2(y-1) - z + 5 = 0.$$

$$z = -5 + 4x + 2y.$$

(ii) at $(z_0, x_0, y_0) = (2,1,3)$

$$\nabla f_x(\cdot\cdot) = \left[8x^3 - 6xy,\ 2y - 3x^2\right] \Rightarrow \nabla f(1,3) = [8-18,\ 3] = [-10, 3]$$

$$z = -10(x-1) + 3(y-3) + 2 = -10x + 3y + 3.$$

Ex (6) (i) $f(0,0) = 1$ $\quad \nabla f(0,0) = [1,\ 0]$ $\quad \left(\left[(1+y)(1+x)^y,\ (1+x)^{1+y}\ln(1+x)\right]\right)$

$$H(x,y) = \begin{bmatrix} (1+y)\,y\,(1+x)^{y-1} & (1+x)^y + (1+y)(1+x)^y\ln(1+x) \\ (1+y)(1+x)^y\ln(1+x) + (1+x)^y & (1+x)^{1+y}\cdot(\ln(1+x))^2 \end{bmatrix} =$$

$$H(0,0) = \begin{bmatrix} 0 & 1+0 \\ 1 & 0 \end{bmatrix}$$ (observe $f \in C^3$ at $(0,0)$).

$\Rightarrow$

$$f(x) \approx 1 + x + \frac{1}{2}(2yx) + E_3(x)$$

(ii) $f(0,0) = 0.$

$$\nabla f(x,y) = \left[\frac{2x}{1+x^2+y^2},\ \frac{2y}{1+x^2+y^2}\right] \qquad \nabla f(0,0) = [0,0]$$

$$H(x,y) = \begin{bmatrix} \dfrac{2+2x^2+2y^2-4x^2}{(1+x^2+y^2)^2} & -\dfrac{4xy}{(1+x^2+y^2)^2} \\ \dfrac{-4xy}{(1+x^2+y^2)^2} & \dfrac{2+2x^2-2y^2}{(1+x^2+y^2)^2} \end{bmatrix}$$ at $(0,0)$ $$H(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow f(x) = 0 + \frac{1}{2}(2x^2 + 2y^2) + E_n = x^2 + y^2 + E_n(x).$$

$(f \in C^3)$

(iii) $f(0,0) = 1$

$$\nabla f(x,y) = \left[\exp\{x\cdot e^y\}\cdot e^y,\ \exp\{x\cdot e^y\}\cdot xe^y\right] \quad f\in C',$$

$$\nabla f(0,0) = [1, 0]$$

$$H(x,y) = \begin{bmatrix} \exp\{xe^y\}\{e^y\}^2 & e^y\exp\{xe^y\}+(e^y)^2x\exp\{xe^y\} \\ e^y\exp\{xe^y\}+x\{e^y\}^2\exp\{e^y\} & xe^y\exp\{xe^y\}+x^2(e^y)^2\exp\{xe^y\} \end{bmatrix}$$

$$H(0,0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad f \text{ is clearly } C^3.$$

$$f(x) = 1 + x + \frac{1}{2}(x^2+xy) + \frac{1}{2}xy + E_3(n) = 1+x+\frac{1}{2}x^2+xy+E_3(x$$

Ex (7) (i) for $\bar{x}=0$, there is a unique y st $y^3+y-x^3=0$. ($y=0$). $f_y(0,0)$ is $3y+1>0$ (for $y > -\frac{1}{3}$).
All the assumptions of the Implicit Function thm are ~~satisfyed~~ satisfied $\Rightarrow \exists!\ \Phi$ st $\phi(x)=y$ $f(x,\Phi(x))=0$ for some nbd of $(0,0)$.

$$\Phi'(x) = -\frac{f_x(x,y)}{f_y(x,y)} = \frac{+3x^2}{1+3y}.$$

(ii) $f_y(x,y) = x^y\ln x - xy^{(x-1)}$. At $\bar{x}=1$,

$f_y(0,y) = 0 - y^0 = -1 \quad \forall y\in\mathbb{R}$. Hence the assumption of the implicit function theorem are satisfied and $\exists!\ \phi \ni$ $\Phi(x)=y$ and $f(x,\Phi(x))=0$ (in some nbd of $(\bar{x},\bar{y})=(1,\bar{y})$).

$$\phi'(x) = -\frac{f_x(x,y)}{f_y(x,y)} = +\frac{y^x\ln y - yx^{(y-1)}}{x^y\ln x - xy^{(x-1)}}.$$

Ex (8) (i) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = y^2 \cdot 1 + (2ts) \cdot (2xy)$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = y^2 \cdot (2s) + (2xy) \cdot t^2$.

(ii) $\frac{\partial z}{\partial t} = \frac{(x+y) - x + y}{(x+y)^2} \cdot e^{t+s} + \frac{-(x+y) - x + y}{(x+y)^2} \cdot s e^{ts}$

$\frac{\partial z}{\partial s} = \frac{2y}{(x+y)^2} \cdot e^{t+s} + \frac{-2x}{(x+y)^2} t e^{st}$.

(Ex 9)(i) See Sydsaeter. Thm 3.13 p 126 ff
(ii) Thm 3.14 p. 128 - 129.

(Ex 10) Pick $(\bar{x},\bar{y}) \in S$. And consider a sequence $(x^n,y^n) \xrightarrow[n\to\infty]{} (\bar{x},\bar{y})$.

Observe that $f \notin C^1$ (or, at least, we do not assume $f \in C^1$).
Hence we cannot apply the general argument used in class.
However, if you look at the proof, you can see that we did use $f \in C^1$ (in the proof of the Thm. $f \in C^1 \Rightarrow f$ continuous) just to show that f_{x^i} is bounded above and below. Now we know that $f_x(x,y)$ and $f_y(x,y)$ are bounded everywhere on $S \Rightarrow$
$\exists\ M \geq |f_x(x,y)|$ $M \geq |f_y(x,y)|$ $\forall (x,y) \in S$.

Then consider

$$f(x^n,y^n) - f(\bar{x},\bar{y}) = [f(x^n,y^n) - f(\bar{x},y^n)] + [f(\bar{x},y^n) - f(\bar{x},\bar{y})]$$

Apply the one dimensional MVT (by assumption f_x, and f_y exist everywhere on S) to the functions $G(x) = f(x,y^n)$ and $\Phi(y) = f(\bar{x},y)$. We obtain

$$f(x^n,y^n) - f(\bar{x},\bar{y}) = f_x(\tilde{x},y^n)(x^n - \bar{x}) + f_y(\bar{x},\tilde{y})(y^n - \bar{y})$$

for some $\tilde{y} \in (y^n,\bar{y})$ $\tilde{x} \in (x^n,\bar{x})$.

Take the absolute value, and apply the triangular inequality and, the Cauchy-Schwartz inequality.

$$|f(x^n,y^n) - f(\bar{x},\bar{y})| \leq |f_x(\tilde{x},y^n)|\,|x^n - \bar{x}| + |f_y(\bar{x},\tilde{y})|\,|y^n - \bar{y}|.$$

By assumption, for each term of the sequence $n \to +\infty$
$|f_x(\tilde{x}_n,y^n)| \leq M$ $|f_y(\bar{x},\tilde{y}_n)| \leq M$

Whence, $\forall n$

$$0 \leq |f(x^n, y^n) - f(\bar{x}, \bar{y})| \leq M|x^n - \bar{x}| + M|y^n - y|$$

Clearly, $\forall$ sequence $(x^n, y^n) \xrightarrow[n\to\infty]{} (\bar{x}, \bar{y})$ it must be the case

$$|f(x^n, y^n) - f(\bar{x}, \bar{y})| \xrightarrow[n\to\infty]{} 0$$

Hence $f(\cdot)$ is continuous at each $(\bar{x}, \bar{y}) \in S$. $\blacksquare$

(7) (11)' $f(x,y,z) = x^2 + yz^2 + y^2x + 1 = 0.$

$f_y(x,y,z) = z^2 + 2yx$ and, at $(x,z) = (-1,1)$ it is equal to $f_y(-1,y,+1) = 1 - 2y.$

It is easy to check that there are two possible values of y st $f(-1,y,1) = 0$, $y = +2$, $y = -1$.

In both cases $f_y(\cdot) \neq 0$.

Then, we can apply the implicit function theorem, and $\exists!\ \varphi = \Phi(x,z)$ st $f(x, \Phi(x,y), z) = 0$. in a nbd of $\bar{y}$,

$$\Phi_z = -\frac{f_z}{f_{xy}} = -\frac{2zy}{z^2+2yx} = \begin{cases} y = 2 \Rightarrow \Phi'_z(-1,\ 1) = . \\ y = -1 \Rightarrow \Phi_z(1,\ 1) = \frac{2}{3} \end{cases}$$

2/3

$$\Phi_x = -\frac{f_x}{f_y} = -\frac{2x+y^2}{z^2+2yx} \Rightarrow \begin{cases} y=2 \Rightarrow \Phi'_x(-1,\ 1) = . \\ y=-1 \Rightarrow \Phi_x(-1,\ 1) = \frac{1}{3} \end{cases}$$

(Φ' is the f. around $(-1,2,1)$, Φ the one around $(-1,-1,1)$.

(12) Clearly, $f(1,1) = 1 \Longrightarrow f(x,f(x,x))$ at $x=1$, is $f(1,f(1,1)) = f(1,1) =$

$$\Phi'_x(x) = f_x(x,f(x,x)) + f_y(x,f(x,x)) \cdot \left[f_x(x,f(x,x)) \cdot f_y(x,f(x,x)) \cdot 1\right].$$

$$= a + b \cdot (ab) = a + ab^2.$$

Essentially we have to apply twice the Chain rule.

In the definition of $\Phi(x)$ we express the second variable as a function of x, $f(x,x)$. However, $f(\cdot,\cdot)$ is a function of (x,y): imposing $y = x$ we are expressing y as a function of x, say $y = \varphi(x)$.

ie

$\Phi(x) = f(x, f(x,\varphi(x))) = f(x,\theta(x))$ Hence to find $\Phi'(x)$.

we compute

$$\Phi'(x) = f_x + f_y \cdot \theta'_x(x) \qquad \text{where } \theta(x) = f(x,\varphi(x)) \text{ implies}$$

$$\theta'_x(x) = f_x + f_y \cdot \varphi'(x) = f_x + f_y$$

Ex (13). $\forall i \quad f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable $\Rightarrow \exists$ a linear map T_{ix} st

$$f_i(x+v) - f_i(x) = T_{ix}(v) + \|v\| E_i(v) \qquad E_i(v) \xrightarrow[\|v\| \to 0]{} 0.$$

$\forall i: f_i : S \subset \mathbb{R}^n \to \mathbb{R}$.

The function $F(x) = \sum_{i=1}^n f_i(x) : S \subset \mathbb{R}^n \to \mathbb{R}$.

I claim that the linear map given by the sum of the linear maps T_{ix} is the required one.

ie $T_x = \sum_{i=1}^n T_{ix}$.

To show: it is sufficient to show that given $v \in \mathbb{R}^n$,

$$F(x+v) - F(x) - T_x(v) = \|v\| E(v) \quad \text{and} \quad E(v) \xrightarrow[\|v\| \to 0]{} 0.$$

Clearly, this expression is equivalent to

$$\sum_{i=1}^n f_i(x+v) - \sum_{i=1}^n f(x) - \sum_{i=1}^n T_{x_i}(v) = \sum_{i=1}^n \left[f_i(x+v) - f(x) \right] - \sum_{i=1}^n \langle \nabla f_i(x), v \rangle =$$

$$= \sum_{i=1}^n \left[f_i(x+v) - f(x) - \langle \nabla f_i(x), v \rangle \right] = \|v\| \sum_{i=1}^n E_i(v).$$

The last equality follows by assumption. and clearly, if $\forall i \; E_i(v) \xrightarrow[\|v\| \to 0]{} 0$ then $\sum_{i=1}^n E_i(v) \xrightarrow[\|v\|]{} 0$. Hence $F(x)$ is differentiable.

The second statement follows immediately by the fact $T_x = \sum_{i=1}^n T_{ix}$.

Ex (14). Look at Apostol, Mathematical Analysis and let $n = 1$.

Ex (15).

(a) $\max U(x)$ st. $\sum_{i=1}^n p^i x^i = y$.

Consider the Lagrangian $\Lambda(x, \lambda) = \prod_{i=1}^n x^{i\alpha} - \lambda \left(\sum_{i=1}^n p^i x^i - y \right)$.

FOC $\quad \Lambda_{x^i} = \dfrac{\alpha \prod_{i=1}^n x^{i\alpha}}{x^i} - \lambda p^i = 0 \qquad \forall i$

$\Lambda_\lambda = -\sum_{i=1}^n p^i x^i - y = 0$

multiply Λ_{x^i} by x^i to obtain $\alpha \prod_{i=1}^{n} x^{i\alpha} = \lambda p^i x^i$

Summing over i, $(n\alpha) \prod_{i=1}^{n} x^{i\alpha} = \lambda \sum_{i=1}^{n} p^i x^i = \lambda y$ (because $\Lambda_\lambda = 0$)

Hence

$\lambda^* = (n\alpha) \prod_{i=1}^{n} x^{i\alpha} / y$ and substituting into the expression for Λ_{x^i} we get

$$x^i = \frac{y}{np^i} \quad \forall i$$

(2) $U(x)$ is well known to be concave, given that $f(x_i, x_2) = x_i$ and $g(x_1, x_2) = 2x^2$ are both linear, hence concave.

$U(x)$ is not differentiable, however, it is self evident that, at a maximum, $x^1 = 2x^2$ ($p >> 0$).

Hence, substitute $2x^2$ to x^1 in the budget constraint to get

$$2p^1x^2 + p^2x^2 = y \implies x^{2*} = \frac{y}{p^2 + 2p^1}$$

and $x^{1*} = \dfrac{2y}{p^2 + 2p^1}$. It is easy to see that (x^{1*}, x^{2*}) is actually a maximum.

(3) $\max \quad u(x) = A\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right)^{-\frac{1}{\varrho}}$. s.t. $p_1 x_1 + p_2 x_2 = y$.

It is well known that the CES function is concave (for $\varrho \geq 1$) and $x_1, x_2 >> 0$). (See Sydsaeter p. 252).

It is clear that if x maximize $F(u(x))$ where F is a strictly increasing map, then x will also maximize $u(x)$. [This fact is quite often usefull].

Let $F(\cdot) = \ln(\cdot)$, and consider the problem.

$$\max \ \ln A - \frac{1}{\varrho} \ln\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right) \quad \text{st. } p_1 x_1 + p_2 x_2 = y.$$

$$\Lambda(x) = \ln A - \frac{1}{\varrho} \ln\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right) - \lambda\left(p_1 x_1 + p_2 x_2 - y\right).$$

FOC

$$\Lambda_{x_1} = \left(-\frac{1}{\varrho}\right) \frac{(-\varrho)\,\delta\, x_1^{-\varrho-1}}{\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right)} - \lambda p_1 = 0$$

$$\Lambda_{x_2} = \left(-\frac{1}{\varrho}\right) \frac{(-\varrho)(1-\delta)\, x_2^{-\varrho-1}}{\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right)} - \lambda p_2 = 0$$

$$\Lambda_\lambda = -(p_1 x_1 + p_2 x_2 - y) = 0$$

$$\Lambda_{x_1} = \Lambda_{x_2} \Rightarrow \frac{\delta\, x_1^{-(\varrho+1)}}{p_1\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right)} = \frac{(1-\delta)\, x_2^{-(\varrho+1)}}{p_2\left(\delta x_1^{-\varrho} + (1-\delta) x_2^{-\varrho}\right)}.$$

$$\Rightarrow \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{p_2}{p_1}\right)\right]^{-\frac{1}{\varrho+1}} x_1 = x_2.$$

Substitute it in $\Lambda_\lambda = 0$ to get.

$$p_1 x_1 + p_2 \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{p_2}{p_1}\right)\right]^{-\frac{1}{\varrho+1}} x_1 = y.$$

and

$$x_1 = \frac{y}{p_1 + p_2\left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{p_2}{p_1}\right)\right]^{-\frac{1}{\varrho+1}}}$$

$$x_2 = \frac{y}{p_2 + p_1\left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{p_2}{p_1}\right)\right]^{\frac{1}{\varrho+1}}}$$

(5) $\max \quad u(x) = \sum_{i=1}^{n} x_i^{\alpha} \quad (\alpha > 0, \alpha \neq 1) \quad \sum p^i x^i = y.$

If $\alpha > 1$ $(x_i \geq 0)$, $u(x)$ is clearly a convex function (strictly convex if $x_i \gg 0$). The Hessian matrix is.

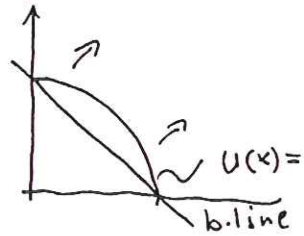
$$U_{xx} = \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2} & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & 0 & \alpha(\alpha-1)x_n^{\alpha-2} \end{bmatrix}.$$

$x \gg 0 \Rightarrow$ U_{xx} positive def if $(\alpha-1) > 0$
U_{xx} negative def. if $(\alpha-1) < 0$.

If $(\alpha-1) > 0$, we will get the corner solution

$$x_i = \begin{cases} 0 & \text{if } p_i > \min_{j=1..n} \{p_j\} \\ 0 \text{ or } \dfrac{y}{p_i} & \text{if } p_i = \min_{j=1,..n} (p_j) \end{cases}$$

It is easy to see that $(n=2)$ the contour of $u(x) = k$ is, for $\alpha > 1$. The result follows immediately.



If $\alpha < 1$, $U(x)$ is strictly concave on $\mathbb{R}^n_{++}$.

Consider the Lagrangian function

$$\Lambda(x) = \sum_{i=1}^{n} x_i^{\alpha} - \lambda\left(\sum_{i=1}^{n} p^i x^i = y\right).$$

$$\Lambda_{x_i} = \alpha x_i^{\alpha-1} - \lambda p^i = 0 \ \forall i.$$

$$\Lambda_\lambda = \left(-\sum p_i^i x_i^{\alpha} + y\right) = 0.$$

Put $\Lambda_{x_i} = \Lambda_{x_1}$ $\forall i$: $\dfrac{p^i}{p^1} x_1^{\alpha-1} = x_i^{\alpha-1}$.

Hence, $\forall i$, $\left(\dfrac{p^i}{p^1}\right)^{\frac{1}{\alpha-1}} x_1 = x_i$. Substitute this expression in Λ_λ.

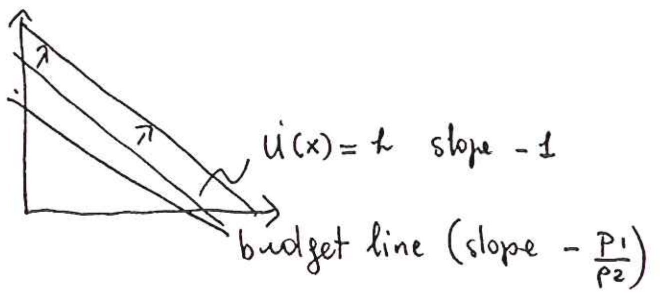
$$\sum_{i \neq 1}^{n} \left[\frac{p^i}{p^1}\right]^{\frac{1}{\alpha-1}} p^i x_1 + \left(\frac{p^1}{p^1}\right)^{\frac{1}{\alpha-1}} p_1 x_1 = y.$$

$$\Rightarrow \left(\sum_{i=1}^{n} (p_i)^{\frac{1+\alpha-1}{\alpha-1}} x_1\right)\left(\frac{1}{p^1}\right)^{\frac{1}{\alpha-1}} = y \Rightarrow x_1 = \frac{y \cdot (p^1)^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n} (p_i^i)^{\frac{\alpha}{\alpha-1}}}$$

for x_i^i :-
$$x_i = \frac{y\,(p^i)^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n} (p^i)^{\alpha/\alpha-1}}.$$

(6) $\max \quad U(x) = \sum_i x^i \quad \text{st} \quad \sum_i p^i x^i = y$

in two dimension



If $p_i/p_J \neq 1$, the only solution is a corner one, hence, we cannot find the maximum using the Lagrange technique.
However it is self evident

$$x^i = \begin{cases} 0 & \text{if } p^i > \min_J \{p^J\} \\ \in \left[0, \frac{y}{p^i}\right] & \text{if } p^i = \min_J \{p^J\}. \end{cases}$$

Clearly the optimal solution is not unique, because $u(x)$ is concave but clearly not strictly concave.