MACROECONOMICS I
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Contents

1 Introduction

The aim of this course is to provide at least a part of the technical tools necessary to understand the macroeconomic literature currently published in professional journals, research reports, institutional documents and the like. Emphasis will be placed more on the methods for constructing models than on their factual premises and implications. Macroeconomics offers a wide variety of models, which means that there is also a wide variety of alternative theories that differ from each other because of their specific assumptions and of the different logical structures used in processing them. In order to make one's own mind in this maze of competing offers one must first of all be able to understand the language of models, and this cannot be done without some basic knowledge of the mathematical and statistical structures that provide the scaffolding used in the construction. Given its closeness to economic policy, and the fact of being frequently used to support political ideologies, macroeconomics has always been a highly controversial field. In these lectures, the actual contents of these controversies will be played down, while emphasis will be placed on the technical operations involved in building the models used to support conflicting views.

Throughout the course the main reference model will be the Dynamic Stochastic General Equilibrium model (DSGE), i.e. a theory of the economy based on a combination of three items: (1) General Economic Equilibrium, defined at the level of macro variables; (2) Dynamics, generated by the fact that agents' choices (consumption and investment) affect both the present and the future of the economy; (3) Uncertainty, due to the fact that the future cannot be entirely known at the time a decision is made, and represented by the interplay between personal choices and impersonal stochastic factors in determining the evolution of the system.

DSGE is neither the most convincing model on offer nor the most used in empirical applications of macroeconomics. After the worldwide financial crisis started in the US in 2007 and subsequent depression it has been heavily criticized. So far, however, critics have not been able to provide better and at least equally tractable alternatives and the DSGE has remained in saddle. In many quarters it is still considered to be the model which embodies the core of modern macroeconomic theory, a sort of doctrinal barycenter reference to which is inevitable in all macroeconomic discussion. This is why it is taken as the core also of this course.

These lectures refer to Wickens, *Macroeconomic Theory*, 2nd edition 2011, especially chapters 2, 4, 6.3, 9, 11 (corresponding to 2, 4, 6.3, 9, 10 of the first edition, 2008). Other readings will be suggested as we proceed with the

course.

As a general introduction to the recent history and present state of macroeconomic theory, the survey contained in chapter 1 of Wickens 2011 may be sufficient. It may be useful also to have a look at some of the following: E. Phelps, *Seven schools of macroeconomic thought*, Oxford 1990; Chari, Kehoe, "Modern macroeconomics in practice", and Mankiw, "The macroeconomist as scientist and engineer", both in *Journal of Economic Perspectives*, 20:4 (Fall 2006); Snowdon, Vane, Wynarczyk, *A modern guide to Macroeconomics*, Elgar 1994 and, more updated, Snowdon, Vane, *Modern Macroeconomics*, Elgar 2005 (the texts indicated so far are available at the Biblioteca di Polo). A recent historical assessment is found in M. De Vroey, *A history of Macroeconomics from Keynes to Lucas and beyond*, Cambridge UP 2016. As to the post-crisis outlook: N. Kocherlakota, "Modern macroeconomic models as tools for economic policy" (Federal Reserve Bank of Minneapolis 2009), and P. Romer, "The trouble with Macroeconomics" (2016), are freely available online; G. Saint-Paul, "The possibility of ideological bias in structural macroeconomic models", *American Economic Journal: Macroeconomics*, 10:1 (2018) is an interesting exercise.

Outline of the contents of this year's (2018) lectures. Basic DGE model; methods of dynamic optimization in a deterministic environment; steady states; exogenous growth; OLG models; sticky prices and adjustment to shocks; uncertainty, stochastic processes, expectations; dynamic optimization in a stochastic environment; applications to portfolio choice; the Capital Asset Pricing Model (CAPM) and the contingent claims approach.

2 The basic DGE model

This whole section refers to Wickens, chs 2 and 4.

As a first approximation to DSGE, we consider an economy set in a deterministic environment. This means that the past determines the future completely and according to perfectly known causal processes, without any chance of unpredictable events ('novelties', or 'innovations') affecting the state of the economy. Having complete knowledge of past history up to the present and being able to understand the causal laws regulating the economy, agents can perfectly predict the future consequences of decisions made in the present. Determinism plus perfect information is equivalent to perfect forecast, or the total absence of uncertainty.

For a description of the components of the model-economy see Wickens 2.3. One of the main features of the model is the strong simplification afforded by the device of the 'representative agent': the behavior of a population of heterogeneous individuals (households or firms) is reduced to the behaviour of one single individual whose characteristics are assumed to be 'representative' – a sort of average – of all the individuals in the population. This is a much criticized assumption, not only because it wipes out all the macroeconomic phenomena ascribable to differences in the characteristics of individuals belonging to different social groups – for example, the macroeconomic effects of inequality in the distribution of wealth – but also for purely theoretical reasons. Assuming that the choices of a population can be studied as if they were choices of a fictitious individual implies that all the aggregate behaviour functions – basically, aggregate demand and supply – are supposed to have the same characteristics as the corresponding individual behaviour functions. This statement contradicts a theorem of microeconomic general equilibrium theory, known as the Sonnenschein-Mantel-Debreu theorem, that says that, in general, not all the characteristics of individual demand functions carry over to the resulting aggregate demand function: the latter may be a function of a different type, so that no individual function can be 'representative' of the behaviour of the whole population. Thus, by assuming that a representative agent exists macroeconomic equilibrium theory comes into conflict with the microeconomic equilibrium theory from which it is alleged to derive. [For further reading on this point see A.P. Kirman, "Whom or what does the representative individual represent?", *Journal of Economic Perspectives*, 6:2 (Spring 1992).]

For all its theoretical and empirical flaws, however, the assumption that the representative agent can be an adequate device to deal with heterogeneous populations is extremely effective in granting that the model is analytically tractable without excessive complication. Theories that do not resort

to this assumption are certainly more realistic, theoretically sound and richer in implications, but unfortunately they also tend to be much more complex, sometimes beyond analytical tractability. An example in point is the class of models called ACE (Agent-based Computable Economics): individual agents are treated as being different from each other but the dynamics generated by their interaction gives rise to mathematical systems that often turn out to be unsolvable. In such cases the only way to explore their properties is by means of computer simulations. We have here an instance of the fact that tractability and plausibility of assumptions conflict with each other quite often.

We follow Wickens in presenting the basic DGE model in two versions: (1) As the model of a centrally planned economy in which there is only one decision maker, the central planner, who decides the consumption-investment path of the economy with the aim of maximizing an index of social welfare based on the representative household's utility function. In this version, equilibrium means that the planner manages to achieve his object by using his total control of the economy, maximization being subject to constraints of compatibility between consumption-investment and production. (2) As the model of a decentralized economy in which individual decisions are coordinated through the markets for labour, commodities and bonds. Here, all agents are supposed to be price-taker, i.e. markets are assumed to be perfectly competitive; and equilibrium means that prices guarantee that all the transactions (purchases and sales) required in order to implement individual decisions can be carried out, or equivalently, that all individual decisions are compatible among themselves. In both cases, provided the social welfare function in (1) and the representative household's utility function in (2) are the same, also the equilibrium path turns out to be the same. It is thus demonstrated that, under the hypotheses of the model, equilibrium competitive prices as in (2) guarantee a socially optimal outcome as in (1). This result extends the propositions known in microeconomics as the "fundamental theorems of welfare economics" to the level of macroeconomic theory.

2.1 DGE in a centrally planned economy

Further simplifications, that can be removed at a more advanced level of the analysis: international trade does not exist or, if it exists, is fully balanced (trade balance is permanently zero); labour (or its complement, leisure) is not an argument of the representative household's utility function; output (GDP) is treated as a versatile commodity that may serve either as capital good or as consumption good.

The economic variables measure events that occur in time, and this is

treated as a discrete variable measured in integer numbers. Take 0 as the current date; any positive or negative integer s refers to any date in the future or the past.

Constraints of the planner are: (i) the technology, represented by the aggregate production function $Y_s = F(K_s)$, which is assumed to have $F' > 0$, $F'' < 0$ and to satisfy the Inada conditions (NB: the latter exclude $F'' = 0$ uniformly, contrary to what is assumed by Wickens on p. 16). Employed labour is assumed to be constant so that it can be treated as a parameter of the function. No technical progress is assumed.

(ii) The budget constraint, which for the whole economy coincides with the national accounting equation (with zero balance of trade)

$$C_s + I_s = Y_s$$

Given a constant rate $\delta \epsilon (0, 1)$ of depreciation of capital goods due to age, we get $K_{s+1} = K_s(1 - \delta) + I_s$, and by replacing I_s in the previous equation

$$C_s + K_{s+1} = Y_s + K_s(1 - \delta)$$

By introducing constraint (i) in the last equation, we finally get

$$C_s + K_{s+1} = F(K_s) + K_s(1 - \delta)$$

which compounds technical and budget constraints in one single equation.

A relaxed form of this overall constraint is

$$F(K_s) + K_s(1 - \delta) - C_s - K_{s+1} \geq 0$$

Relaxation allows for the possibility of inefficiency in production ($Y_s < F(K_s)$) and/or waste in the use of final product ($C_s + I_s < Y_s$).

On date 0 the capital stock of the economy K_o is given, being the result of past choices of consumption-investment. The optimal plan from 0 to any future date T consists in choosing a consumption-accumulation sequence $\{C_s, K_{s+1} : s = 0, 1, 2...T\}$ so as to maximize the utility of the representative consumer (household), written as a generic function of the expected consumption profile

$$U = U(C_o, C_1...C_T)$$

In this form the utility function is more general than the standard sum of discounted utilities used by Wickens and by and large in macroeconomic literature. We shall restrict to that more specific form when necessary. By now we only assume that the function has positive first derivative and negative second derivative with respect to each argument C_s, and that it satisfies the Inada condition for $C_s \to 0$.

Summing up, and taking the constraints in the relaxed form, the DGE of the planned economy is a plan $\{C_s, K_{s+1} : s = 0, ...T\}$ that solves the following problem: given the current K_o, and a planning period from 0 to T (in a later generalization we shall admit T to be infinite), maximize U by choosing a plan that at each $s = 0, ...T$ satisfies the restriction

$$\begin{aligned} F(K_s) + K_s(1-\delta) - C_s - K_{s+1} &\geq 0 \\ C_s, K_{s+1} &\geq 0 \end{aligned}$$

Notice that if $K_o = 0$ the only feasible plan is $C_s = K_{s+1} = 0$ at all s. Thus, K_o is always assumed to be strictly positive for the problem to make sense.

The solution(s) to the problem, if these exist, can be found by means of several methods, some of which are especially devised for dynamic problems of this kind. The Lagrangean method is not particularly well-suited for dynamic problems but in the case of finite horizon it works as efficiently as any other. We will examine some of these methods in sequence, starting with Lagrange.

2.2 Optimization via Lagrange

I assume the basics of the method to be known from microeconomics. Briefly, the steps are: (i) construct the Lagrange function $\mathcal{L}$ out of the data of the original constrained maximization problem; (ii) check that the objective function and the constraints satisfy concavity and 'constraint qualification'; (iii) look for the "saddle points" (or min max) of $\mathcal{L}$ (if any); (iv) use the properties of the saddle points to characterize the properties of the corresponding solutions to the original problem.

Here, the Lagrange function can be written as

$$\begin{aligned} \mathcal{L}(C, K; \lambda_o; \lambda, \mu, \gamma) \equiv \lambda_o U(C_o, ...C_T) + \\ + \sum_{s=0}^{T} \left\{ \lambda_{s+1} \left[F(K_s) + K_s(1-\delta) - C_s - K_{s+1} \right] + \mu_s C_s + \gamma_{s+1} K_{s+1} \right\} \end{aligned}$$

where $\lambda_o, \lambda = (\lambda_1, ...\lambda_{T+1}), \mu = (\mu_o, ...\mu_T), \gamma = \left(\gamma_1, ...\gamma_{T+1}\right)$ are the Lagrange multipliers, all of them definitionally constrained to be non-negative, with λ_o strictly positive.

Exercise 1 *Check that the conditions of concavity and 'constraint qualification' hold (provided $K_o > 0$).*

A saddle point of $\mathcal{L}(C, K; \lambda_o; \lambda, \mu, \gamma)$ is a maximum with respect to the variables $C = (C_o, ...C_T), K = (K_1, ...K_{T+1})$, and a minimum with respect to λ, μ, γ (λ_o is treated as an arbitrarily given parameter). According to a proposition not difficult to demonstrate, if $(C, K; \lambda_o; \lambda, \mu, \gamma)$ are the coordinates of a saddle point of $\mathcal{L}$, then also $(C, K; a\lambda_o; a\lambda, a\mu, a\gamma)$, where a is an arbitrary positive real number, are coordinates of a saddle point. In other words, the vector $(\lambda_o, \lambda, \mu, \gamma)$ is determined by the solution but only up to its scale. The Lagrange multipliers can therefore be taken at a scale a such that $a\lambda_o = 1$.

Notice that $\mathcal{L}$ is concave with respect to the arguments C and K (see exercise 1) and linear with respect to λ, μ and γ. As a consequence, the FOCs of maximum with respect to the former, and minimum with respect to the latter, provide both necessary and sufficient conditions of max and min. Notice also that in maximizing $\mathcal{L}$ with respect to C and K, these variables must be treated as free (the constraints of the original problem will take care that in the solution these variables turn out to be non-negative), while in minimizing with respect to λ, μ and γ the non-negativity constraint must be taken into account explicitly.

Exercise 2 *Set $\lambda_o = 1$ and check that the FOCs characterizing the saddle points of $\mathcal{L}$ are the following*

$$U'_s - \lambda_{s+1} + \mu_s = 0 \qquad s = 0, ...T \tag{1}$$

$$\lambda_{s+1}\left[F'_s + 1 - \delta\right] - \lambda_s + \gamma_s = 0 \qquad s = 1, ...T \tag{2}$$

$$-\lambda_{T+1} + \gamma_{T+1} = 0 \tag{3}$$

$$F(K_s) + K_s(1-\delta) - C_s - K_{s+1} \geq 0 \text{ and} \tag{4}$$

$$\lambda_{s+1}\left[F(K_s) + K_s(1-\delta) - C_s - K_{s+1}\right] = 0 \qquad s = 0, ...T$$

$$C_s \geq 0 \text{ and } \mu_s C_s = 0 \qquad s = 0, ...T \tag{5}$$

$$K_{s+1} \geq 0 \text{ and } \gamma_{s+1} K_{s+1} = 0 \qquad s = 0, ...T \tag{6}$$

From an analysis of the FOCs found in solving exercise 2 we can derive the main characteristics of the optimal consumption-investment plan.

From condition (1) and the fact that U' is always positive, $\lambda_{s+1} > 0$ for all $s = 0, ...T$; hence, from (4)

$$F(K_s) + K_s(1-\delta) = C_s + K_{s+1} \qquad s = 0, ...T \tag{7}$$

Besides, from (3) and $\lambda_{T+1} > 0$ it follows that $\gamma_{T+1} > 0$ and therefore, from (6),

$$K_{T+1} = 0 \tag{8}$$

Next we prove that the consumption sequence $\{C_s : s = 0, ...T\}$ must be strictly positive. Briefly, if at any t it turns out that $C_t = 0$, then (i) $C_s = 0$ at all $s < t$ (prove this by using conditions (1) and (2)), and also (ii) $C_s = 0$ at all $s > t$ (again, prove this by means of (1), (2), (i) and the hypothesis that $K_o > 0$). Thus, the consumption sequence is either strictly positive or strictly zero. The latter case is impossible because it would contradict $K_o > 0$ and (8) $K_{T+1} = 0$. Hence the sequence must be strictly positive.

From the last result it follows that the sequence $\{K_s : s = 0, ...T\}$ must be strictly positive as well (prove this by means of (7) and $K_o > 0$). The two results together imply $\mu_s = \gamma_{s+1} = 0$ for all s from 0 to $T - 1$ and $\mu_T = 0$.

We thus pass from the conditions in general form (1) to (6) to the following reduced set of conditions

$$\begin{aligned} U'_s &= \lambda_{s+1} && s = 0, ...T \\ \lambda_{s+1}\left[F'_s + 1 - \delta\right] &= \lambda_s && s = 1, ...T \\ F\left(K_s\right) + K_s\left(1 - \delta\right) &= C_s + K_{s+1} && s = 0, ...T \\ K_{T+1} &= 0 \end{aligned}$$

By replacing the first into the second condition we further reduce the necessary and sufficient characterization of an optimal path to the following three conditions

$$U'_s\left[F'_s + 1 - \delta\right] = U'_{s-1} \qquad s = 1, ...T \tag{9}$$

$$F\left(K_s\right) + K_s\left(1 - \delta\right) = C_s + K_{s+1} \qquad s = 0, ...T \tag{10}$$

$$K_{T+1} = 0 \tag{11}$$

The first of the three conditions is called Euler condition or Euler equation. The economic interpretation is in terms of the equality between marginal rate of substitution in consumption and marginal rate of transformation in production. In this case substitution and transformation applies to consumption/production of the same commodity at two different dates considered as if it were consumption/production of two different commodities. The fact that Euler condition amounts to an equality between rates of intertemporal substitution/transformation can be checked by solving the next

Exercise 3 *Prove that* $\frac{U'_{s-1}}{U'_s}$ *measures the representative household's marginal rate of substitution between consumption at* $s - 1$ *and consumption at* s*, and that* $F'_s + 1 - \delta$ *measures the technical rate of transformation between the same.*

2.3 Dynamic properties of the optimal plan

We now examine the dynamic implications of the equations (9) and (10) that characterize the optimum consumption-investment plan. For the sake of simplicity we shall restrict the utility function to the case of a function U with separable marginal utilities, in the sense that each partial derivative U'_s is assumed to be a function of C_s alone, independently of consumption levels at dates different from s.

Given this restriction, we rewrite the two equations as follows

$$\begin{aligned} U'_{s+1}\left[F'\left(K_{s+1}\right)+1-\delta\right]-U'_s &= 0 \qquad (12)\\ F\left(K_s\right)+K_s\left(1-\delta\right)-C_s-K_{s+1} &= 0 \qquad (13)\end{aligned}$$

where U'_t (with $t = s, s+1$)means partial derivative of U with respect to C_t and is a function of C_t alone.

As s ranges from 0 to $T-1$in (12), and from 0 to T in (13), we have T couples of equations like (12)-(13) plus the residual equation (13) that links C_T, K_T to K_{T+1}. It is not difficult to check that each couple of equations links the pair (C_s, K_s) to (C_{s+1}, K_{s+1}) in a one-to-one, invertible way. It thus defines a *rule of transition* (a "law of motion" in classical mechanics terminology) connecting two successive states of the economy characterized by means of the levels of consumption and capital stock. The rule can be used either forwards, knowing (C_s, K_s) determine the next state (C_{s+1}, K_{s+1}); or backwards, knowing (C_{s+1}, K_{s+1}) determine the preceding state (C_s, K_s).

If one starts from a given initial state (C_o, K_o), repeated forward applications of the rule of transition T times determine a unique sequence of states $\{(C_s, K_s) : s = 1, ...T\}$, plus K_{T+1} by applying equation (13) with $s = T$. Starting from the other end, take a pair (C_T, K_{T+1}) and apply the rule of transition backwards T times: the result is a unique sequence $\{(C_{s-1}, K_s) : s = 1, ...T\}$, plus K_o by applying equation (13) with $s = 0$. Thus, again, one gets a unique sequence of states from T to 0.

More in general, it is possible to demonstrate the following result: if we assign given values to two variables arbitrarily chosen from the set $\{C_o, ..., C_T, K_o, ..., K_{T+1}\}$, then there is one and only one set of values of the remaining $2T+1$ variables such that the $2T+1$ equations (12)-(13) are satisfied. In other words, system (12)-(13) has a unique solution provided two "degrees of freedom" (dof) are saturated assigning arbitrary values to two of its unknowns. This can be done in as many ways as there are elements in $\mathbb{R}^2$. But of course, in order for it to be economically meaningful, the solution must not contain negative terms. Therefore, not all arbitrary assignments will turn out to be economically meaningful.

The solution of the system identifies a sequence of states $\{(C_s, K_s) : s = 0, ...T\}$ followed by a unique level of capital stock K_{T+1} remaining at the end. The sequence can be seen as a function mapping the integer variable $s = 0, ...T$ into $\mathbb{R}^2$ and describing the position of the economy at each date, its 'trajectory' in the $\mathbb{R}^2$ space: the 'story' or the 'dynamics' of the economy over the $[0, T]$ interval. It is usual to take the initial capital stock K_o as given. We also know, from equation (11) in section 2.2, that the final stock K_{T+1} is bound to be zero if the trajectory has to be optimal. The two conditions concerning the initial and terminal capital stock over the planning period are called 'boundary conditions'. Once these have been fixed, the two dof of the system (12)-(13) are saturated and the optimal trajectory is uniquely determined, as can be seen by solving the following exercise.

Exercise 4 *Argue that equations (12)-(13), together with boundary conditions $K_o = \overline{K}_o$ (a given value) and $K_{T+1} = 0$, identify one and only one optimal sequence in the finite T-horizon problem of optimal planning. [Hint: show that, given $\overline{K}_o$, C_T and K_T are continuous functions of C_o, the former strictly increasing, the latter strictly decreasing. Find the relationship between C_T and K_T that the optimal plan must satisfy. Now you have one equation with the only unknown C_o. Show that if the equation has a solution, this is unique...]*

As in all finite horizon optimal planning problems, the second boundary condition in the form $K_{T+1} = 0$ implies that the planner is not interested in the survival of the economy beyond the end of the planning period. This unrealistic implication can be removed simply by assuming that the planner has a second target concerning the capital stock at the end of the planning period: for example, to avoid that the final stock is lower than the initial one, $K_{T+1} \geq \overline{K}_o > 0$. The changes in the characteristics of the optimal plan are shown by the solution to the next exercise.

Exercise 5 *Like the previous exercise, but now assume that the second boundary condition is given by $K_{T+1} = \overline{K}_{T+1}$, a given positive value. Show that there is an upper bound on $\overline{K}_{T+1}$ beyond which the problem has no economically meaningful solution, and that the utility associated to the optimum accumulation plan is inversely related to $\overline{K}_{T+1}$.*

Finally, an important implication of the uniqueness of the optimal plan associated with given boundary conditions is the following: different optimal plans, i.e. plans associated with different boundary conditions, cannot have

any state (C_s, K_s) in common. This is demonstrated easily. If two different plans had a common state at some $s \in (0,T)$, then they would also have common states (C_t, K_t) at all $0 \leq t < s$ and $s < t \leq T$. But then they would also have the same boundary conditions, and therefore they would be the same plan. Thus, if a state belongs to an optimal plan for given boundary conditions, there are no other boundary conditions for which that state can be optimal.

2.4 Infinite horizon: recursive utility

Now we extend the analysis of optimal planning to the case of an infinite horizon $T = \infty$. We shall see that, thanks to convenient restrictions concerning the shape of the utility function, the problem takes on a recursive structure that enables us to use simple mathematical methods.

With $T = \infty$ utility turns out to be a function of an infinite number of arguments. In order to simplify formal manipulations we assume that it has the so-called 'recursive property' explained here below.

Generally speaking, a function with an infinite number of arguments of the type $y = f(x_0, x_1, x_2, ...)$ is said to be 'recursive' if there exist two functions, call them $\varphi : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R}^2 \to \mathbb{R}$, such that the value of f can be written in the following (separable) way

$$f(x_0, x_1, x_2, ...) = W(\varphi(x_0), f(x_1, x_2, ...))$$

The W function is called the "aggregator function". The function indicated with φ isolates the first argument of the sequence from all the others. The argument of f in the right hand side of the formula is the same sequence as in f in the left hand side, but for the elimination of its first term ('truncated' sequence).

As f appears on the right of the formula as second argument of the aggregator, by applying the same kind of decomposition a second time we get

$$f(x_0, x_1, ...) = W(\varphi(x_0), W(\varphi(x_1), f(x_2, ...)))$$

an so on. After n iterations we get

$$f(x_0, x_1, ...) = W(\varphi(x_0), W(\varphi(x_1), W(...W(\varphi(x_n), f(x_{n+1}, ...))...)))$$

Iterations may continue to any extent; in general, it is always possible to write

$$f(x_0, x_1, ...) = \lim_{n \to \infty} W(\varphi(x_0), W(\varphi(x_1), W(...W(\varphi(x_n), f(x_{n+1}, ...))...)))$$

Notice that the last formula isolates each of the first n arguments from all the others by means of the φ function, while the residual arguments in the tail from $n+1$ to infinity remain attached to each other. An additional hypothesis, that is usually assumed to hold in economic applications, is the hypothesis called 'complete separability'. This means assuming that the function can be written in the form

$$\begin{aligned} f(x_0, x_1, ...) = \\ = \lim_{n\to\infty} W(\varphi(x_0), W(\varphi(x_1), W(...W(\varphi(x_n), f(x_{n+1}, ...))...))) \\ = W(\varphi(x_0), W(\varphi(x_1), W(... \quad ...))) \end{aligned}$$

with the tail of unseparated arguments disappearing entirely from the arguments of W.

A recursive utility function is a function that can be written as

$$U(C_0, C_1, C_2, ...) = W(u(C_0), U(C_1, C_2, ...))$$

In the case of utility the first argument of the aggregator is called the 'instantaneous utility' of a flow of consumption. Total utility is thus decomposed into two components, the instantaneous utility determined by C_o and the total utility of the truncated sequence $C_1, C_2, ...$ Iterating the decomposition n times we get

$$\begin{aligned} U(C_0, C_1, C_2, ...) = \\ = W(u(C_0), W(u(C_1), W(u(C_2), ...W(u(C_n), U(C_{n+1}, C_{n+2}, ...))...))) \end{aligned}$$

a function of a series of instantaneous utilities and the total utility of the residual tail of consumption levels. Here as follows u will always be assumed to be increasing, concave, with first derivative tending to infinity for $C_s \to 0$. It is natural to assume that W is an increasing function with respect to both its arguments.

A particularly simple and much used case of recursive utility is that in which the aggregator is a linear function of the type $W(u, U) = u + \beta U$. The parameter β is usually assumed to be positive and less than one and is interpreted as a subjective *factor* of discount, measuring the representative consumer's time preference or degree of 'impatience' (the more impatient individual has β nearer to zero). Alternatively, if we call θ the subjective *rate* of discount, we have

$$\beta = \frac{1}{1+\theta}$$

and impatience grows as θ increases from zero to infinity.

By operating iteratively with the linear aggregator

$$\begin{aligned} U(C_0, C_1, ...) &= u(C_0) + \beta U(C_1, C_2, ...) \\ U(C_1, C_2, ...) &= u(C_1) + \beta U(C_2, C_3, ...) \\ U(C_2, C_3, ...) &= u(C_2) + \beta U(C_3, C_4, ...) \end{aligned}$$

we get, by way of substitution,

$$\begin{aligned} U(C_0, C_1, ...) = u(C_0) + \beta u(C_1) + \beta^2 u(C_2) + ... \\ ... + \beta^n u(C_n) + \beta^{n+1} U(C_{n+1}, C_{n+2}, ...) \end{aligned}$$

In this case the hypothesis of complete separability (see above) takes on a particularly simple form. From

$$\begin{aligned} U(C_0, C_1, ...) &= \lim_{n\to\infty} \left[\sum_{s=0}^{n} \beta^s u(C_s) + \beta^{n+1} U(C_{n+1}, C_{n+2}, ...) \right] \\ &= \lim_{n\to\infty} \sum_{s=0}^{n} \beta^s u(C_s) + \lim_{n\to\infty} \beta^{n+1} U(C_{n+1}, C_{n+2}, ...) \end{aligned}$$

it is clear that complete separability, i.e. the hypothesis

$$U(C_0, C_1, ...) = \lim_{n\to\infty} \sum_{s=0}^{n} \beta^s u(C_s) = \sum_{s=0}^{\infty} \beta^s u(C_s)$$

according to which U resolves entirely into a sum of instantaneous utilities independent of each other, is equivalent to

$$\lim_{n\to\infty} \beta^{n+1} U(C_{n+1}, C_{n+2}, ...) = 0$$

Since $\beta < 1$, a sufficient condition for this is the boundedness of total utility.

2.5 Infinite horizon: dynamics

Although the Lagrange method is not the most qualified method for solving problems of dynamic optimization with infinite horizon, there is no obstacle to continuing to use it either. Apart from the specific form of the utility function, the only difference in constructing the Lagrange function with $T = \infty$ (see section 2.2) is in the fact that, the number of constraints being infinite, also the sums that appear in the formula have an infinite number of addends. It may turn out that for some values of the variables and of the Lagrange multipliers these sums add up to infinity, so that $\mathcal{L}(\cdot) = \infty$, but this raises no problem: since we are looking for saddle or maxmin points of the function,

it is easy to see that when the function is at a minimum with respect to all the Lagrange multipliers, all the sums in the formula equal zero and (for λ_o normalized to 1) $\mathcal{L}(\cdot)$ is finite and equal to U.

With infinite horizon the FOCs of minimum and maximum are mainly the same as displayed in the solution to exercise 2, with ∞ in the place of T. The only changes are those due to the introduction of recursive utility, and to the fact that a condition like (3) no longer exists because there are no longer left-overs of capital to be managed at the end of the planning period – indeed, there is no end to the planning period.

After the FOCs (12) and (13) have been transformed according to recursivity, they take the form

$$\begin{aligned} \beta u'(C_{s+1})\left[F'(K_{s+1})+1-\delta\right]-u'(C_s) &= 0 \qquad (14) \\ F(K_s)+K_s(1-\delta)-C_s-K_{s+1} &= 0 \qquad (15) \end{aligned}$$

which differs from the previous system for the presence of the discount factor and marginal instantaneous utilities in Euler equation (14). We still have a one-to-one, reversible rule of transition between consecutive states, but the important difference is that here the rule is independent of the date: indeed, the marginal utility function in (14) is the same at all dates, while in equation (12) it could be different from one date to another.

This fact establishes the recursive character of the dynamics generated by the (14)-(15) system. Starting from a given (C_o, K_o), for example, the system determines the position (C_s, K_s) at any date s, either $s > 0$ or $s < 0$ – hence possibly the entire trajectory from $-\infty$ to $+\infty$ – by applying the same transition rule s times, either forwards or backwards.

In mathematical terms, a recursive system such as (14)-(15) is a system of two first-order difference equations. Just to recall some of the main properties of such systems: (i) a solution is a time sequence ranging from $-\infty$ to $+\infty$, in our case a function mapping the set integers into $\mathbb{R}^2$, such that replacing the values of the function into the equations the latter are satisfied at all s. (ii) The general solution to an n-order difference equation has n dof, i.e. n free parameters, with a unique solution for each arbitrary assignment of values to them. In our case of two first-order equations, we have two times one dof. We have an idea of what this means from the discussion in section 2.3.

But differently from the finite-horizon case, here there is a problem. We do not have the terminal capital stock to provide us with a second boundary condition (in addition to the given initial capital stock K_o) to make the optimal sequence uniquely determined. Conditions (14) and (15), together with the given K_o, leave one dof open: to close it we need one further condition,

but how to make sure that this picks up exactly the optimal path out of the infinite number of paths that still satisfy (14) and (15)?

The transversality condition mentioned all of a sudden by Wickens (chapt. 2 p. 21), provides the answer to the problem but why this is so remains unexplained – on the basis of the Lagrange method alone. There is no direct way through leading from the logic of that method to transversality. In fact the necessity of the latter condition when the problem has no terminal boundary condition becomes evident if dynamic optimization is approached through other methods, notably those based on variational calculus that we shall see in section 3.

To have an intuition of how it is that transversality is able to sort the optimum sequence from all those that satisfy (14) and (15), we can use the graphical representation of the dynamics of the system provided by the phase-diagram in Wickens figure 2.10 on p. 28. This is reproduced as figure 1 here. The picture has been completed by inserting the curve of equation $F(K) + K(1-\delta) = C$. This curve (line 0a in the figure), together with the horizontal axis, encompasses the economically feasible part of the state-space (states in the positive orthant but above the curve cannot be reached). Besides, we extend the graph of the curve $F(K) - \delta K = C$ up to its intersection (point B) with the abscissa, and call K^{**} the stock of capital such that $F(K^{**}) - \delta K^{**} = 0$.

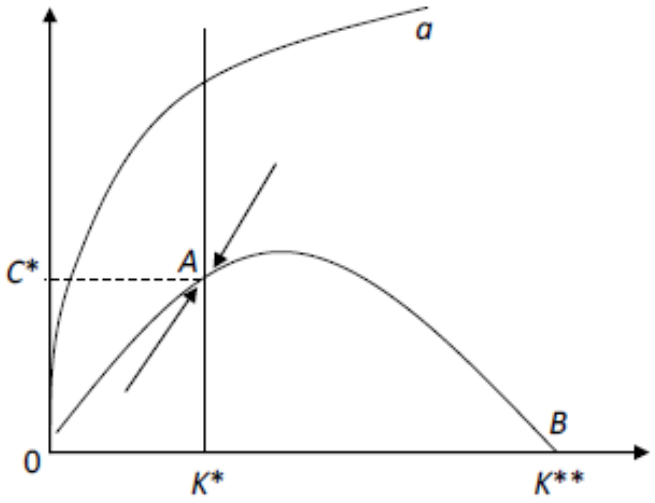

Figure 1

From the analysis of figure 1 we reach the following conclusions.

(i) All the sequences having at least one state placed above the saddlepath cannot represent solutions to the (14) and (15) dynamic system. Why? Just consider the following argument. All the sequences of this type will sooner or later cross the 0a curve that encompasses the feasible states of the economy. But if the next state to a given (C_s, K_s) according to equations (14) and (15) is a state (C_{s+1}, K_{s+1}) lying outside the curve, i.e. with $C_{s+1} > F(K_{s+1}) - \delta K_{s+1}$, what the economy can at most do is to reach a state (C, K) such that $C = F(K) - \delta K$. Consequently, between s and $s+1$ equation (15) does not hold and this sequence is not consistent with the system.

(ii) Sequences consistent with (14) and (15) must therefore lie entirely in the area bounded from above by the saddlepath. Those having at least one state in the saddlepath are entirely included in the latter and converge to the stationary sequence represented by the state (C^*, K^*) from below or from above. Those having at least one state placed below the saddlepath lie entirely below it and converge to the stationary sequence $(0, K^{**})$. Notice that convergence must be asymptotic because, as seen above, different sequences cannot have any state in common.

(iii) Consider a sequence identified by an initial state $\left(\hat{C}_o, \hat{K}_o\right)$ placed below the saddlepath. Compare it with a sequence identified by an initial state $\left(C_o, \hat{K}_o\right)$ placed in the saddlepath. Obviously $\hat{C}_o < C_o$. Working on the system (14)-(15) it is not difficult to prove that, at all $s > 0$, states $\left(\hat{C}_s, \hat{K}_s\right)$ in the former sequence and states (C_s, K_s) in the latter are characterized by $\hat{C}_s < C_s$ and $\hat{K}_s > K_s$. The proof of this statement comes from the hint given in exercise 4: according to (14)-(15), C_{s+1} is increasing with respect to C_s, decreasing with respect to K_s, while K_{s+1} is decreasing with respect to C_s and increasing with respect to K_s. Apply (14) and (15) repeatedly: you will get the statement to be proved.

From the last point, it follows directly that, given an initial capital stock K_o, the optimal sequence is the one which starts and remains forever in the saddlepath, converging to (C^*, K^*) in infinite time. Of course the optimal sequence coincides with the stationary (C^*, K^*) if K_o happens to equal K^*.

It is also easy to check that all optimal sequences satisfy the condition of transversality because

$$\lim_{s\to\infty} \beta^s u'(C_s) K_s = u'(C^*) K^* \lim_{s\to\infty} \beta^s = 0$$

The last step consists in proving that, in our case, transversality is not only a necessary condition of optimality but also a sufficient one, in the sense that no non-optimal sequence can satisfy it. This follows from the fact that all non-optimal sequences converge to the stationary sequence $(0, K^{**})$.

Look at the series of the terms $\beta^s u'(C_s) K_s$ with K_s approaching K^{**} and C_s monotonically declining to 0. From Euler equation (14) it is easy to see that the series is positive and monotonically increasing with respect to s, therefore its limit cannot be zero.

2.6 Steady states and convergence

Let us reconsider the transition rule defined by equations (14)-(15). We rewrite it as an one-to-one map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(C_s, K_s) = \varphi(C_{s-1}, K_{s-1})$$

or inversely

$$(C_{s-1}, K_{s-1}) = \varphi^{-1}(C_s, K_s)$$

Given an initial state (C_o, K_o), the corresponding (unique) infinite sequence is described by

$$(C_s, K_s) = \varphi^s(C_o, K_o)$$

with s, the number of (forward or backward) applications of the transition rule, ranging from $-\infty$ to $+\infty$.

As we have seen from figure 1, all the sequences generated by φ for a given initial state converge to some state, either the one represented by A or by B in the figure. Notice that the continuity of the u', F and F' functions implies that the function φ (and its inverse) is continuous. Now recall the theorem that says that if a function $f(x)$ is continuous and

$$\lim_{x \to x^*} f(x) = a$$

then $a = f(x^*)$. In our case, call $x_s \equiv (C_s, K_s)$: from

$$\lim_{s \to \infty} \varphi^s(x_o) = x^*$$

(where x^* is one of the limit states of the system), and from

$$\lim_{s \to \infty} \varphi^s(x_o) = \lim_{s \to \infty} \varphi(x_{s-1}) = \lim_{x_{s-1} \to x^*} \varphi(x_{s-1})$$

it follows that

$$\lim_{x_{s-1} \to x^*} \varphi(x_{s-1}) = x^*$$

As φ is continuous, the theorem recalled above implies

$$\varphi(x^*) = x^*$$

i.e., the limit states of all the sequences generated by φ are *fixed points* of φ.

A fixed point of a transition rule corresponds to a stationary infinite sequence: from $\varphi(x^*) = x^*$ it follows that, taking $x_o = x^*$, $x_s = \varphi^s(x^*) = x^*$ for any s. As we know, no two different sequences can have states in common. Thus, even if an infinite sequence $\varphi^s(x)$ converges to a fixed point x^*, the sequence can never 'reach' the state x^*.

In economic terminology, states (sequences) corresponding to fixed points are called "steady states". Not all transition rules have steady states. There are various theorems of existence of fixed points that state sufficient conditions to that effect. In our case, existence descends from the convergence of all the sequences of the type $\{\varphi^s(x) : s = 1, 2...\}$ with x chosen from an appropriate subspace of $\mathbb{R}^2$. Here we don't inquire further into the matter.

The steady state marked with A in figure 1 and corresponding to (C^*, K^*) is the optimal sequence of the economy for $K_o = K^*$, and the limit of all optimal sequences for $K_o \neq K^*$. Therefore stationary levels of consumption C^* and capital stock K^* are either the actual or the tendential (in the longrun) state of the optimally planned economy. The absence of any longrun growth here descends from the assumptions of the model, that rule out technical progress and changes in the labouring population (see section 2.7).

Our system has two other fixed points, corresponding to points 0 and B in figure 1. The last one is the limit of all the sequences that have no states in the saddlepath, while 0 is the limit of none (apart from itself).

From equation (14) we see that a characteristic of this steady state is

$$F'(K^*) - \delta = \frac{1}{\beta} - 1 = \theta$$

or in other words the marginal productivity of capital net of scrapping equals the subjective rate of discount of the representative consumer. This last characteristic establishes an inverse relationship between θ, the degree of impatience, and the steady state levels (C^*, K^*). Notice that a population of absolutely non-impatient consumers, with $\theta = 0$, will place its steady state at the point of maximum of the $F(K) - \delta K$ curve, a point at which also the steady state level of consumption is the maximum attainable in the economy. This is called the 'Golden Rule' steady state.

The ability of fixed points to attract sequences is called *stability*. A stable point x^* has a proper basin of attraction, i.e. a set of points x (including itself) such that

$$\lim_{s \to \infty} \varphi^s(x) = x^*$$

The basin of an unstable fixed point like 0 in figure 1 contains the point itself and nothing else. Point B is stable and its basin covers almost the whole

relevant space with the only exclusion of the saddlepath. The basin of the optimal steady state of the system coincides with the saddlepath, which is a one-dimensional manifold in $\mathbb{R}^2$. Getting there, therefore, seems more a matter of luck than the preordered course of events of the optimally planned economy. If the economy does not manage to enter the saddlepath, the model predicts either unsustainable overconsumption leading to self-destruction in a finite time, or unrelenting accumulation with starvation looming in the far future.

2.7 Optimal exogenous growth

This section refers to Wickens chapter 3, sections from 1 to 4, where the basics of growth models and the meaning of the exogenous/endogenous growth distinction are made clear. The primary source of all these models and of many others that we don't have the time to discuss is Robert Solow, "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics* 70 (1956).

In the model of exogenous optimal growth presented in sect. 3.4 Wickens assumes that two factors of growth, population increase and technological progress, are simultaneously at work. The result is a rather bulky mathematical formulation that obscures how simple an extension of the DGE model of optimal plan this growth model is. In order to get rid of useless mathematical complexity, we consider the two factors of growth separately by presenting the model in two distinct steps: first, an economy with demographic growth ($n > 0$) without technical progress ($\mu = 0$); second, the same economy with technical progress and a stationary working population ($n = 0$, $\mu > 0$).

Before we develop the model, we must introduce two simplifying hypotheses regarding the shape of the functions of production and of instantaneous utility.

(i) Production function. Since the input of labour in the economy is now treated as a variable, production will be assumed explicitly to be a function of both production factors, capital and labour K, L. What kind of function? In the literature on macroeconomic growth the standard assumption is that the aggregate technology in the economy is characterized by constant returns to the scale of utilization of both factors. This corresponds to the mathematical property $F(\lambda K, \lambda L) = \lambda F(K, L)$, where λ, a positive scalar, is a measure of expansion or contraction in the utilization of the factors and in the corresponding GDP. Notice that this property is more general than linearity: in mathematics the property is called homogeneity of degree 1. All linear functions are homogeneous degree 1 but not viceversa. A particularly

convenient special case of non-linear but homogeneous degree 1 production function is the so-called Cobb-Douglas function, of the form

$$Y = K^{\alpha} L^{1-\alpha}$$

The parameter α is taken in the open $(0, 1)$ interval.

Exercise 6 *Demonstrate the following facts concerning the above Cobb-Douglas production function: (i) it is degree 1 homogeneous, (ii) the exponents α and $1-\alpha$ represent the elasticity of output with respect to capital and to labour, (iii) each exponent being less than one implies that the marginal productivity of each factor is decreasing, (iv) the isoquants of the function in K, L space are concave.*

(ii) Instantaneous utility function. A convenient class of utility functions is defined by the property of 'constant elasticity of marginal utility', also known as the property of 'constant relative risk aversion' (CRRA). Given a generic increasing, concave, twice differentiable instantaneous utility function $u = u(C)$, the elasticity of marginal utility with respect to consumption – that in models of choice under uncertainty provides also a measure of relative risk aversion – is defined by

$$\sigma \equiv -\frac{C}{u'}\frac{du'}{dC} = -C\frac{u''}{u'}$$

where σ must be positive from $u' > 0 > u''$. Since we know that

$$C\frac{u''}{u'} = \frac{d \ln u'(C)}{d \ln C}$$

then

$$\ln u'(C) = a - \sigma \ln C$$

with a an arbitrary constant. Let us call $e^a \equiv A$. The property of constant elasticity therefore requires that

$$u'(C) = AC^{-\sigma}$$

For $\sigma < 1$ this implies that the constant elasticity utility function must be a function of the kind

$$u(C) = A\frac{C^{1-\sigma}}{1-\sigma}$$

which is also called the 'power' utility function. For $\sigma = 1$ the constant elasticity function must be a logarithmic function of the type

$$u(C) = A \ln C + b$$

with b another (positive) arbitrary constant. (The case $\sigma > 1$ is excluded by the general properties of utility – why?). In the following we will in general set $A = 1$ and $b = 0$ for the sake of simplicity.

(I) Demographic growth

Let us now consider an infinite horizon optimal plan for an economy characterized by a constant rate of growth $n > 0$ of the labouring population, with production and instantaneous utility functions as described above. Let L_o be the labouring population at the initial date 0: then $L_s = (1+n)^s L_o$, and in general $L_s = (1+n) L_{s-1}$.

The constraint of the economy becomes

$$K_s^\alpha L_s^{1-\alpha} + K_s (1-\delta) - C_s - K_{s+1} \geq 0$$

Let us use small type letters for indicating capital, output, and consumption ratios per-head, as follows

$$k_s \equiv \frac{K_s}{L_s}, \quad y_s \equiv \frac{Y_s}{L_s} = \frac{K_s^\alpha L_s^{1-\alpha}}{L_s} = k_s^\alpha, \quad c_s \equiv \frac{C_s}{L_s}$$

Notice that output per-head is a decreasing returns function ($\alpha < 1$) of capital per-head. As is easy to check, the budget constraint can be restated in per-head terms by dividing it by L_s

$$k_s^\alpha + k_s (1-\delta) - c_s - (1+n) k_{s+1} \geq 0$$

An optimal growth plan is an infinite-horizon sequence (c_s, k_s) with $s = 0, 1, ...$ such that discounted total utility is maximized compatibly with the budget constraint and all the non-negativity constraints seen in 2.2 being satisfied. Expressed in per-head terms, total utility is given by (we take the case $\sigma < 1$)

$$\sum_{s=0}^{\infty} \beta^s \frac{C_s^{1-\sigma}}{1-\sigma} = \sum_{s=0}^{\infty} \beta^s \frac{(L_s c_s)^{1-\sigma}}{1-\sigma}$$

Maximization by means of Lagrange method, as in section 2.2, is left as an exercise. At the end of the procedure the two necessary conditions of optimum, Euler and budget condition, turn out to be similar to equations (14) and (15)

$$\begin{aligned} \beta (L_{s+1} c_{s+1})^{-\sigma} \left[\alpha k_{s+1}^{\alpha-1} + 1 - \delta\right] - (L_s c_s)^{-\sigma} &= 0 \\ k_s^\alpha + k_s (1-\delta) - c_s - (1+n) k_{s+1} &= 0 \end{aligned} \tag{16}$$

Through simple manipulation, Euler equation can also be written as

$$\alpha k_{s+1}^{\alpha-1} + 1 - \delta = \frac{1}{\beta}\left(\frac{L_{s+1}c_{s+1}}{L_s c_s}\right)^{\sigma} = \frac{1}{\beta}(1+n)^{\sigma}\left(\frac{c_{s+1}}{c_s}\right)^{\sigma}$$

Remember that $1/\beta = 1 + \theta$. Besides, let us use the linear approximation

$$(1+n)^{\sigma} \simeq 1 + \sigma n$$

(this is found by taking a truncated Taylor expansion of the function $(1+n)^{\sigma}$ in a neighborhood of $n = 0$). We then get

$$\frac{1}{\beta}(1+n)^{\sigma} \simeq (1+\theta)(1+\sigma n) \simeq 1 + \theta + \sigma n$$

and Euler condition takes on the form

$$\alpha k_{s+1}^{\alpha-1} + 1 - \delta = (1 + \theta + \sigma n)\left(\frac{c_{s+1}}{c_s}\right)^{\sigma} \tag{17}$$

Graphic analysis of the dynamic system constituted by equations (17) and (16) gives the same results as the one carried out in section 2.5, see figure 2 here below

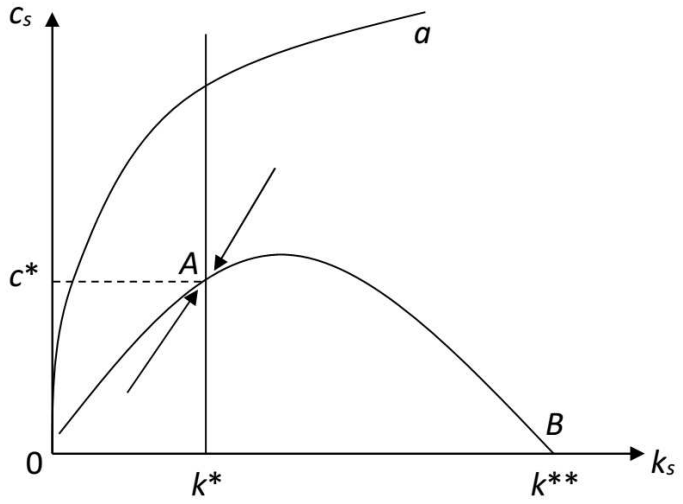

Figure 2

Here, the variables measured along the axes are per-head capital and consumption. The curve 0a represents the function $k^{\alpha} + (1-\delta)k$ and defines

the boundary of the feasible states of the economy. Curve 0AB represents the function $k^\alpha - (\delta + n)\, k$ which, from the budget equation (16), equals $c_s + (1 + n)\,(k_{s+1} - k_s)$, so that if

$$c_s = k_s^\alpha - (\delta + n)\, k_s$$

the capital-labour ratio k_s must be constant. The capital-labour ratio k^* is chosen so as to satisfy the equation

$$\alpha k_{s+1}^{\alpha-1} - \delta = \theta + \sigma n$$

so that, if $k_s = k^*$ consumption per-head c_s must be constant (look at Euler equation (17)).

The dynamics of (c_s, k_s) in the feasible area is exactly like the dynamics in figure 1 of section 2.5: the optimal paths are those that are located along the saddle-path converging to the steady state path represented by point A. In the very long run therefore any optimal path will come to resemble the steady state path more and more closely. The steady state is characterized by

$$\begin{aligned} c^* &= (k^*)^\alpha - (\delta + n)\, k^* \\ \alpha\,(k^*)^{\alpha-1} &= \delta + \theta + \sigma n \end{aligned}$$

stationary capital-labour and consumption-labour ratios. The absolute levels of capital and consumption, $K_s = L_s k^*$ and $C_s = L_s c^*$ will therefore be increasing at the same rate n as the labouring population. All other characteristic ratios in the economy (output per-head, consumption to output, etc.) will also remain constant because everything is growing at the same rate ("balanced growth").

Exercise 7 *The steady state of an economy is said to satisfy the "Golden Rule" (GR) when consumption per-head is the maximum feasible consumption per-head available in the economy. Show that, in order to have GR steady state, the rate of discount of the economy θ must be equal to $(1 - \sigma)\, n$ while, with $\theta > (1 - \sigma)\, n$, steady state consumption per-head is less than the available maximum.*

(II) Growth with technological progress

We now turn to an economy with stationary population $L_s = L$ $(n = 0)$ but a steady flow of exogenous innovations which raises the productivity of both factors of production. Following Wickens sect. 3.2, the pace of technical

progress is supposed to be given by a constant positive rate of shift μ of the technology such that, taking $s = 0$ as the base-year for the technology, at each successive date $s > 0$

$$Y_s = (1+\mu)^s K_s^\alpha L^{1-\alpha}$$

The last equation can also be written as

$$Y_s = K_s^\alpha \left[(1+\mu)^{\frac{s}{1-\alpha}} L\right]^{1-\alpha}$$

From this, define

$$L_s \equiv (1+\mu)^{\frac{s}{1-\alpha}} L$$

where L_s can be interpreted as labour measured in "efficiency units", i.e. the equivalent in base-year labour of L labourers equipped with the technology available at s. If we use the usual linear approximation

$$(1+\mu)^{\frac{s}{1-\alpha}} = \left[(1+\mu)^{\frac{1}{1-\alpha}}\right]^s \simeq \left(1 + \frac{\mu}{1-\alpha}\right)^s$$

the rate $\eta \equiv \frac{\mu}{1-\alpha}$ represents the (per-period) rate of increase of labour efficiency due to technical progress at (per-period) rate μ.

Let us now redefine capital, output and consumption in per-efficiency-head terms exactly as we did in the model of demographic growth. In the technological progress model the rate of increase of efficiency η replaces the rate of demographic growth n, but all the equations of the two models are formally identical. Therefore, the optimal growth path will be characterized by the two necessary (Euler and budget) conditions represented by equations (17) and (16) with only η in the place of n. Convergence to steady state and properties of the latter remain unchanged, in particular

$$\begin{aligned} c^* &= (k^*)^\alpha - (\delta+\eta)\, k^* \\ \alpha\,(k^*)^{\alpha-1} &= \delta + \theta + \sigma\eta \end{aligned}$$

There is a difference, however, regarding consumption and capital per (physical) head. Indeed, from

$$\begin{aligned} C_s &= c^* L_s = c^* L\,(1+\eta)^s \\ K_s &= k^* L_s = k^* L\,(1+\eta)^s \end{aligned}$$

it turns out that consumption, capital (and implicitly output) per labourer grow at the constant rate η determined by technical progress.

Exercise 8 *Combine demographic and technological growth in a single model as in Wickens sect. 3.4.1.*

3 Tools for dynamic maximization

In this chapter we consider how to analyze dynamic systems by means of linear approximations in neighbourhoods of particular points. We recall some of the mathematics necessary to deal with at least the simplest types of systems of linear difference equations. Lastly, we introduce methods of maximization especially designed for recursive dynamic problems.

3.1 Linear approximations to nonlinear dynamics

The dynamic system defined by equations (14) and (15) of section 2.5 is nonlinear. While an intuitive graphic analysis of the dynamics is still possible and reasonably simple, non-linearity may complicate the algebraic analysis a lot. In order to make it simpler we follow Wickens, section 2.4.7, and take advantage of the general possibility of approximating the behavior of a given nonlinear system by means of a different, linear system that in limited neighborhoods of particular states behaves 'almost in the same way'. In particular we focus on the neighborhood of the steady state (C^*, K^*) because it indicates the long run tendency of the optimal paths of the economy.

The possibility of building a satisfactory local linear approximation to a given nonlinear dynamics in the neighborhood of its fixed points is granted by a theorem known as Hartman-Grobman (H-G) theorem. [A user-friendly introduction to the theorem, Scott Zimmerman "An undergraduate's guide to the Hartman-Grobman and Poincaré-Bendixon theorems", referred to continuous-time dynamic systems, is freely downloadable from the web.]

Consider a generic first-order dynamic system in $\mathbb{R}^n$ of the type

$$x_{s+1} = \varphi(x_s)$$

where the x are n-dimensional vectors and the transition rule φ is a continuously differentiable nonlinear function with domain and range in some open subset of $\mathbb{R}^n$. Take a fixed point x^* of φ, $x^* = \varphi(x^*)$, and approximate φ in the neighbourhood of x^* by taking the first two terms of the Taylor expansion

$$\varphi(x) \simeq \varphi(x^*) + \nabla\varphi^*(x - x^*) = x^* + \nabla\varphi^*(x - x^*)$$

where $\nabla\varphi$ indicates the $n \times n$ Jacobian matrix of the partial derivatives of φ, and $\nabla\varphi^*$ the Jacobian at steady state x^*.

The $\mathbb{R}^n \to \mathbb{R}^n$ function

$$\psi(x) = x^* + \nabla\varphi^*(x - x^*)$$

is an affine approximation of φ with the same fixed point $x^* = \psi(x^*) = \varphi(x^*)$. The H-G theorem affirms that, provided the Jacobian $\nabla\varphi$ at steady state x^* has no purely imaginary eigenvalues, there is an open neighborhood of x^* in which the dynamics according to φ and the dynamics according to ψ are 'topologically conjugate' or, translated into non-technical language, can be considered to be qualitatively equivalent.

Now we apply this technique to the study of the dynamic properties of the (14)-(15) system near the steady state (C^*, K^*). First, rewrite the system with (C_{s+1}, K_{s+1}) as an explicit function of (C_s, K_s) by taking advantage of the fact that the marginal utility function u' is invertible. Call $v \equiv (u')^{-1}$ the inverse of marginal utility and rewrite the equations (14)-(15) in the form

$$\begin{aligned} C_{s+1} &= v\left(\frac{u'(C_s)}{\beta\left[F'\left(F(K_s)+K_s(1-\delta)-C_s\right)+1-\delta\right]}\right) \\ K_{s+1} &= F(K_s)+K_s(1-\delta)-C_s \end{aligned}$$

Now compute the Jacobian of the function at the steady state point, where we know that the following equations hold

$$\begin{aligned} \beta\left(F'(K^*)+1-\delta\right) &= 1 \\ F(K^*)-\delta K^*-C^* &= 0 \end{aligned}$$

Recall that, from our definition, $v' = \frac{1}{u''}$. By way of calculus we get the Jacobian

$$\begin{bmatrix} 1+\beta\frac{u'(C^*)}{u''(C^*)}F''(K^*) & -\frac{u'(C^*)}{u''(C^*)}F''(K^*) \\ -1 & \frac{1}{\beta} \end{bmatrix}$$

Using the abbreviation $B^* \equiv \frac{u'(C^*)}{u''(C^*)}F''(K^*) > 0$ we rewrite it as

$$\begin{bmatrix} 1+\beta B^* & -B^* \\ -1 & \frac{1}{\beta} \end{bmatrix}$$

Exercise 9 *Prove that the Jacobian matrix has two distinct real and positive eigenvalues.*

The solution to exercise 9 makes sure that our system satisfies the conditions of the H-G theorem. We therefore proceed to define the approximating affine system (in vector-matrix form)

$$\begin{bmatrix} C_{s+1} \\ K_{s+1} \end{bmatrix} = \begin{bmatrix} C^* \\ K^* \end{bmatrix} + \begin{bmatrix} 1+\beta B^* & -B^* \\ -1 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} C_s - C^* \\ K_s - K^* \end{bmatrix}$$

By taking the deviations of C and K from their steady state values

$$\begin{aligned} \zeta_s &= C_s - C^* \\ \eta_s &= K_s - K^* \end{aligned}$$

as the variables of our dynamic system, the approximating transition rule is given by the linear system

$$\begin{bmatrix} \zeta_{s+1} \\ \eta_{s+1} \end{bmatrix} = \begin{bmatrix} 1 + \beta B^* & -B^* \\ -1 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \zeta_s \\ \eta_s \end{bmatrix} \tag{18}$$

with the transition rule identified by a fixed-coefficient matrix. Since the determinant is non-zero, the matrix is non-singular, which means that the trasition rule can be operated in the reverse direction, from $s+1$ to s.

As is easy to see, if the state of the system at a certain date s is known, the state at any other previous or successive date can be computed simply by applying the transition rule, i.e. the Jacobian matrix, the required number of times. Call B the matrix in formula (18), and f_s the bi-dimensional vector of deviations at s. Then

$$f_{s+t} = B^t f_s$$

with t either positive or negative. Thus, the dynamics of (18), and in particular its tendency to converge to, or diverge from the steady state (now identified by $f = 0$), can be investigated simply by studying the behavior of successive (positive or negative) powers of a matrix.

3.2 Solving a system of linear fixed-coefficient first-order difference equations: the homogeneous case

The two-equation system (18) is an example of a system of linear fixed-coefficient first-order difference equations. Given the state of the system at a certain date, f_s, the corresponding particular solution to the system is a sequence of points in $\mathbb{R}^2$ extending from $-\infty$ to $+\infty$.

In general, a system of the kind

$$f_{s+1} = B f_s \quad \text{or} \quad f_s = B^{-1} f_{s+1}$$

with the f vectors in $\mathbb{R}^n$ and B an $n \times n$ non-singular matrix is called a homogeneous linear system. If the transition from/to s to/from $s+1$ is also affected by some other known vector of n variables, as in the following example

$$f_{s+1} = B\left(f_s + g_{s+1}\right) \quad \text{or} \quad f_s = B^{-1} f_{s+1} - g_{s+1}$$

where the sequence of the terms $g_{s+1} \in \mathbb{R}^n$ is known, then the (linear, fixed-coefficient, first-order) system is said to be non-homogeneous. Every non-homogeneous system has an 'auxiliary' homogeneous system which is obtained simply by putting the known sequence of vectors g_s equal to zero.

We study the homogeneous case first. As anticipated in 3.1, from $f_{s+t} = B^t f_s$ for any positive or negative integer t, it follows that the dynamics of f depends entirely on the behavior of the successive powers of B. This behavior is better understood if we exploit the possibility of 'diagonalizing' B, i. e. of transforming it into a block-diagonal matrix J called its 'Jordan normal form'. We know from theorems of linear algebra that, given the non-singular $n \times n$ matrix B, there exists a non-singular $n \times n$ matrix V that allows the two similarity transformations $V^{-1}BV = J$ and $V^{-1}B^{-1}V = J^{-1}$. The entries of the main diagonal of J and J^{-1} are the characteristic roots, or eigenvalues, of B and their inverses, repeated according to their multiplicities if there are multiple roots. The columns of V are called the generalized characteristic vectors, or eigenvectors, of B and constitute a basis in $\mathbb{R}^n$. Once the matrix has been diagonalized, from $B = VJV^{-1}$ and $B^{-1} = VJ^{-1}V^{-1}$ it follows that all positive and negative powers of B can be computed from

$$B^s = VJ^sV^{-1}$$

In the case in which B has n distinct roots (like the GDE case analyzed in 3.1), J is the diagonal matrix of the roots and the columns of V are the corresponding eigenvectors. In this case the convenience of resorting to this transformation of the original B matrix is particularly evident . The sequence of the powers of B comes to depend on the sequence of the powers of its n eigenvalues, a sequence of numbers rather than matrices, something which is much simpler to analyze.

Let us consider this case more in detail in the hypothesis that all the eigenvalues (and therefore, all the eigenvectors) are real numbers λ_i, $i = 1, ...n$, with $v^{(i)}$ to indicate the eigenvector associated with λ_i. Call Λ the diagonal matrix of the eigenvectors. Assume that at $s = 0$ the position of the system is f_o. The general solution $f_s = B^s f_o$ can be rewritten as

$$f_s = V\Lambda^s V^{-1} f_o$$

or, equivalently

$$V^{-1} f_s = \Lambda^s V^{-1} f_o \tag{19}$$

Recall that the columns of V represent a basis for $\mathbb{R}^n$ alternative to the standard orthogonal unit basis. If the vectors f_s represent the coordinates of points in $\mathbb{R}^n$ with respect to the standard basis, their coordinates with

respect to the alternative V basis are given by vectors x_s such that

$$Vx_s = f_s$$

Thus, the product $V^{-1}f_s = x_s$ is simply the same vector that f_s identifies by means of orthogonal unit coordinates, but expressed by means of a different system of coordinates, corresponding to the eigenvectors in V.

If we rewrite equation (19) in the x-coordinates

$$x_s = \Lambda^s x_o$$

the analysis of long-run behavior of the system becomes quite simple and intuitive. Each of the n coordinates $x_{is} = \lambda_i^s x_{io}$ depends entirely on the powers of the corresponding root λ_i. The main considerations for the purposes of stability analysis are: (i) whether λ_i is positive or negative (in the latter case the powers alternate in sign); (ii) whether the norm of λ_i is greater or smaller than one (the norm of the successive positive powers converge to infinity or to zero respectively). For powers $s > 0$ a root λ_i is said to be stable if $|\lambda_i| < 1$, unstable in the opposite case. If λ_i is stable, the i-th component of x_s tends to 0 as $s \to \infty$, in other words the "projection" of x_s on the eigenvector $v^{(i)}$ gets nearer and nearer 0, a fixed point of the dynamics; in that case $v^{(i)}$ is also said to be stable. If λ_i is unstable, the reverse holds.

For powers $s < 0$ (s going backwards, from 0 to $-\infty$) the above discussion and terminology must be reversed because, if λ_i is an eigenvalue of B, the eigenvalue of B^{-1} associated with the same eigenvector $v^{(i)}$ is $(\lambda_i)^{-1}$. Thus, if λ_i is stable for $s > 0$ it must be unstable for $s < 0$ (and vice versa). As for the associated eigenvector $v^{(i)}$, if it is stable for s moving forward it must be unstable for s moving backwards (and vice versa).

Just to consider two extreme cases, take the globally stable case in which $\max_i |\lambda_i| < 1$: in the forward dynamics x_s converges asymptotically to the fixed point 0 no matter what the initial position x_o is; in backwards dynamics instead x_s (with $s < 0$) diverges from 0 in directions that depend on x_o. In the opposite, globally unstable case in which $\min_i |\lambda_i| > 1$, the forward dynamics is divergent while, looked at backwards, all the paths come from a smaller and smaller neighborhood of 0.

Mixed cases with both stable and unstable roots present a variety of types. As an example consider the following exercise.

Exercise 10 *Go back to system (18) in 3.1. Show that the system has one stable and one unstable root. Construct the two-dimensional phase diagram showing the 'saddle' with the corresponding stable and unstable path.*

3.3 The same: non-homogeneous case

Consider now the non-homogeneous system

$$f_s = B\left(f_{s-1} + g_s\right)$$

or in the opposite direction

$$f_{s-1} = B^{-1} f_s - g_s$$

where $g_s \in \mathbb{R}^n$ is a vector whose behavior from $-\infty$ to $+\infty$ is known. We assume the sequence of the g_s to be bounded in the sense that there exists a finite positive scalar G such that at all s the condition $|g_s| \leq G$ holds. This sequence is also called the 'driver' of the system. The homogeneous system

$$f_s = B f_{s-1}$$

in which g_s is assumed to be uniformly zero is called the 'auxiliary' system of the original one.

Let the state of the system at $s = 0$ be f_o. Then, by simply operating on the system $1, 2...$ s times either forward or backward, we see that the position at any later or earlier date s or $-s$ is given by, respectively, the first or the second of the following formulas

$$\begin{aligned} f_s &= B^s f_o + \sum_{h=0}^{s-1} B^{s-h} g_{h+1} \\ f_{-s} &= B^{-s} f_o + \sum_{h=0}^{s-1} \left(-B\right)^{h+1-s} g_{-h} \end{aligned}$$

This is called the general solution to the system. According to a theorem which applies to linear non-homogeneous difference and differential systems alike, general solutions consist of the sum of two parts: a particular solution for a particular choice of f_o, added to the general solution to the homogeneous auxiliary system associated with the given one. We know that the latter is given by $f_s = B^s f_o$ for s both positive and negative. As to the particular solution, take $f_o = 0$: resorting to iterated computation again, we find that the second part of the right-hand side of the two formulas represent the particular sequence (entirely ruled by the driver g) that satisfies the system.

After diagonalization of the matrix B the general solution can be rewritten as

$$\begin{aligned} f_s &= V\left[\Lambda^s V^{-1} f_o + \sum_{h=0}^{s-1} \Lambda^{s-h} V^{-1} g_{h+1}\right] \\ f_{-s} &= V\left[\Lambda^{-s} V^{-1} f_o + \sum_{h=0}^{s-1} \left(-\Lambda\right)^{h+1-s} V^{-1} g_{-h}\right] \end{aligned}$$

or, equivalently

$$\begin{aligned} V^{-1}f_s &= \Lambda^s V^{-1}f_o + \sum_{h=0}^{s-1}\Lambda^{s-h}V^{-1}g_{h+1} \\ V^{-1}f_{-s} &= \Lambda^{-s}V^{-1}f_o + \sum_{h=0}^{s-1}(-\Lambda)^{h+1-s}\,V^{-1}g_{-h} \end{aligned}$$

Now rename the coordinates of the f and g vectors with respect to the eigenvectors in V as

$$Vx_s = f_s \qquad and \qquad Vy_s = g_s$$

so that the last two equations can be rewritten as

$$\begin{aligned} x_s &= \Lambda^s x_o + \sum_{h=0}^{s-1}\Lambda^{s-h}y_{h+1} \\ x_{-s} &= \Lambda^{-s}x_o + \sum_{h=0}^{s-1}(-\Lambda)^{h+1-s}\,y_{-h} \end{aligned}$$

The analysis of the system follows along the lines of section 3.2. Let us consider only the forward dynamics for brevity. If all the roots are stable the part of the forward solution that depends on x_o and is independent of the dynamics of the driver tends to 0 for $s \to \infty$. So, in the long run the driver takes the lead and comes to govern x_s almost entirely. Notice however that the influence of the driver decreases with the passing of time, in the sense that a vector y_s has the maximum effect on x_s, a bit less on x_{s+1} and so on, with its influence on x_{s+t} tending to zero as $t \to \infty$.

In case of unstable roots instead both the initial state x_o and the oldest realizations of the driver tend to prevail in determining the current state. The influence of a given y_s is smallest on x_s but increases with the passing of time and tends to infinity on x_{s+t} as $t \to \infty$.

3.4 Basics of dynamic programming

Problems of dynamic maximization like those we have been considering so far can be solved by means of methods of static constrained maximization such as the Lagrangean method. They are treated more appropriately, however, by using completely different methods devised for explicitly dynamic applications like the methods elaborated in the theory called 'optimum control theory'. The foundations of optimal control were laid by Richard Bellman in *The Theory of Dynamic Programming*, 1957. An essential feature of this

theory is that it applies to problems in which the objective function is recursive. The methods based on it are therefore called 'recursive methods'. The classic reference for economic applications is the book by Nancy Stokey and Robert Lucas, *Recursive Methods in Economic Dynamics*, 1989.

In general, an optimum control problem (OCP) consists of the following items. A vector of n 'state' variables, call it x, and a vector of m other variables called 'controls', call it y. A 'law of motion' connecting the state at a given date to the state-control pair at the previous date, represented by a system of n first-order difference equations

$$x_{s+1} = g(x_s, y_s)$$

where $g : \mathbb{R}^{n+m} \to \mathbb{R}^n$ indicates the transition rule. A sequence $\{x_s, y_s\}_0^\infty$ beginning at a given state x_o is said to be 'feasible' in the system if it satisfies the law of motion at every step s.

Control variables are so called because they are supposed to be under the control of the planner or manager of the system: given x_s, the planner has the power to determine x_{s+1} by choosing the level of controls y_s. The aim is to operate the controls from, say, $s = 0$ to ∞, so as to maximize an objective function that depends on the infinite state-control sequence.

Let the objective of the planner be represented by the function $F(\{x_s, y_s\}_0^\infty)$. This is supposed to be a recursive function (see the definition in sect. 2.4) of the kind

$$F(\{x_s, y_s\}_0^\infty) = W(\varphi(x_o, y_o), F(\{x_s, y_s\}_1^\infty))$$

where φ represents the instantaneous return (or benefit, or utility or ...) at each date. The aggregator W is supposed to be strictly increasing with respect to both its arguments. In addition, we will assume that the F function satisfies the hypothesis called 'complete separability' in section 2.4

$$\begin{aligned} &F(\{x_s, y_s\}_0^\infty) = \\ &= \lim_{t\to\infty} W\left(\varphi(x_o, y_o), W\left(\varphi(x_1, y_1), ...W\left(\varphi(x_t, y_t), F\left(\{x_s, y_s\}_{t+1}^\infty\right)\right)...\right)\right) = \\ &\qquad = W(\varphi(x_o, y_o), W(\varphi(x_1, y_1), W(... \quad ...))) \end{aligned}$$

This allows us to consider the total return F as entirely determined by the infinite sequence of the instantaneous returns $\varphi(x_o, y_o), \varphi(x_1, y_1), \varphi(x_2, y_2)....$

Given an initial state $x = x_o$, and given that the dynamics is constrained by the law of motion at each step, the OCP problem is solved by finding the infinite sequence $\{y_o, y_1, ...\}$ that maximizes F. Instead of setting out directly to maximize under constraint, dynamic programming methods follow

an indirect route and try to identify the solution starting from the properties that, if it exists, the solution must necessarily satisfy.

Thus, assume that one particular sequence $\{\hat{x}, \hat{y}\}$ solves our OCP for given x_o; then, $\hat{x}_{s+1} = g(\hat{x}_s, \hat{y}_s)$ for all $s = 0, 1, ...$ and $F(\{\hat{x}_s, \hat{y}_s\}_0^\infty)$ is a maximum with respect to all the feasible sequences that start from x_o. Define 'value of the problem', and call it $V(x_o)$, the maximum return one can get by solving the OCP with initial state x_o

$$V(x_o) \equiv F(\{\hat{x}_s, \hat{y}_s\}_0^\infty)$$

If the set up of the model to which we want to apply optimal control methods is such that for all the initial states x belonging to a set $X \subset \mathbb{R}^n$ the OCP has a solution; and if, moreover, X is closed with respect to the law of motion g, in the sense that

$$\textit{for any } y, \textit{ if } x \in X \textit{ then } g(x, y) \in X$$

then there exists a 'value function' $V : X \to \mathbb{R}$ that for each state $x \in X$ indicates the maximum return obtainable from the optimal plan having x as initial state. Closure with respect to g guarantees that this function is defined also at all the states any plan may meet at some stage.

A first property of the value function is recursivity according to the same aggregator W that applies to F, i. e. the property

$$V(x_s) = W(\varphi(x_s, \hat{y}_s), V(\hat{x}_{s+1})) \tag{20}$$

where the 'hat' above the variables indicates that these belong to an optimal path starting at x_s. To see that this must be the case, consider that, on the basis of our definitions and hypotheses,

$$\begin{aligned} V(x_s) &= F(\{\hat{x}_t, \hat{y}_t\}_s^\infty) \\ &= W\left(\varphi(x_s, \hat{y}_s), F\left(\{\hat{x}_t, \hat{y}_t\}_{s+1}^\infty\right)\right) \end{aligned}$$

Now, if $\{\hat{x}_t, \hat{y}_t\}_s^\infty$ solves the OCP with initial state x_s, then $\{\hat{x}_t, \hat{y}_t\}_{s+1}^\infty$ must solve the same OCP with initial state $\hat{x}_{s+1} = g(x_s, \hat{y}_s)$ (just reflect: were it not so, the former problem would have a better solution than $\{\hat{x}_t, \hat{y}_t\}_s^\infty$, against the hypothesis). For this reason, we can also write

$$F\left(\{\hat{x}_t, \hat{y}_t\}_{s+1}^\infty\right) = V(\hat{x}_{s+1})$$

replace it in the previous formula, and thus get property (20).

A second property of the value function is its step-by-step optimization with respect to the control y, as we are now going to show. Take again x_s as given, and consider W as a function of y – given x_s – as follows

$$W(y \mid x_s) \equiv W(\varphi(x_s, y), V(g(x_s, y)))$$

From (20) it clearly follows that, for any control vector y,

$$V(x_s) = W(\hat{y}_s \mid x_s) \geq W(y \mid x_s)$$

i. e. an optimal plan chooses y at each state x_s so as to maximize $W(y \mid x_s)$. Hence the property of step-by-step optimization, according to which, at all x_s,

$$V(x_s) = \max_y W(\varphi(x_s, y), V(g(x_s, y))) \tag{21}$$

The last property introduces a *necessary* characterization of an optimal plan – a plan that solves an OCP for given x_o – as a plan that must follow a 'policy rule' consisting in maximizing the $W(y \mid x_s)$ function at each date, starting at $s = 0$ from x_o and continuing up to infinity.

We now prove that the reverse also holds, that is that the policy rule of step-by-step optimization expressed by (21) provides also a *sufficient* characterization of an optimal plan, in the following sense: if we can identify a function $U : X \to \mathbb{R}$ such that

$$U(x) = \max_y W(\varphi(x, y), U(g(x, y)))$$

then, starting from any given x_o and maximizing step-by-step, we can assemble a plan $\{x_s, y_s\}_0^\infty$ that is optimal and such that $U(x) = V(x)$ at all the states met by the plan itself.

Here is a proof of this last proposition. Let $\{\tilde{x}_s, \tilde{y}_s\}_0^\infty$ be a plan beginning at x_o and such that (21) is satisfied at all s for some function $U(x)$. And consider another plan $\{\hat{x}_s, \hat{y}_s\}_0^\infty$ that also starts at x_o and that we assume to be optimal. From recursivity and the hypothesis of complete separability (section 2.4, recalled above) we know that we can write

$$\begin{aligned} U(x_o) &= W(\varphi(x_o, \tilde{y}_o), U(g(x_o, \tilde{y}_o))) \\ &= W(\varphi(x_o, \tilde{y}_o), W(\varphi(\tilde{x}_1, \tilde{y}_1), W(... \quad ...))) \\ &= F(\{\tilde{x}_s, \tilde{y}_s\}_0^\infty) \end{aligned}$$

Since the other plan $\{\hat{x}_s, \hat{y}_s\}_0^\infty$ is optimal by assumption, given $V(x_o) = F(\{\hat{x}_s, \hat{y}_s\}_0^\infty)$ we must have

$$V(x_o) \geq U(x_o)$$

But from the fact that $\{\tilde{x}_s, \tilde{y}_s\}_0^\infty$ satisfies (21) at all s, we also have

$$\begin{aligned} U(x_o) &\geq W(\varphi(x_o, \hat{y}_o), U(g(x_o, \hat{y}_o))) = W(\varphi(x_o, \hat{y}_o), U(\hat{x}_1)) \\ U(\hat{x}_1) &\geq W(\varphi(\hat{x}_1, \hat{y}_1), U(g(\hat{x}_1, \hat{y}_1))) = W(\varphi(\hat{x}_1, \hat{y}_1), U(\hat{x}_2)) \end{aligned}$$

and so on. And since W is increasing in both its arguments, this implies

$$\begin{aligned} U(x_o) &\geq W(\varphi(x_o, \hat{y}_o), W(\varphi(\hat{x}_1, \hat{y}_1), W(... \quad ...))) \\ &= F(\{\hat{x}_s, \hat{y}_s\}_0^\infty) = V(x_o) \end{aligned}$$

In conclusion, we have proved that

$$U(x_o) \geq V(x_o) \geq U(x_o)$$

which of course can hold only if $U(x_o) = V(x_o)$. Therefore, either $\{\tilde{x}_s, \tilde{y}_s\}_0^\infty$ and $\{\hat{x}_s, \hat{y}_s\}_0^\infty$ coincide, or they are different plans but both solve the same OCP with initial state x_o.

It thus turns out that a plan satisfies the property (21) if and only if it is optimal: in other words, optimality of a plan as a whole or piecewise optimality as expressed by (21) are equivalent. This proposition is known as Bellman's 'Principle of Optimality'.

3.5 Bellman equations, Euler conditions and transversality

From now on, we restrict ourselves for simplicity to an OCP with $n = m = 1$, one state and one control variable only (extension to multivariate functions does not create conceptual difficulties, only longer formulas). In addition, we take the particular case of a linear aggregator of the type $W = \varphi + \beta F$ with $\beta < 1$ (subjective discount factor). Finally, we assume the objective function to be a function of the control variable alone. As we know, this means an objective function in the form

$$\begin{aligned} F(\{y_s\}_0^\infty) &= \varphi(y_o) + \beta F(\{y_s\}_1^\infty) \\ &= \varphi(y_o) + \beta\varphi(y_1) + \beta^2 F(\{y_s\}_2^\infty) \\ &\quad ... \\ &= \lim_{n\to\infty} \left[\sum_{s=0}^{n} \beta^s \varphi(y_s) + \beta^{n+1} F\left(\{y_s\}_{n+1}^\infty\right) \right] \end{aligned}$$

If $F(\{y\})$ is bounded, the term $\beta^{n+1} F(\cdot)$ approaches 0 for $n \to \infty$ and therefore

$$F(\{y_s\}_0^\infty) = \lim_{n\to\infty} \sum_{s=0}^{n} \beta^s \varphi(y_s)$$

This is equivalent to saying that the function is completely separable, as explained in section 2.4.

Condition (20) takes the form

$$V(x_s) = \varphi(\hat{y}_s) + \beta V(\hat{x}_{s+1})$$

and step-by-step optimization (21) requires

$$V(x_s) = \max_y \left[\varphi(y) + \beta V(g(x_s, y))\right]$$

In all the cases in which $\varphi + \beta V$ is differentiable and concave with respect to y, an equivalent formulation of condition (21) is in terms of f.o.c. of maximum

$$\varphi'(y) + \beta \frac{dV(x_{s+1})}{dx_{s+1}} g_2(x_s, y) = 0 \qquad s = 0, 1, ... \tag{22}$$

where $g_2(x_s, y)$ indicates the partial derivative of g with respect to y. This equation is called Bellman equation and is used as an alternative characterization of solutions to OCP whenever differentiability and concavity occur. Written in the form

$$\beta \frac{dV(x_{s+1})}{dx_{s+1}} = -\frac{\varphi'(y)}{g_2(x_s, y)}$$

equation (22) can be read in economic terms as: the discounted marginal benefit/cost of a change in x_{s+1} (left-hand side of the equation, depending on whether positive/negative) must equal its instantaneous marginal cost/benefit measured by

$$-\frac{d\varphi_s}{dx_{s+1}} = -\frac{\varphi' dy}{g_2 dy} = -\frac{\varphi'}{g_2}$$

If the state variable x affects positively the value of the optimal plan, as in all the economic applications we shall meet, having more of x must involve a positive cost, therefore the optimal plan must have negative $\frac{\varphi'}{g_2}$ ratio at all dates.

It may be useful to compare Bellman's characterization of optimal plans with the conditions that we know by the name 'Euler conditions'. We found the latter as a result of the application of Lagrange method to a dynamic maximization problem, but they actually owe their name to a different maximization procedure devised by the XVIIIth Century Swiss mathematician Leonhard Euler in order to solve problems of the optimal control type. Differently from Lagrange and Bellman, Euler's method is based on the so-called 'calculus of variations'. We briefly review it here, with reference to the OCP set up at the beginning of the section.

Euler elaborated on the simple idea that a sequence $\{\hat{x}, \hat{y}\}_0^\infty$ starting from given x_o is optimal if and only if there is nothing to gain from deviating from

it in all possible directions (but keeping the initial state x_o unchanged). In order to model the effects of deviations he introduced the notion of 'variation' from a given sequence, i.e. a sequence $\{\hat{x}+\zeta, \hat{y}+\eta\}_0^\infty$ such that $\{\zeta, \eta\}_0^\infty$, with $\zeta_o = 0$, represents a sequence of deviations, arbitrary except for the fact of satisfying the law of motion g. Euler's method consists in (i) computing the variation of the objective function, ΔF, associated with each variation $\{\zeta, \eta\}_0^\infty$; (ii) finding necessary and sufficient conditions that rule out the possibility of obtaining $\Delta F > 0$.

Step (i). Using the usual linear approximation

$$\varphi(\hat{y}_s + \eta_s) - \varphi(\hat{y}_s) \simeq \varphi'(\hat{y}_s)\,\eta_s$$

we can write

$$\Delta F = \lim_{n\to\infty} \sum_{s=0}^{n} \beta^s \varphi'(\hat{y}_s)\,\eta_s \tag{23}$$

The deviations η_s must be linked to the deviations ζ_s through the law of motion g, because in general

$$\hat{x}_{s+1} + \zeta_{s+1} = g(\hat{x}_s + \zeta_s, \hat{y}_s + \eta_s)$$

By using the same type of approximation, we have (g_1 and g_2 indicate partial derivatives of g with respect to first and second argument)

$$\begin{aligned} \zeta_{s+1} &= g(\hat{x}_s + \zeta_s, \hat{y}_s + \eta_s) - g(\hat{x}_s, \hat{y}_s) \\ &\simeq g_1(\hat{x}_s, \hat{y}_s)\,\zeta_s + g_2(\hat{x}_s, \hat{y}_s)\,\eta_s \end{aligned}$$

from which (assuming always $g_2(\hat{x}_s, \hat{y}_s) \neq 0$)

$$\eta_s \simeq \frac{1}{g_2(\hat{x}_s, \hat{y}_s)}\zeta_{s+1} - \frac{g_1(\hat{x}_s, \hat{y}_s)}{g_2(\hat{x}_s, \hat{y}_s)}\zeta_s$$

By replacing this formula in (23) we get

$$\Delta F \simeq \lim_{n\to\infty} \sum_{s=0}^{n} \beta^s \varphi'(\hat{y}_s) \left[\frac{1}{g_2(\hat{x}_s, \hat{y}_s)}\zeta_{s+1} - \frac{g_1(\hat{x}_s, \hat{y}_s)}{g_2(\hat{x}_s, \hat{y}_s)}\zeta_s\right]$$

It takes some patience to check it, but by disassembling and reassembling its terms (remember that $\zeta_o = 0$) the summation in the last formula can also be written as

$$\sum_{s=0}^{n-1} \beta^s \zeta_{s+1} \left[\frac{\varphi'(\hat{y}_s)}{g_2(\hat{x}_s, \hat{y}_s)} - \frac{g_1(\hat{x}_{s+1}, \hat{y}_{s+1})}{g_2(\hat{x}_{s+1}, \hat{y}_{s+1})}\beta\varphi'(\hat{y}_{s+1})\right] + \zeta_{n+1}\frac{\beta^n \varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)}$$

so that, finally

$$\Delta F \simeq \lim_{n\to\infty} \sum_{s=0}^{n-1} \beta^s \zeta_{s+1} \left[\frac{\varphi'(\hat{y}_s)}{g_2(\hat{x}_s, \hat{y}_s)} - \frac{g_1(\hat{x}_{s+1}, \hat{y}_{s+1})}{g_2(\hat{x}_{s+1}, \hat{y}_{s+1})} \beta \varphi'(\hat{y}_{s+1}) \right] + \\ + \lim_{n\to\infty} \zeta_{n+1} \frac{\beta^n \varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} \quad (24)$$

Now on to step (ii). It is easily seen that necessary and sufficient conditions for ΔF to be non-positive, no matter what the choice of the deviations ζ_s may be, are the following: first, for all s from 0 to infinity

$$\frac{\varphi'(\hat{y}_s)}{g_2(\hat{x}_s, \hat{y}_s)} - \frac{g_1(\hat{x}_{s+1}, \hat{y}_{s+1})}{g_2(\hat{x}_{s+1}, \hat{y}_{s+1})} \beta \varphi'(\hat{y}_{s+1}) = 0 \quad (25)$$

These are called Euler conditions in generalized form and are necessary conditions of optimality. As is easy to check, if these do not hold, it is always possibile to find deviations ζ_s such that $\Delta F > 0$. If they do hold, then for any deviation $\{\zeta\}$ the summation in formula (24) equals zero and ΔF reduces to

$$\Delta F \simeq \lim_{n\to\infty} \zeta_{n+1} \frac{\beta^n \varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} = \lim_{n\to\infty} \beta^n \frac{\varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} (x_{n+1} - \hat{x}_{n+1})$$

However, without one further condition concerning this limit we cannot rule out the possibility of finding deviations such that the limit turns out to be positive. Here is why Euler conditions (25) are generally not sufficient. The additional condition needed at this point is the one usually called 'transversality'.

The specific form of transversality may vary depending on the characteristics of the problem. With negative $\frac{\varphi'}{g_2}$ (see above, a necessary characterization of the optimal plans in all the problems in which the state variable affects positively the value of the plan), and with x bound to be non-negative, we have

$$\beta^n \frac{\varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} (x_{n+1} - \hat{x}_{n+1}) \leq -\beta^n \frac{\varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} \hat{x}_{n+1}$$

therefore

$$\Delta F \leq -\lim_{n\to\infty} \beta^n \frac{\varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} \hat{x}_{n+1}$$

the expression on the right-hand-side being non-negative. The transversality condition must in this case take the form

$$-\lim_{n\to\infty} \beta^n \frac{\varphi'(\hat{y}_n)}{g_2(\hat{x}_n, \hat{y}_n)} \hat{x}_{n+1} = 0 \quad (26)$$

As $\lim_{n\to\infty} \beta^n = 0$, transversality here simply requires that the 'cost' of increasing the state variable, $-\frac{\varphi'}{g_2}x$, be bounded.

Finally, as to the relationship between Bellman and Euler conditions. If a sequence $\{\hat{x}, \hat{y}\}_0^\infty$ solves an OCP then both Bellman equations (22) and Euler equations (25) must be satisfied at all s. This clearly implies

$$\frac{dV\left(\hat{x}_{s+1}\right)}{dx_{s+1}} = -\frac{g_1\left(\hat{x}_{s+1}, \hat{y}_{s+1}\right)}{g_2\left(\hat{x}_{s+1}, \hat{y}_{s+1}\right)}\varphi'\left(\hat{y}_{s+1}\right) \tag{27}$$

or in other words, if in Bellman equation we replace the term $\frac{dV}{dx}$ with the right-hand-side of this equation, Bellman is transformed into Euler.

In spite of this apparent substitutability, however, the two conditions are not equivalent. We have just seen that Bellman provides necessary and sufficient conditions for optimality, while Euler without transversality is only necessary (therefore weaker). To see the reason for this lack of equivalence, consider that in formula (27) the ratio

$$-\frac{g_1\left(\hat{x}_{s+1}, \hat{y}_{s+1}\right)}{g_2\left(\hat{x}_{s+1}, \hat{y}_{s+1}\right)}$$

measures at all dates the marginal rate of substitution between x and y, i.e. the variation in y_{s+1} that compensates for a unit variation in x_{s+1} so as to leave x_{s+2} unchanged. Writing $\frac{dV}{dx}$ as in (27) therefore means considering the effect of a variation in x_{s+1} on the value of the optimal plan beginning at $s+1$ in the particular hypothesis that the variation is entirely offset by a variation in y_{s+1} such that from $s+2$ on things continue as if nothing had changed. The whole effect of the variation is contained in period $s+1$.

Looking at it in this light, the interpretation of Euler equation is: if at s there is a variation in y_s from a given plan, and the effect of this variation on x_{s+1} is entirely absorbed in period $s+1$ so that from $s+2$ on we are back to the original plan, the effect of this two-period deviation must be nil. Thus, there is nothing to gain from such a particular deviation, and this is clearly a necessary condition of optimality of the given plan. The condition is not sufficient, however, for the simple reason that one could deviate from the original plan in many other ways, different from the one tested by means of Euler's method.

Bellman equation confirms Euler's necessary condition if $\frac{dV}{dx}$ is interpreted in the particular way we saw above. But Bellman equation as such tells much more than this. What it says is: there is nothing to gain if you deviate at s by choosing a different y_s and then, from $s+1$ on, you follow an optimal plan starting from the x_{s+1} resulting from the previous deviation. This may be a

two-period deviation, as with Euler, or any other possible kind of deviation. The proposition is clearly much stronger than Euler's, stronger enough to turn into a sufficient condition with no need to resort to transversality.

3.6 Application to the optimal path of consumption and accumulation

Just to see how Euler and Bellman conditions work in a simple case, we go back to the optimal consumption-accumulation plan seen in section 2.5. Take K in the role of the state variable we have called x and C in the role of the control y. The instantaneous return function φ coincides with instantaneous utility $u(C)$ and is independent of K. Finally, the law of motion is given by the budget equation $K_{s+1} = F(K_s) + K_s(1-\delta) - C_s$, with $g_1 = F' + 1 - \delta$, $g_2 = -1$.

In this particular case equation (25) becomes

$$u'\left(\hat{C}_s\right) - \beta u'\left(\hat{C}_{s+1}\right)\left(F'\left(\hat{K}_{s+1}\right) + 1 - \delta\right) = 0 \qquad s = 0, 1, ...$$

in which we recognize Euler condition (14) of 2.5.

Bellman equation instead derives from condition (21), that in this case takes the form

$$V\left(\hat{K}_s\right) = \max_{C_s}\left[u(C_s) + \beta V\left(F\left(\hat{K}_s\right) + \hat{K}_s(1-\delta) - C_s\right)\right]$$

The expression maximized on the right-hand side of the formula is differentiable and concave, so we get directly Bellman equation in the form

$$u'\left(\hat{C}_s\right) - \beta\frac{dV\left(\hat{K}_{s+1}\right)}{dK_{s+1}} = 0 \qquad s = 0, 1, ...$$

Comparison between this equation and Euler condition shows that

$$\frac{dV\left(\hat{K}_{s+1}\right)}{dK_{s+1}} = u'\left(\hat{C}_{s+1}\right)\left(F'\left(\hat{K}_{s+1}\right) + 1 - \delta\right) \tag{28}$$

as in (27). The change in the value of the optimal sequence due to a variation dK_{s+1} is measured by the marginal utility of the variation in consumption that leaves the stock of capital from K_{s+2} on unchanged.

Just to confirm – from a different point of view – that Euler and Bellman are conditions of different force, consider the following. Rewrite the two types

of conditions as difference equations in one of the two variables, for example in terms of the state variable K. Bellman equation gives

$$u'\left(F\left(\hat{K}_s\right)+\hat{K}_s\left(1-\delta\right)-\hat{K}_{s+1}\right)-\beta\frac{dV\left(\hat{K}_{s+1}\right)}{dK_{s+1}}=0$$

which is a first-order difference equation with one degree of freedom; Euler condition instead gives

$$\begin{aligned}&u'\left(F\left(\hat{K}_s\right)+\hat{K}_s\left(1-\delta\right)-\hat{K}_{s+1}\right)-\\&\quad-\beta u'\left(F\left(\hat{K}_{s+1}\right)+\hat{K}_{s+1}\left(1-\delta\right)-\hat{K}_{s+2}\right)\left(F'\left(\hat{K}_{s+1}\right)+1-\delta\right)=0\end{aligned}$$

a second-order difference equation with two degrees of freedom. Thus, fix one state arbitrarily by taking the initial state as given, $K_o=\hat{K}_o$: the solution to the Bellman equation determines a unique sequence, while the solution to the Euler condition is still undetermined and in order to identify the optimum sequence some other condition is needed. This confirms that Euler condition alone is insufficient to characterize the solution and must be accompanied by the transversality condition.

3.7 Dynamic maximization through Hamiltonians

Hamiltonian functions provide still another method to deal with an OCP of the kind we have seen in section 3.5. The underlying mathematical theory (Pontryagin et al., *The Mathematical Theory of Optimal Processes*, Wiley, N.Y. 1962) needn't concern us here. We start from the principle, demonstrated by Pontryagin, that if an OCP has a solution, that solution maximizes also an 'auxiliary' function called Hamiltonian that depends on the state and control variables of the problem plus other auxiliary variables called 'costate' variables. We take this principle for granted and simply illustrate how to put it in operation in working out the solution to an OCP.

Take an OCP with all the characteristics listed in 3.5. Only one state variable x and one control variable y, objective function in the form

$$F\left(\{y\}_0^\infty\right)=\lim_{n\to\infty}\left[\sum_{s=0}^{n}\beta^s\varphi\left(y_s\right)\right]$$

law of motion defined by

$$x_{s+1}=g\left(x_s,y_s\right)$$

and initial state $x_o=\hat{x}_o$.

The first step consists in defining the 'costate' auxiliary variables, say p_s, one for each date, with the only restriction that $p_s \geq 0$. The Hamiltonian function is a function in the three arguments x, y, p defined as follows

$$H(x_s, y_s, p_s) \equiv \varphi(y_s) + p_s g(x_s, y_s) \tag{29}$$

As is easy to check, if a sequence $\{x, y\}$ is feasible, i.e. satisfies the law of motion at all s, then

$$\varphi(y_s) = H(x_s, y_s, p_s) - p_s x_{s+1}$$

and therefore the problem of maximizing the objective function of the original OCP is the same as maximizing

$$\lim_{n\to\infty} \left[\sum_{s=0}^{n} \beta^s \left(H(x_s, y_s, p_s) - p_s x_{s+1} \right) \right]$$

with respect to x, y, p and with the only constraint $x_o = \hat{x}_o$. This is because the dynamic constraint represented by the function g has been embodied into the objective function.

We deal with this modified OCP problem by means of Euler's variational method. If a sequence $\{\hat{x}_s, \hat{y}_s, \hat{p}_s\}_o^\infty$ solves the problem, then there must be no gain $\Delta > 0$ in deviating from it. Let us indicate variations with the symbols $\eta_s = x_s - \hat{x}_s$, $\theta_s = y_s - \hat{y}_s$, and $\zeta_s = p_s - \hat{p}_s$. The optimal sequence will be characterized by the impossibility of creating a positive total differential

$$\Delta = \lim_{n\to\infty} \left[\sum_{s=0}^{n} \beta^s \left(H_{1s}\eta_s + H_{2s}\theta_s + H_{3s}\zeta_s - x_{s+1}\zeta_s - p_s\eta_{s+1} \right) \right]$$

(notation: the second subscript of H refers to date, the first subscript to partial derivative with respect to which argument) by arbitrarily choosing a sequence of variations $\{\eta_s, \theta_s, \zeta_s\}$ with the only restriction $\eta_o = 0$ (the deviation must in any case take $\hat{x}_o$ as given). Besides, from the definitional non-negativity of the costate variable, if p_s happens to be zero the only admitted variation is $\zeta_s > 0$.

By rearranging the terms under summation in the total differential above we get the following form

$$\Delta = \lim_{n\to\infty} \left[\sum_{s=0}^{n-1} \beta^s \left(-p_s + \beta H_{1,s+1} \right) \eta_s + \sum_{s=0}^{n} \beta^s \left[H_{2s}\theta_s + \left(H_{3s} - x_{s+1} \right) \zeta_s \right] \right] - \\ - \lim_{n\to\infty} \beta^n p_n \eta_{n+1} \tag{30}$$

In order to exclude the possibility of creating a positive Δ by varying either the x or the y sequence it is both necessary and sufficient that the

multipliers of all the η_s and θ_s in this expression are zero. The analogous condition for the multipliers of the ζ_s would be sufficient but not necessary: we know that ζ_s can only be positive when $p_s = 0$, therefore the only necessary condition in such a case would be that the multiplier of ζ_s is non positive.

Summing up, the necessary and sufficient conditions that grant that the sequence $\{\hat{x}_s, \hat{y}_s, \hat{p}_s\}$ maximizes the objective function are the following:

$$\begin{aligned} -p_s + \beta H_{1,s+1} &= 0 \\ H_{2s} &= 0 \\ H_{3s} - x_{s+1} \leq 0 \ &\textit{and} \ \ (H_{3s} - x_{s+1})\, p_s = 0 \\ -\lim_{n\to\infty} \beta^n p_n x_{n+1} &= 0 \end{aligned}$$

The third group of conditions is weaker for the reason explained above (if $H_{3s} - x_{s+1}$ is negative, p_s must be zero; if $p_s > 0$, $H_{3s} - x_{s+1}$ must be zero). The last condition, transversality, is specified in the hypothesis that x is bound to be non-negative as seen in 3.5.

Now we go back to definition (29) and rewrite the partial derivatives of the Hamiltonian as

$$\begin{aligned} H_{1s} &= p_s g_{1s} \\ H_{2s} &= \varphi'_s + p_s g_{2s} \\ H_{3s} &= g(x_s, y_s) \end{aligned}$$

Substitute from the latter into the conditions of maximum and get

$$-p_s + \beta p_{s+1} g_{1,s+1} = 0 \tag{31}$$

$$\varphi'_s + p_s g_{2s} = 0 \tag{32}$$

$$g(x_s, y_s) \leq x_{s+1} \textit{ with equality if } p_s > 0. \tag{33}$$

From (32) we derive an interpretation of the costate variables as the marginal cost of increasing x_{s+1} in terms of instantaneous benefit

$$p_s = -\frac{\varphi'_s}{g_{2s}}$$

as in 3.5. The non-negativity of p_s implies that in the optimal plan there must be a trade-off between future state and instantaneous benefit, given the present state x_s.

By substitution from (32) into (31), and dividing by g_{2s}, we get

$$\frac{\varphi'_s}{g_{2s}} - \beta \varphi'_{s+1} \frac{g_{1,s+1}}{g_{2,s+1}} = 0$$

which corresponds exactly to the generalized Euler condition (25) of 3.5.

Condition (32) also implies that (unless $\varphi'_s = 0$) costate p_s is generally positive and therefore, from (33), $g(x_s, y_s) = x_{s+1}$.

Lastly, from the interpretation of p_s as the marginal cost of x_{s+1}, we rewrite transversality as

$$-\lim_{n\to\infty} \beta^n p_n x_{n+1} = \lim_{n\to\infty} \beta^n \frac{\varphi'_n}{g_{2n}} x_{n+1} = 0$$

Just to see this method at work, apply it to the usual optimal consumption-accumulation path. The Hamiltonian function of the problem is

$$H(K_s, C_s, p_s) = u(C_s) + p_s (F(K_s) + (1-\delta) K_s - C_s)$$

with partial derivatives

$$\begin{aligned} H_{1s} &= p_s (F'(K_s) + 1 - \delta) \\ H_{2s} &= u'(C_s) - p_s \\ H_{3s} &= F(K_s) + (1-\delta) K_s - C_s \end{aligned}$$

The first two maximum conditions give

$$\begin{aligned} -p_s + \beta p_{s+1} (F'(K_{s+1}) + 1 - \delta) &= 0 \\ p_s &= u'(C_s) \end{aligned}$$

from which Euler condition is immediately derived. Transversality is given by the usual asymptotic condition

$$-\lim_{n\to\infty} \beta^n p_n K_{n+1} = -\lim_{n\to\infty} \beta^n u'(C_n) K_{n+1} = 0$$

4 Introduction to uncertainty

Economic agents make intertemporal choices but the actual context of these choices is not a world of perfect knowledge of the future ('perfect forecast') of the kind assumed in all the models presented so far. The latter are mere didactic devices used for introducing the basic features of theories, such as DGE, based on optimization and coordination of plans in as simple as possible idealized environments. But there comes a stage at which theory must shed the fiction of perfect forecast and come to grips with the inescapable uncertainty that surrounds the agent at the moment of choosing alternative courses of action.

Uncertainty as a subjective state of mind is represented in economic models by means of probability distributions on events that may be true or false, without the agent knowing which one is the case. Probability of an event is supposed to measure the degree of subjective belief in the truth of it – its actual occurring. The fact that belief is less than complete may derive from mere ignorance on the part of the agent, or reflect the presence of stochastic factors in the causal processes by which the event is determined. In all cases, probabilities allow the subject to form expectations on the basis of which to make decisions, and also to appreciate the level of risk involved in relying on expected values that are accompanied by higher or lower variance. In case the theoretical model of the economy takes the presence of objective stochastic factors into explicit account there is the further issue of the correspondence, or the lack of it, between subjective beliefs and objective probabilities. This opens a range of models, with expectations being assumed to be more or less 'rational' depending on their closeness to the actual stochastic processes affecting the state of the economy.

Most of the basic notions on probability utilised in macroeconomic models are explained in intermediate textbooks of statistics, econometrics or probability theory. The following sections contain a quick summing up of the main concepts with no claim of completeness.

4.1 Probability spaces and distribution functions

A probability measure is a special case of measure, i.e. a function with values in $\mathbb{R}$ and the domain of which is a set endowed with a particular mathematical structure called sigma-algebra or sigma-field. To constitute a sigma-algebra we need a set of states ('state space'), call it Ω, and a collection of subsets of Ω, call it $\mathcal{A}$, such that (i) Ω belongs to $\mathcal{A}$, and (ii) $\mathcal{A}$ is closed with respect to the set-theoretic operations of complementation ($a \in \mathcal{A}$ iff $a^c \in \mathcal{A}$, the superscript c standing for 'complement of') and countable union (if all the

terms of the countable sequence $\{a_i : i = 1, 2, ...\}$ belong to $\mathcal{A}$, their union belongs to $\mathcal{A}$ as well). A pair such as $(\Omega, \mathcal{A})$, composed of a state space and a sigma-algebra based on it, is called a measurable space.

In applications to probability measures the elements of $\mathcal{A}$ are interpreted as events. Set-theoretic operations have obvious counterparts in linguistic operations ('negation', 'conjunction', 'disjunction') on propositions. Closure with respect to complementation and countable union is easily seen to imply closure with respect to countable intersection; and from $\Omega \in \mathcal{A}$ it obviously follows $\phi \in \mathcal{A}$, where ϕ, the empty set, is interpreted as 'impossibility', the opposite of 'necessity' represented by Ω.

A probability function P is a function mapping $\mathcal{A}$ into the unit real interval $[0, 1]$, $P : \mathcal{A} \rightarrow [0, 1]$, endowed with two properties, (i) normalization to 1, and (ii) additivity over countable unions of disjoint sets. Property (i) means that the function is scaled so as to have $P(\Omega) = 1$ (no event can have higher probability than Ω, called the 'universal event'). Property (ii) means that if $\{a_i \in \mathcal{A} : i = 1, 2, ...\}$ is a collection of events pairwise disjoint, then

$$P\left(\bigcup_i a_i\right) = \sum_i P(a_i)$$

A triple such as $(\Omega, \mathcal{A}, P)$, composed of a measurable space joined to a probability measure, is called a probability space.

In discrete time macrodynamics the points of the state space, $\omega \in \Omega$, are typically interpreted as sequences or time series, covering a stretch of time of any length, $\{X_s : s \in \boldsymbol{T}\}$ where X is an economic variable or vector of variables and $\boldsymbol{T}$ a set of integers, possibly extended from $-\infty$ to $+\infty$, providing all the needed time indexes. Events of the sigma-algebra $\mathcal{A}$ may take shapes such as $(X_s = x)$, meaning 'the set of all $\omega \in \Omega$ having $X_s = x$ at the s-th place'; or $(X_s = x \text{ } and \text{ } X_v = y)$, meaning the intersection of $(X_s = x)$ and $(X_v = y)$; or $(X_s \in [q, r])$, meaning the union of all the $(X_s = x)$ where $x \in [q, r]$, and so on. A singleton $\{\omega\} \in \mathcal{A}$ will consist of a complete list of realisations of X_s, one for each $s \in \boldsymbol{T}$, i.e. a fully specified time series.

X_s is a random variable, or random vector if it is a vector. In this presentation I consider the case of a variable for simplicity. The probabilities of its possible realisations can be derived from $(\Omega, \mathcal{A}, P)$ by means of a convenient mathematical device called distribution function, as follows. Let $S \subseteq \mathbb{R}$ represent the set of all the values that X_s may possibly take on. Since in each state $\omega \in \Omega$ (fully specified sequence) X_s will take a definite numerical value, we can consider X_s as a function that maps the points of Ω onto the set of real numbers S, $X_s : \Omega \rightarrow S$, with $S = \{x \in \mathbb{R} : \exists \omega \in \Omega (x = X_s(\omega))\}$.

In order to construct a distribution function we have now to assume that the function X_s has a property called 'measurability with respect to $\mathcal{A}$',

meaning the following: for a suitably defined sigma-algebra on S ('suitably defined' in a sense that will be specified below), call it $\mathcal{B}$, the pre-image of each numerical event $b \in \mathcal{B}$ according to X_s, i.e. the set defined as

$$X_s^{-1}(b) \equiv \{\omega : X_s(\omega) \in b\}$$

is an event that belongs to $\mathcal{A}$. If this property is granted, then the probability measure P can be transferred from $\mathcal{A}$ to $\mathcal{B}$ simply by associating the probability $P(X_s^{-1}(b))$ to each event $b \epsilon \mathcal{B}$. The composite function just defined

$$P \circ X_s^{-1} : \mathcal{B} \rightarrow [0,1]$$

is called a 'distribution function' and serves the purpose of attaching well-defined probabilities to all kinds of events included in $\mathcal{B}$. In fact, we have constructed a new probability space $(S, \mathcal{B}, P \circ X_s^{-1})$ derived from $(\Omega, \mathcal{A}, P)$ and tailored to represent the probabilities of numerical events concerning our random variable X_s. For each random variable we can construct one such probability space, all of these being derived from the same original space $(\Omega, \mathcal{A}, P)$.

Exercise 11 *Show that $P \circ X_s^{-1}$ is a true probability function, i.e. satisfies the properties of (i) normalization to 1, and (ii) countable additivity over unions of disjoint events.*

A few words of explanation about $\mathcal{B}$. A natural candidate for $\mathcal{B}$ would seem to be the set of all subsets of S, i.e. the power set of S, which is always the largest (meaning 'most inclusive') sigma-algebra that can be based on any set. However, when X_s is a continuous random variable and S is an interval or the whole of $\mathbb{R}$, referring to such a large set may create problems with measurability. To avoid these, the most suitable choice is a 'smaller' sigma-algebra known as Borel algebra. If defined on $\mathbb{R}$, the Borel algebra $\mathcal{B}(\mathbb{R})$ is the set of all the subsets of $\mathbb{R}$ that can be generated by way of complementation and countable union starting from semi-closed intervals of the kind $(-\infty, x]$. These can be thought of as the basic building materials, the elementary 'bricks' that serve to construct all the elements of $\mathcal{B}(\mathbb{R})$. As can be checked simply by trying, practically all kinds of intervals in the real line, including degenerate closed ones (such as $[x, x]$), can be constructed by way of complementation and countable union of the elementary intervals, so that $\mathcal{B}(\mathbb{R})$ comes to coincide with the set of almost all the real intervals in $\mathbb{R}$. If S is a proper subset of $\mathbb{R}$, its Borel algebra $\mathcal{B}$ will be the so-called restriction of $\mathcal{B}(\mathbb{R})$ to S, that is the set of the intersections of all the sets of $\mathcal{B}(\mathbb{R})$ with S. Since $\mathcal{B}(\mathbb{R})$ is a sigma-algebra in $\mathbb{R}$, it can be proved that its

restriction to S, $\mathcal{B} \subset \mathcal{B}(\mathbb{R})$, is a sigma-algebra on S. This obviously covers the cases in which X_s is a discrete variable taking on finitely many values, with S being a countable subset of $\mathbb{R}$ too.

For simplicity, from now on we reason in the hypothesis that $S = \mathbb{R}$ unless otherwise specified. If we restrict the distribution function to the elementary 'bricks' in $\mathcal{B}(\mathbb{R})$, i.e. to the events of the kind $b_x = (-\infty, x]$, we obtain what is commonly known as the 'cumulative distribution function' of X_s. Indeed, $X_s^{-1}(b_x)$ is the event $\{\omega : X_s(\omega) \leq x\}$; its probability $P \circ X_s^{-1}(b_x)$ can be written as an ordinary algebraic function from $\mathbb{R}$ into $[0,1]$ as follows

$$F_s(x) \equiv P \circ X_s^{-1}(b_x)$$

The properties of cumulative distribution functions, especially its positive monotonicity, are known from statistics and probability theory. To recall, once we have the cumulative distribution function F_s, the probability of all types of numerical non-elementary events can be reconstructed starting from it. For example, take an event such as $b = (y, x]$ with $y < x$. Since, as is easy to check, b is equal to the complement of the union of two disjoints, $(-\infty, y]$ (whose probability is $F_s(y)$) and $(x, +\infty)$ (probability $1 - F_s(x)$), then simple computation gives

$$\begin{aligned} P \circ X_s^{-1}(b) &= 1 - [F_s(y) + 1 - F_s(x)] \\ &= F_s(x) - F_s(y) \end{aligned}$$

In general, a distribution function may or may not be continuous, depending on the properties of X_s. This makes a big difference when it comes to attributing probabilities to single values of X_s, i.e. to numerical events corresponding to degenerate intervals of the kind $[x, x]$. A degenerate interval can be seen as a countable intersection of the kind

$$\begin{aligned} [x, x] &= \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x \right] \\ &= \lim_{t \to \infty} \bigcap_{n=1}^{t} \left(x - \frac{1}{n}, x \right] \end{aligned}$$

In terms of probability, for each term of the intersection we have

$$P \circ X_s^{-1} \left(\bigcap_{n=1}^{t} \left(x - \frac{1}{n}, x \right] \right) = F_s(x) - F_s\left(x - \frac{1}{t} \right)$$

Continuous functions have the property $f(\lim \cdot) = \lim f(\cdot)$. Thus, if the distribution function is continuous, we have

$$P \circ X_s^{-1}([x, x]) = F_s(x) - \lim_{t \to \infty} F_s\left(x - \frac{1}{t} \right) = 0$$

i.e., with continuous distributions the probability of single values is always zero. For $[x, x]$ to have positive probability, F_s must have a discontinuity at x. The latter is certainly the case when X_s is a discrete random variable, its range S being a finite set of 'isolated' points of $\mathbb{R}$. In that case, as is known, F_s is a step function, each 'jump' representing the probability of the value at which the jump occurs. When X_s is non-discrete, instead, dealing with probabilities of single values requires attention (see the problems with conditional probability in the next section).

In the case of a distribution function F_s continuous and differentiable the first derivative is called 'density function'

$$f_s(x) \equiv \lim_{\delta \to 0} \frac{F_s(x+\delta) - F_s(x)}{\delta}$$

Density indicates the limit of the average probability mass concentrated in intervals $(x, x+\delta]$ as the interval collapses to a point. The mass in point x is of course zero, but the limit of the average mass can be any non-negative value. From the definition of density it obviously follows that $F_s(x) = \int^x f_s(u)\, du$, from which the designation of F_s as a 'cumulative' function.

The expected value (or mean value, or first moment) of X_s, usually indicated with the notation EX_s can be computed as a sum or as an integral depending on whether the distribution of X_s is discrete or continuous and differentiable. The same as regards variance (second moment of $X_s - EX_s$) and covariance between X_s and any other random variable that can be extracted out of Ω, such as X_v where v is a date different from s ('autocovariance'), or Y_s where Y measures a different phenomenon from X, and the like. The formulas are well-known and needn't be recalled here.

Recalling that X_s has been defined as a measurable function from the state space Ω onto $S \subseteq \mathbb{R}$, it is also possible to integrate X_s directly on Ω by means of a notion of integral, called Lebesgue integral, that turns out to be more general than the usual linear or Riemann integral (in the sense that Lebesgue and Riemann integrals coincide when the latter happen to exist, while Lebesgue may exist when Riemann don't, for example with non-continuous functions). We cannot expand on this subject here, the present hint is only for information and reference. Suffice it to say that if in constructing the Lebesgue integral of X_s on Ω the events in $\mathcal{A}$ are measured by means of the probability function P, the resulting integral, variously indicated with the notation

$$\int_\Omega X_s(\omega)\, P(d\omega) \quad \text{or} \quad \int_\Omega X_s dP$$

coincides with the expected value EX_s. For further examination of this

subject see Stokey and Lucas, cit., chapter 7; John Stachurski, *Economic Dynamics: Theory and Computation*, MIT Press, 2009, chapter 7.

4.2 Information and conditioning

With the passing of time, time series turn uncertainty into certainty in the sense that one X_s after another takes on a definite value and randomness is progressively confined to the dates that have yet to come. Consequently, at each date t the (unconditional, or 'a priori', or simply 'prior') probability function P must be updated by making it conditional to what has become certain at t according to the ordinary rules of probabilistic conditionalization.

Let $\{x_s : s \leq t\}$ be the sequence of realisations of all the random variables X_s up to t. This identifies an event in $\mathcal{A}$, corresponding to the set of all the time series $\omega \in \Omega$ characterized by the given $\{x_s : s \leq t\}$ sequence up to t. Let us call $D_{ti} \in \mathcal{A}$ this particular event. Had the sequence up to t been different for just one realisation, or more than one, the corresponding event in $\mathcal{A}$ would have been a different one, call it D_{tj}, and of course no time series $\omega \in \Omega$ could possibly belong to both events, i.e. $D_{ti} \cap D_{tj} = \phi$. Considering now all the possible different sequencies up to t, it is not difficult to convince oneself that the corresponding events must constitute a partition of Ω, call it $\mathcal{D}_t \subset \mathcal{A}$. Now repeat the exercise taking a later date $v > t$. The longer sequences of realisations up to v will generate a 'finer' (or 'less coarse') partition $\mathcal{D}_v \subset \mathcal{A}$, in the sense that each event $D \in \mathcal{D}_t$ will now correspond to a union of more specific events $D' \in \mathcal{D}_v$ (these will have the same sequence up to t in common, and differ from each other for the sequence from $t+1$ to v). And so on, as the date index slides forward.

At any date t, given a particular realization $D \in \mathcal{D}_t$, the state space Ω shrinks to D and the algebra of events $\mathcal{A}$ to

$$\mathcal{A}_D := \{a \cap D : a \in \mathcal{A}\} \subset \mathcal{A}$$

Together with a conditional probability function $P_D : \mathcal{A}_D \to [0,1]$ (see below), this defines a conditional probability space $(D, \mathcal{A}_D, P_D)$, one for each $D \in \mathcal{D}_s$.

If $P(D) > 0$, the conditional probability P_D is derived from P through the well-known Bayes rule: for any $a \in \mathcal{A}$, $a \cap D \in \mathcal{A}_D$ and

$$P_D(a \cap D) := \frac{P(a \cap D)}{P(D)} = P(a \mid D) \tag{34}$$

The corresponding conditional distribution and cumulative conditional distribution of a random variable X_s with date index $s > t$ come directly

from these definitions. The domain of X_s is now restricted to D but the range S is unchanged: X_s^{-1} now maps $\mathcal{B}$ into $\mathcal{A}_D$, with distribution function $P_D \circ X_s^{-1}$. The cumulative conditional distribution function is given by

$$F_s(x \mid D) \equiv P_D \circ X_s^{-1}(b_x)$$

If instead X_s is a random variable with $s \leq t$, at t it has become a degenerate random variable in the following sense: let $\bar{x}$ be the (now uniquely determined) value of X_s, then the event $X_s = \bar{x}$ coincides with D and of course $P_D(D) = 1$, while for any $x \neq \bar{x}$ the event $X_s = x$ coincides with the nil event ϕ with $P_D(\phi) = 0$. Moreover, $F_s(x \mid D) = 0$ for $x < \bar{x}$, and $F_s(x \mid D) = 1$ for $x \geq \bar{x}$.

The application of these definitions, in particular of the definition of conditional probability (34), to the case of continuous random variables is not banal since in this case the prior probability P of events such as $X_s = \bar{x}$ may easily turn out to be zero and therefore, with $P(D) = 0$, formula (34) does not apply, although the concept of conditional probability may continue to make sense. See the example here below.

Example 12 *The state space is the set of two-period sequences of the type* $\{X_1, X_2\}$*, where each* X_s *is a random variable taking values in the* $[0,1]$ *interval. The cumulative joint distribution function* $F : ([0,1])^2 \to [0,1]$ *is continuous. At date 1 information about* X_1 *induces a partition* $\mathcal{D}$ *of the domain of the function,* $([0,1])^2$*, into events of the kind* $D_x \equiv \{x_1, x_2 : x_1 = x, x_2 \in [0,1]\}$*, with* x *a real number in the* $[0,1]$ *interval. As recalled in the previous section, with continuous* F *the probability of events (pieces of information) such as* D_x *is zero. Thus, take a generic event* $a \equiv \{x_1, x_2 : x_1 \in [0,1], x_2 \leq y\}$ *with prior probability* $F(1, y)$*. The intersection* $a \cap D_x$ *has probability zero as well, so that the conditional probability of* a *given* D_x *cannot be determined. Yet it may make sense to search for such a probability. As soon as we consider a 'perturbation' of the information represented by* D_x*, for example the slightly different event*

$$D_{x,\varepsilon} \equiv \{x_1, x_2 : x_1 \in (x - \varepsilon, x + \varepsilon], x_2 \in [0,1]\}$$

with $\varepsilon > 0$ *no matter how small (the smaller* ε*, the better the approximation to* D_x*), both* $D_{x,\varepsilon}$ *and* $a \cap D_{x,\varepsilon}$ *have generally non-nul probability, respectively* $F(x + \varepsilon, 1) - F(x - \varepsilon, 1)$ *and* $F(x + \varepsilon, y) - F(x - \varepsilon, y)$*. If these are both positive and* ε *sufficiently small, the ratio*

$$\frac{F(x + \varepsilon, y) - F(x - \varepsilon, y)}{F(x + \varepsilon, 1) - F(x - \varepsilon, 1)}$$

may be considered a satisfactory estimate of the conditional probability of a given D_x.

The example provides an intuitive sketch of the general procedure that may be followed in order to obviate the difficulty of conditioning on events that, because of continuity, have necessarily zero prior probability. Let $\mathcal{D}_t$ be the information partition at t, with each $D \in \mathcal{D}_t$ having nil probability. Call $\mathcal{A}_t$ the sigma-algebra generated by $\mathcal{D}_t$ by way of countable union and complementation of its members, with $\mathcal{D}_t \subset \mathcal{A}_t \subset \mathcal{A}$. Take a generic event $b \in \mathcal{A}$ and for each $a \in \mathcal{A}_t$ write the equation

$$P(a) P(b \mid a) = P(b \cap a)$$

It can be shown, on the basis of measure-theoretic arguments that here we must omit, that the equation allows to calculate a well-defined value $P(b \mid a)$ for 'almost all' the $a \in \mathcal{A}_t$, in the precise sense that the events a for which $P(b \mid a)$ cannot be determined (typically, but not only, all the events $D \in \mathcal{D}_t$) have probability zero.

Now form a partition of Ω by assembling a collection of events with positive probability in $\mathcal{A}_t$. This can always be done, and can be done in such a way as to approximate the unusable original partition $\mathcal{D}_t$ as much as is needed. In the case of example 12 above, one may take a finite set of $n+1$ numbers in $[0,1]$, $\{i_0, i_1, ...i_n\}$ with $i_0 = 0$, $i_n = 1$ and such that the $[0,1]$ interval is subdivided into n intervals of equal length $\frac{1}{n}$. The collection of events

$$D_h \equiv \{x_1, x_2 : x_1 \in (i_h, i_{h+1}], x_2 \in [0,1]\}$$

with $h = 1...n-1$, together with the event

$$D_o \equiv \{x_1, x_2 : x_1 \in [i_0, i_1], x_2 \in [0,1]\}$$

make up a partition of the domain of events $([0,1])^2$. This is coarser than the $\mathcal{D}$ partition of our example, but the latter can be approximated more and more closely simply by increasing the number n of slices into which $([0,1])^2$ is subdivided. In addition, we may make so that all the slices have positive prior probability.

In general, by constructing 'surrogate' partitions of this kind it is always possible to define a conditional probability function that can be interpreted as an estimate of what the probability conditional on the available information would be if it were not indeterminate. In the following, we shall always refer to conditional probabilities at some date t with reference to some partition $\mathcal{D}_t$, either directly suggested by the setup of the model or suitably constructed with the approximating procedure seen here.

Considered from an 'a priori' point of view, i.e. before the uncertainty concerning information available at t is resolved, conditional probability is as uncertain as the information on which it depends. Take a generic event $a \in \mathcal{A}$, and call $P(a \mid \mathcal{D}_t)$ the function "conditional probability of a at t", where $\mathcal{D}_t \subset \mathcal{A}$ is the partition whose members represent pieces of information that may possibly be known at t. This is a function, actually a random variable, from Ω into $[0,1]$ that can be computed starting from the unconditional probability function P by means of formula (34). As a consequence, the random variable $P(a \mid \mathcal{D}_t)$ must possess two characteristics:

(i) For each $D \in \mathcal{D}_t$ and all the $\omega \in D$, $P(a \mid \mathcal{D}_t)(\omega)$ has constant value.

(ii) The unconditional expected value $EP(a \mid \mathcal{D}_t)$ coincides with the unconditional $P(a)$.

Property (i) is obvious. Property (ii) can be proved easily from formula (34), as can be seen solving the following

Exercise 13 *Assume that the partition $\mathcal{D}_t$ is countable and that $P(D) > 0$ for all $D \in \mathcal{D}_t$. By using formula (34) demonstrate that for any $a \in \mathcal{A}$*

$$P(a) = EP(a \mid \mathcal{D}_t)$$

[Hint: take into account the fact that

$$a = \bigcup_{D \in \mathcal{D}_t} (a \cap D)$$

and the property of countable additivity of P.]

Notice that property (ii), that connects the prior probability of a given event to its conditional probability at a date t, can be generalized so as to connect conditional probabilities at any two different dates. Take two dates t and $t+n$, with n a positive integer. Consider that the relevant partition at $t+n$ will generally be finer than the partition at t. Following the same track as in exercise 13, prove that for each $D \in \mathcal{D}_t$ and $a \in \mathcal{A}$

$$P(a \mid D) = E(P(a \mid \mathcal{D}_{t+n}) \mid D) \tag{35}$$

where the expected value on the right hand side is computed using probabilities conditional on D. The last formula shows that an important property of Bayesian expectations – expectations based on probabilities updated with respect to information according to Bayes rule (34) – is their "consistency" through time. From t to $t+n$ probabilities will most likely change as time

usually brings new information, but the way they may change is already taken into account at t. Indeed, one can see, directly from formula (35), that

$$E\left(P\left(a \mid \mathcal{D}_{t+n}\right) - P\left(a \mid D\right) \mid D\right) = 0$$

i.e., future corrections of probability are expected on average to amount to zero.

4.3 Restriction to weakly stationary processes

In macroeconomic applications we are interested in sequences of random variables (time series) of the type $\{X_s : s = 0, 1, ...\}$ with X standing for a macroeconomic variable or vector of variables. Assuming that the probability distribution of X_s remains the same at all dates s seems excessively restrictive and at odds with economic experience. On the other hand, leaving it free to change without restrictions of any sort seems also scarcely plausible, beyond introducing excessive volatility into the model. What is needed is an intermediate position which avoids the extreme of absolute stationarity of the probability distributions while admitting that their variability is subject to restrictions that grant a certain degree of time-homogeneity. The latter serves to keep the complexity of the theory within reasonable limits, and also to design a context in which agents can learn from experience by estimating the probabilities ruling their economic environment from the statistics of past realizations.

This aim can be achieved in various ways. The most common consists in assuming that X_s is ruled by a law of motion of the kind

$$X_{s+1} = g\left(X_s, \varepsilon_{s+1}\right) \tag{36}$$

where g is a deterministic function and ε_s a random variable ('shock') of the 'White Noise' (WN) kind, $\varepsilon \sim WN$. To recall, this means that ε has positive variance but zero mean and autocovariance across intervals of any length.

Such a law of motion generates a first-order autoregressive process (AR), the underlying probabilities of which depend exclusively on the distribution of ε. At each s the conditional distribution of X_{s+1} changes according to the realisation of X_s but, as is easily seen, the probability of transition from any given value x to any y between two successive dates

$$P\left(X_{s+1} = y \mid X_s = x\right) = P\left(\{\varepsilon_{s+1} : g\left(x, \varepsilon_{s+1}\right) = y\}\right)$$

is always the same independently of the date s, unless the distribution of ε changes as well. This property is common to all so-called 'Markov processes' or 'stochastic recursive sequences'.

Frequently the function g in the process (36) is assumed to be linear, in the form

$$X_{s+1} = \alpha X_s + \varepsilon_{s+1}$$

If $|\alpha| = 1$ the process is a random walk. Let us take the case $|\alpha| < 1$: admitting that the process is bounded and extends backwards to $-\infty$, we can write it in the moving average (MA) form

$$X_s = \lim_{n\to\infty} \left[\alpha^n X_{s-n} + \sum_{h=0}^{n-1} \alpha^h \varepsilon_{s-h} \right] = \sum_{h=0}^{\infty} \alpha^h \varepsilon_{s-h}$$

(this can be checked simply by substituting X_s backwards indefinitely, or by means of the methods illustrated in the next section).

As can be seen from the MA representation the process has unconditional first moment $EX_s = 0$, and all the second order unconditional moments for $n = 0, 1, ...$

$$Cov\left(X_{s+n}, X_s\right) = \frac{\alpha^n}{1-\alpha^2} Var\varepsilon$$

(again, check this as an exercise).

A process like this, with all its moments up to second order constant independently of the date s, is called 'weakly' or 'second-order' stationary. Notice that in the case $|\alpha| = 1$ the process, a random walk, does not belong to the weakly stationary class.

In order to make sure that the stochastic processes we shall deal with satisfy this mild condition of time-homogeneity it is not strictly necessary to assume that the probability distribution of the basic chance factor ε remains unchanged through time. Changes in the distribution of ε may be reflected in the probabilities of transition as defined above: but as far as ε remains WN and its variance does not change, mean variance and autocovariances of X_s remain unchanged too. From the point of view of macroeconomic applications this is almost always all that matters.

4.4 The 'lag' operator

Before we proceed to apply these notions to economic models, we use this last section to introduce a new operator typically used by econometricians to analyze time series and to solve linear difference equations that include stochastic factors. This is the operator called 'lag' or 'backshift' and denoted with the symbol L.

By definition, the application of L to a term x_t of a time series transforms it into its immediate predecessor, $Lx_t = x_{t-1}$. Applying L to the whole series

$\{x_t\}$ generates the series itself shifted one position backwards (if the series goes from $-\infty$ to $+\infty$ nothing changes). If the series is multiplied by a constant a, then lag and product commute with each other $Lax_t = aLx_t$; the same in the case of addition, $L(x_t + y_t) = Lx_t + Ly_t$. The last two properties taken together show that L is a linear operator.

In general, the power of an operator is the number of times it is applied: in the case of L, $L^h x_t$ means taking x_{t-h}. Power zero coincides with identity (doing nothing) $L^0 x_t = 1x_t = x_t$. From this, there follows the definition of the inverse operator for any $h = 1, 2, ...$

$$L^{-h}\left(L^h x_t\right) = L^0 x_t = x_t$$

which of course implies $L^{-h}x_t = x_{t+h}$.

Combinations of operations on time series can be defined simply by forming polinomial expressions in L. See for example the following, corresponding to first, second, third order differences (Δ) between the terms of a time series

$$\begin{aligned} \Delta &= 1 - L \\ \Delta^2 &= (1-L)^2 = 1 - 2L + L^2 \\ \Delta^3 &= (1-L)^3 = 1 - 3L + 3L^2 - L^3 \qquad etc \end{aligned}$$

In general, applying a 'polinomial lag' to a time series generates a time series of weighted moving averages. For example, from the polinomial lag $a_0 + a_1 L + a_2 L^2$ applied to the series $\{x_t\}$ with t running from $-\infty$ to $+\infty$, obtain the series $\{a_0 x_t + a_1 x_{t-1} + a_2 x_{t-2}\}$.

A notable polinomial lag, particularly useful in solving linear systems of difference equations, is the following

$$P = \sum_{h=0}^{\infty} a^h L^h$$

Applied to each x_t of a series extending backwards to $-\infty$ it results in the infinite expansion

$$Px_t = \sum_{h=0}^{\infty} a^h x_{t-h}$$

If this summation is finite, then it is also true that

$$aLPx_t = \sum_{h=1}^{\infty} a^h x_{t-h} = Px_t - 1x_t$$

from which it follows that $(1 - aL)\,Px_t = x_t$, and thus

$$Px_t = (1 - aL)^{-1}\, x_t$$

When the sequence $\{x_t\}$ is bounded, meaning that there exists a finite positive X such that $X \geq |x_t|$ at all t, a sufficient condition for P to be equal to the inverse of $(1 - aL)$ is $|a| < 1$. Under this condition

$$Px_t \leq X \sum_{h=0}^{\infty} a^h = \frac{X}{1-a}$$

which guarantees that Px_t is finite.

(I) Use of L in solving a system of first-order difference equations

As a first example of the use of L in solving a system of non-homogeneous linear difference equations we apply it to the system used as an illustration in sections 3.2 and 3.3. Let $f_t = B\,(f_{t-1} + g_t)$ be the system on n equations, with the sequence of the known terms $\{g_t\}$ going back to $-\infty$ and forward to $+\infty$.

Here the lag operator L must be understood as the square matrix of dimension n having L on the main diagonal and zero elsewhere, so that applying L to an n-vector f_t transforms it into f_{t-1}. Thus $Bf_{t-1} = BLf_t$ and the system becomes

$$(I - BL)\, f_t = Bg_t \tag{37}$$

Now we apply the operator $(I - BL)^{-1}$ to both members of (37) and get

$$\begin{aligned} f_t &= (I - BL)^{-1} Bg_t \\ &= \sum_{h=0}^{\infty} B^{h+1} g_{t-h} \end{aligned} \tag{38}$$

If the sequence $\{g_t\}$ is bounded, for the summation on the right-hand side to have a finite value it is enough to assume that

$$\lim_{h \to \infty} B^h = 0$$

This requires as a necessary and sufficient condition that all the characteristic roots of B have absolute value smaller than one (all 'stable' roots).

As is easy to check, if it exists, the function on the right-hand side of (38) is a particular backward solution to the original system. As with all non-homogeneous linear systems, the general solution is found by adding a particular solution to the general solution to the auxiliary homogeneous system

$$(I - BL)\, f_t = 0$$

The latter is easily solved by substitution: from

$$f_t = BLf_t = (BL)^2 f_t = (BL)^3 f_t = ...$$

choosing an arbitrary vector f_o at $t = 0$ (or any other date) we get the solution

$$f_t = (BL)^t f_t = B^t f_o$$

The general backward solution to (37) can therefore be written as

$$\begin{aligned} \varphi_t &= B^t f_o + \sum_{h=0}^{\infty} B^{h+1} g_{t-h} \\ &= B^t \left(f_o + \sum_{h=0}^{\infty} B^{h+1} g_{-h} \right) + \sum_{h=0}^{t-1} B^{h+1} g_{t-h} \\ &= B^t \varphi_o + \sum_{h=0}^{t-1} B^{h+1} g_{t-h} \end{aligned}$$

For the particular choice $f_o = 0$ this coincides with the particular solution (38).

The backward solution is non viable if the value of the summation in formula (38) is infinite. Even in that case, however, we can write a 'truncated' backward solution such as, for example (truncating at date $s = 0$)

$$\begin{aligned} \varphi_t &= B^t f_o + \sum_{h=0}^{t} B^{h+1} g_{t-h} \\ &= B^t \varphi_o + \sum_{h=0}^{t-1} B^{h+1} g_{t-h} \end{aligned}$$

with $\varphi_o = f_o + Bg_o$ as a free parameter (compare similar formulas in section 3.3).

In order to get the forward solution rewrite (37) as

$$\left(I - B^{-1}L^{-1}\right) f_t = -g_{t+1}$$

from which obtain

$$\begin{aligned} f_t &= -\left(I - B^{-1}L^{-1}\right)^{-1} g_{t+1} \\ &= -\sum_{h=0}^{\infty} B^{-h} g_{t+1+h} \end{aligned}$$

Recalling that the characteristic roots of B^{-1} are the inverse of the characteristic roots of B, the summation on the right-hand side converges to a finite value if and only if all the roots of B are unstable. In that case, the function on the right-hand side provides a particular forward solution.

The corresponding general forward solution (check as an exercise) will be, for an arbitrary choice f_{t+n} at any future date $t+n$

$$\psi_t = B^{-n} f_{t+n} + \sum_{h=0}^{\infty} B^{-h} g_{t+1+h}$$

In case not all roots of B are unstable this solution is not viable, but we can always resort to a truncated solution such as

$$\begin{aligned} \psi_t &= B^{-n} f_{t+n} + \sum_{h=0}^{n} B^{-h} g_{t+1+h} \\ &= B^{-n} \psi_{t+n} + \sum_{h=0}^{n-1} B^{-h} g_{t+1+h} \end{aligned}$$

with $\psi_{t+n} = f_{t+n} + g_{t+n+1}$ as a free parameter.

(II) Use of L in solving a second-order difference equation

As a second example, we apply L to the solution of a (linear, fixed coefficients) second order difference equation. We take a single equation for simplicity.

Let the generic form of the equation be

$$af_t + bf_{t-1} + f_{t-2} = g_t$$

with a and b fixed coefficients and $a \neq 0$. Rewritten by means of L the equation becomes

$$\left(a + bL + L^2\right) f_t = g_t \tag{39}$$

Notice that the coefficients a and b of the polinomial lag in parentheses can be expressed as functions of the roots of the characteristic equation (the 'auxiliary' characteristic equation) $a + bx + x^2 = 0$. The roots are

$$\begin{aligned} \rho_1 &= \frac{-b + \sqrt{b^2 - 4a}}{2} \\ \rho_2 &= \frac{-b - \sqrt{b^2 - 4a}}{2} \end{aligned}$$

from which

$$\rho_1 + \rho_2 = -b \qquad \textit{and} \qquad \rho_1 \rho_2 = a$$

By replacing the latter in (39) we get

$$\left(\rho_1\rho_2 - (\rho_1 + \rho_2)\, L + L^2\right) f_t = g_t$$

and, through obvious steps,

$$\left(1 - \frac{1}{\rho_1}L\right)\left(1 - \frac{1}{\rho_2}L\right) f_t = \frac{g_t}{\rho_1\rho_2} \tag{40}$$

Now take the auxiliary homogeneous equation associated with (39), $(a + bL + L^2)\, f_t = 0$. By operating the same transformations on it we get

$$\left(1 - \frac{1}{\rho_1}L\right)\left(1 - \frac{1}{\rho_2}L\right) f_t = 0$$

It is easy to check that this equation is satisfied by the functions $f_t = \left(\frac{1}{\rho_1}\right)^t$, $f_t = \left(\frac{1}{\rho_2}\right)^t$, and therefore also by any function

$$f_t = \alpha\left(\frac{1}{\rho_1}\right)^t + \beta\left(\frac{1}{\rho_2}\right)^t$$

with α and β arbitrarily chosen. This is the general solution to the homogeneous auxiliary equation. The two dof are well in evidence in the formula.

At this point, insert

$$\left(1 - \frac{1}{\rho_1}L\right)\left(1 - \frac{1}{\rho_2}L\right)\left(\alpha\left(\frac{1}{\rho_1}\right)^t + \beta\left(\frac{1}{\rho_2}\right)^t\right) = 0$$

into (40), which is rewritten as

$$\begin{aligned}\left(1 - \frac{1}{\rho_1}L\right)\left(1 - \frac{1}{\rho_2}L\right) f_t = \\ = \frac{g_t}{\rho_1\rho_2} + \left(1 - \frac{1}{\rho_1}L\right)\left(1 - \frac{1}{\rho_2}L\right)\left(\alpha\left(\frac{1}{\rho_1}\right)^t + \beta\left(\frac{1}{\rho_2}\right)^t\right)\end{aligned}$$

By applying the double operator $\left(1 - \frac{1}{\rho_2}L\right)^{-1}\left(1 - \frac{1}{\rho_1}L\right)^{-1}$ to both sides of this equation we get the general backward solution to (39)

$$\psi_t = \left(1 - \frac{1}{\rho_2}L\right)^{-1}\left(1 - \frac{1}{\rho_1}L\right)^{-1}\frac{g_t}{\rho_1\rho_2} + \alpha\left(\frac{1}{\rho_1}\right)^t + \beta\left(\frac{1}{\rho_2}\right)^t$$

The free parameters of the solution are α and β, while the roots ρ_1 and ρ_2 derive from the parameters a and b of the original equation (39).

The polinomial lags $\left(1-\frac{1}{\rho_i}L\right)^{-1}$ converge to finite values if the sequence $\{g_t\}$ is bounded and the two roots ρ_i have modulus greater than one (i.e. $\frac{1}{|\rho_i|}<1$). This is called the 'stable' case. If both roots have modulus smaller than one we have the 'unstable' case, which is however stable in the forward solution (see below). The mixed case, with the modulus of one root greater and the other smaller than one, is called a 'saddle'.

Let us consider the stable case. By developing the double polinomial lag sequentially, at the first step we get

$$\left(1-\frac{1}{\rho_1}L\right)^{-1}\frac{g_t}{\rho_1\rho_2}=\frac{1}{\rho_2}\sum_{h=0}^{\infty}\left(\frac{1}{\rho_1}\right)^{h+1}g_{t-h}$$

and at the second

$$\begin{aligned}\left(1-\frac{1}{\rho_2}L\right)^{-1}\left(1-\frac{1}{\rho_1}L\right)^{-1}\frac{g_t}{\rho_1\rho_2} &= \left(1-\frac{1}{\rho_2}L\right)^{-1}\frac{1}{\rho_2}\sum_{h=0}^{\infty}\left(\frac{1}{\rho_1}\right)^{h+1}g_{t-h}\\ &= \sum_{k=0}^{\infty}\left(\frac{1}{\rho_2}\right)^{k+1}\sum_{h=0}^{\infty}\left(\frac{1}{\rho_1}\right)^{h+1}g_{t-h-k}\end{aligned}$$

or, after rearranging

$$\left(1-\frac{1}{\rho_2}L\right)^{-1}\left(1-\frac{1}{\rho_1}L\right)^{-1}\frac{g_t}{\rho_1\rho_2}=\sum_{h=0}^{\infty}g_{t-h}\sum_{j=0}^{h}\left(\frac{1}{\rho_1}\right)^{j+1}\left(\frac{1}{\rho_2}\right)^{(h-j)+1}$$

By using the last formula we rewrite the general solution as

$$\psi_t=\sum_{h=0}^{\infty}g_{t-h}\sum_{j=0}^{h}\left(\frac{1}{\rho_1}\right)^{j+1}\left(\frac{1}{\rho_2}\right)^{(h-j)+1}+\alpha\left(\frac{1}{\rho_1}\right)^{t}+\beta\left(\frac{1}{\rho_2}\right)^{t}\qquad(41)$$

In case the modulus of one, or of both characteristic roots is smaller than one the first summation on the right-hand side is infinite and the backward solution is available only in truncated form. With both roots unstable, however, we can resort to the forward solution.

Exercise 14 *Work out the general forward solution to equation (39) [Hint: rewrite the equation in the form*

$$\begin{aligned}L^{-2}g_t &= \left(\rho_1\rho_2L^{-2}-(\rho_1+\rho_2)L^{-1}+1\right)f_t\\ &= \left(1-\rho_1L^{-1}\right)\left(1-\rho_2L^{-1}\right)f_t\end{aligned}$$

Finally, a form of mixed solution, partly forward and partly backward, called 'partial adjustment' can be applied in the case of a saddle. This is used for example by Wickens in the study of a dynamic equation of prices in section 9.4.4 (formula 9.28 on p 236 of the first edition).

Call ρ_1 the stable root, ρ_2 the unstable one

$$|\rho_1| > 1, |\rho_2| < 1$$

Rewrite equation (39) as

$$L^{-1}g_t = \left(\rho_1\rho_2 L^{-1} - (\rho_1 + \rho_2) + L\right) f_t$$

and transform it as follows

$$\begin{aligned} g_{t+1} &= \left(\rho_1\left(\rho_2 L^{-1} - 1\right) - L\left(\rho_2 L^{-1} - 1\right)\right) f_t \\ &= -\rho_1\left(1 - \frac{L}{\rho_1}\right)\left(1 - \rho_2 L^{-1}\right) f_t \end{aligned}$$

As $|\rho_2|$ is smaller than one we can write

$$\begin{aligned} \left(1 - \frac{L}{\rho_1}\right) f_t &= -\frac{1}{\rho_1}\left(1 - \rho_2 L^{-1}\right)^{-1} g_{t+1} \\ &= -\frac{1}{\rho_1}\sum_{h=0}^{\infty}\rho_2^h g_{t+1+h} \end{aligned}$$

with the summation giving a finite value.

The left-hand side of the last equation is

$$\left(1 - \frac{L}{\rho_1}\right) f_t = f_t - \frac{f_{t-1}}{\rho_1}$$

hence

$$\begin{aligned} f_t - f_{t-1} &= \frac{1-\rho_1}{\rho_1} f_{t-1} - \frac{1}{\rho_1}\sum_{h=0}^{\infty}\rho_2^h g_{t+1+h} \\ &= \frac{\rho_1 - 1}{\rho_1}\left[\frac{1}{1-\rho_1}\sum_{h=0}^{\infty}\rho_2^h g_{t+1+h} - f_{t-1}\right] \end{aligned} \tag{42}$$

Define the new variable

$$\gamma_t \equiv \frac{1}{1-\rho_1}\sum_{h=0}^{\infty}\rho_2^h g_{t+1+h}$$

According to this definition, γ_t can be taken as a moving average of the future behavior of g_t, an index that charges the realizations from g_{t+1} on with weights that decrease as the distance from t increases, tending to zero as the distance tends to infinity. Equation (42) now becomes

$$f_t - f_{t-1} = \frac{\rho_1 - 1}{\rho_1} (\gamma_t - f_{t-1})$$

which explains each variation in f between consecutive dates as a fraction of the difference between f and the indicator γ, with f being increasing, stationary or decreasing depending on whether γ is greater than, equal to or smaller than f. Since the ratio $\frac{\rho_1 - 1}{\rho_1}$ is always positive (from $|\rho_1| > 1$), the formula shows that γ works as a moving attractor of f.

5 Expectations

Expectation formation is a most controversial topic in macroeconomic modelling. Theories dealing with expectations range from the extremely rational to the extremely empirical and descriptive of professional practices and rules of thumb. In all cases, it is acknowledged that expectations affect economic behavior and consequently the movements of the variables with which expectations are concerned. In the opposite direction, systematic gaps between actual and expected movements of economic variables will lead to revisions in the manner in which expectations are formed. Consequently, until some kind of equilibrium between the two sides, expectations and realizations, is reached the economy will be in a more or less volatile state of flux. Equilibrium requires expectations to induce behavior the consequences of which to a certain degree confirm expectations themselves. But not all methods for expectation formation have such self-confirming power. This is why economic theory focuses on a limited number of models of expectation, those that possess it to some degree.

Here we examine two of the most commonly used models in macroeconomic applications: the one called 'rational expectations', and the model called 'adaptive expectations'. The former assumes expectations to be specified in the form of probabilities, and updated with respect to information according to Bayes rule: it is therefore a case of Bayesian expectations. The latter, although entirely based on information, is instead non-Bayesian, and not only so but also non-probabilistic. It may be considered the extreme opposite to rational expectations. But before entering into the differences between the two, we discuss briefly what it means to have Bayesian expectations even if not necessarily of the 'rational' kind.

5.1 Bayesian expectations

If we assume, first, that the economic environment is one in which agents can learn about current events as soon as these occur, so that at each s the realisation of X_s becomes a piece of information; and second, that whatever their prior beliefs may have been, agents update these to current information consistently with Bayes rule of conditionalization of probabilities; then subjective beliefs will evolve with time in a way that parallels the evolution of objective probabilities as seen above in section 4.2. Let the probability measure P represent the a priori subjective beliefs of the representative agent on the measurable space $(\Omega, \mathcal{A})$ described in 4.1; and let us pay no attention to whether P does or does not reflect the objective probability P defined on the same space. The representative agent may have wrong beliefs. But

information at s will in any case be represented by the same partition $\mathcal{D}_s$ of Ω seen above, with the partition becoming finer and finer as the time index s goes up. Thus, passing from s to $t > s$, the subjective probability of the same event $a \in \mathcal{A}$ will change from $\mathsf{P}(a \mid D)$, where $D \in \mathcal{D}_s$ represents what the agent knows at s, to any $\mathsf{P}(a \mid D')$ where $D' \in \mathcal{D}_t$ is one of the subsets into which D will be partitioned due to additional information made available by the passing of time from s to t. As in formula (34), the relationship between the two is given by

$$\mathsf{P}(a \mid D') = \frac{\mathsf{P}(a \cap D' \mid D)}{\mathsf{P}(D' \mid D)}$$

Although subjective beliefs P are allowed to diverge from objective probability P, updating according to this formula implies that P and P should not diverge to such an extent that an objectively possible piece of information D' receives a priori subjective probability $\mathsf{P}(D' \mid D) = 0$. In other words, this kind of Bayesian updating requires that what one learns through experience is never in absolute contradiction with previous beliefs, as it might be the case if objective possibilities were subjectively deemed to be impossible.

Besides, the same demonstration used in solving exercise 13 may be applied here to show that

$$\mathsf{P}(a \mid D) = E(\mathsf{P}(a \mid \mathcal{D}_t) \mid D)$$

Present probability equals the (presently) expected value of the random variable future probability. This characteristic of Bayesian updating has the two implications that can be seen by solving the following two exercises.

Exercise 15 *(Second order expectations) Let $E(X_{s+n} \mid D)$ represent the subjective conditional expectation of random variable X_{s+n} at s; and $E(X_{s+n} \mid D')$ the corresponding conditional expectation for each $D' \in \mathcal{D}_t$, with $s + n > t > s$. Show that*

$$E(X_{s+n} \mid D) = E(E(X_{s+n} \mid \mathcal{D}_t) \mid D)$$

Exercise 16 *(Consistency) Call 'expectation revision' the random variable $\delta_{s+n} \equiv E(X_{s+n} \mid \mathcal{D}_t) - E(X_{s+n} \mid D)$ where $D \in \mathcal{D}_s$. Demonstrate that Bayesian updating implies*

$$E(\delta_{s+n} \mid D) = 0$$

The second proposition parallels the similar property of conditional probabilities expressed by formula (35) in section 4.2. Here too, while acknowledging that expectations are going in general to change due to new information, expectation revision is expected to be on average zero.

In macroeconomic modelling of the DSGE kind the representative agent's subjective expectations are relevant in that they affect the agents' choices, and these in turn affect the objective probabilities underlying the stochastic processes that determine the dynamics of the economy. This being the case, however, a representative rational economic agent should be aware of the influence of his beliefs on the very events with which beliefs are concerned, and adjust these in accordance with this principle of mutual influence or, as is often called, self-referentiality. Beyond the minimal consistency between subjective beliefs and objective probabilities required by Bayesian updating, therefore, the assumption of economic rationality compels the theory to consider a stronger notion of consistency in order to take the problem of self-referentiality of expectations into consideration. Perhaps the strongest in this line of consistency notions is the one embodied in the model of rational expectations, still one of the most popular models although the financial crisis of 2007 has somewhat shaken confidence in its validity.

5.2 Rational expectations and actual dynamics

In order to discuss definition, characteristics and implications of rational expectations we refer to a simple model of a dynamic process driven by the expectations themselves and by an independent stochastic process. The stochastic process provides the objective standard necessary to assess the 'rationality' of expectations.

Assume that a macroeconomic variable x is determined by the expectations of economic agents and by an external causal factor z according to the law

$$x_t = \alpha x_t^e + \beta z_t \tag{43}$$

where x_t^e indicates the expected value of x_t on the basis of information available at $t-1$. Randomness enters the system through the law of motion that regulates z. This is described by the first order AR process

$$z_t = a + bz_{t-1} + \varepsilon_t \tag{44}$$

where $\varepsilon_t \sim$WN, a and b are positive parameters and b is supposed to be smaller than 1.

A possible interpretation of the model is: x represents aggregate investment, positively dependent on expectations on aggregate investment (investment creates demand which provides further incentive to invest) and on technical progress. The latter is represented by z, the flow of innovations applied to productive processes. This in turn is affected by a systematic structural component represented by the constant parameter a, by innovations realized

in the recent past, and by a random factor which may play either positive or negative role.

Equation (44) is a first order, linear, fixed coefficients difference equation with a stochastic component. It can be solved directly by using the L operator as seen in section 4.4. Write the equation in the form $(1 - bL) z_t = a + \varepsilon_t$. The auxiliary characteristic equation $1 - bx = 0$ has the unique root $x = \frac{1}{b}$. In the hypothesis $0 < b < 1$ the operator $(1 - bL)^{-1}$ is convergent and therefore

$$\begin{aligned} z_t &= (1 - bL)^{-1} (a + \varepsilon_t) \\ &= a \sum_{h=0}^{\infty} b^h + \sum_{h=0}^{\infty} b^h \varepsilon_{t-h} \\ &= \frac{a}{1 - b} + \sum_{h=0}^{\infty} b^h \varepsilon_{t-h} \end{aligned} \tag{45}$$

This backward solution provides the moving average (MA) representation of the AR process from which we started. It can be checked that mean, variance and autocovariance of any order are constant with respect to t, with autocovariances of order h tending to zero as $h \to \infty$. The process is thus weakly stationary and asymptotically independent. In particular, as can be checked from (45), $EZ_t = \frac{a}{1-b}$ and $varZ_t = \frac{Var\varepsilon}{1-b^2}$. Autocovariance of order h is given by

$$Cov(Z_t, Z_{t+h}) = \frac{Var\varepsilon}{1 - b^2} b^{|h|}$$

which converges to zero as $h \to \infty$. The intuition is that, in the long period, the independence of the shocks ε_t tends to outweigh the correlation between each z_t and z_{t-1}.

Notice that, if one abstracts from the stochastic component ε_t, the process described by equation (44) has a unique steady state, i.e. a unique value z such that $z_{t+1} = a + bz_t = z_t$. This coincides with the constant trend $\frac{a}{1-b}$ around which z_t oscillates according to equation (45). Thus, defining the steady state value of z as

$$z^* = \frac{a}{1 - b}$$

equation (45) becomes

$$z_t = z^* + \sum_{h=0}^{\infty} b^h \varepsilon_{t-h} \tag{46}$$

Strictly defined, rational expectations means the hypothesis that x_t^e coincides with $E(X_t \mid D_{t-1})$, the conditional mean value of the random variable X_t computed on the basis of the model represented by equations (43) and (44), with $x_t^e = E(X_t \mid D_{t-1})$ in the former equation and D_{t-1} being the element of the information partition at $t - 1$ identified by the sequence of

realisations x_s, z_s and ε_s up to $s = t-1$. In other words, $E(X_t \mid D_{t-1})$ is the best estimate that a statistician having full knowledge of the structural features and past history of the system would be capable of doing.

According to this definition,

$$\begin{aligned} E(X_t \mid D_{t-1}) &= \alpha E(X_t \mid D_{t-1}) + \beta(a + bz_{t-1}) + E(\varepsilon_t \mid D_{t-1}) \\ &= \frac{\beta}{1-\alpha}(a + bz_{t-1}) \end{aligned} \tag{47}$$

Alternatively, by substituting the MA representation (46) for z_{t-1} and performing obvious steps, we can write

$$\begin{aligned} E(X_t \mid D_{t-1}) &= \frac{\beta}{1-\alpha}a + \frac{\beta}{1-\alpha}b\left(z^* + \sum_{h=0}^{\infty} b^h \varepsilon_{t-1-h}\right) \\ &= \frac{\beta}{1-\alpha}z^* + \frac{\beta}{1-\alpha}\sum_{h=1}^{\infty} b^h \varepsilon_{t-h} \end{aligned}$$

If we now define the steady state value of x as the value that, if correctly predicted, would be realised when $z_t = z^*$,

$$x^* = \alpha x^* + \beta z^* = \frac{\beta}{1-\alpha}z^*$$

we have

$$E(X_t \mid D_{t-1}) = x^* + \frac{\beta}{1-\alpha}\sum_{h=1}^{\infty} b^h \varepsilon_{t-h} \tag{48}$$

which shows that the expectation of x_t differs from its steady state value only to an extent justified by the observed past fluctuations of z around its own steady state. The formulas (48) and (47) are obviously equivalent.

If expectations are formed in the way described by (48), the actual dynamics of x is determined as follows:

$$\begin{aligned} x_t &= \alpha E(X_t \mid D_{t-1}) + \beta z_t \\ &= \alpha\left(x^* + \frac{\beta}{1-\alpha}\sum_{h=1}^{\infty} b^h \varepsilon_{t-h}\right) + \beta\left(z^* + \sum_{h=0}^{\infty} b^h \varepsilon_{t-h}\right) \\ &= x^* + \frac{\beta}{1-\alpha}\sum_{h=1}^{\infty} b^h \varepsilon_{t-h} + \beta\varepsilon_t \qquad (49) \\ &= E(X_t \mid D_{t-1}) + \beta\varepsilon_t \qquad (50) \end{aligned}$$

From (49) we learn that x fluctuates around its steady state trend in line with the fluctuations of z around its own steady state. The unconditional mean of the process is x^*, constant through time. The same is true of its unconditional variance and autocovariances of all orders, the latter converging

to zero as the order increases. The process ruling x under rational expectations is therefore weakly stationary and asymptotically independent, exactly like the process ruling z.

Equation (50) indicates that the expectation error equals $\beta\varepsilon_t$, i.e. replicates the white noise that generates uncertainty. This implies that rational expectations will almost always be wrong, but the expected error is zero and errors at different dates are uncorrelated. This means that with these expectations error has no systematic bias, predictions do not fail to take all systematic factors into account. Intuitively, rational expectations take advantage of all the autoregressivity existing in the system and surrender only in the face of the unpredictable residuum.

From (50) we can also verify that rational expectations are Bayesian in the sense explained in section 5.1. Indeed, if expectations are rational, they must be consistent with all the implications of the model including equation (50). Thus, in forming expectations on X_{t+n} $(n = 1, 2, ...)$ on the basis of information available at any previous date, take for example date $t-1$, from (50) it follows

$$E\left(X_{t+n} \mid D_{t-1}\right) = E\left(E\left(X_{t+n} \mid \mathcal{D}_{t+n-1}\right) \mid D_{t-1}\right) \tag{51}$$

i.e. the rule concerning second order expectations that characterizes expectations based on Bayesian updating, as showed by the formula found in solving exercise 15 above. Remember that this implies that the expected expectation revision in the sense specified in the next exercise 16 is always zero.

In the last equation $\mathcal{D}_{t+n-1}$ indicates all the possible pieces of information the subject expects to come to acquire from $t-1$ to $t+n-1$, i.e. his/her expectations about $z_t, ...z_{t+n-1}$. Thus, by using formula (47) we can transform the right-hand side of the equation as follows

$$\begin{aligned} E\left(X_{t+n} \mid D_{t-1}\right) &= E\left(\frac{\beta}{1-\alpha}\left(a + bZ_{t+n-1}\right) \mid D_{t-1}\right) \\ &= \frac{\beta}{1-\alpha}\left(a + bE\left(Z_{t+n-1} \mid D_{t-1}\right)\right) \end{aligned}$$

Now notice that expectations consistent with knowledge of equation (44) imply the following

$$E\left(Z_{t+n-1} \mid D_{t-1}\right) = a + bE\left(Z_{t+n-2} \mid D_{t-1}\right)$$

and so on recursively until, after going backwards n times, we arrive at

$$E\left(Z_t \mid D_{t-1}\right) = a + bz_{t-1}$$

By substituting in the original equation, then, we get

$$\begin{aligned} E\left(X_{t+n} \mid D_{t-1}\right) &= \frac{\beta}{1-\alpha}\left(a + ab + ab^2 + ... + ab^n + b^{n+1} z_{t-1}\right) \\ &= \frac{\beta}{1-\alpha}\frac{a}{1-b}\left(1 - b^n\right) + \frac{\beta}{1-\alpha}\left(a + bz_{t-1}\right) b^n \end{aligned}$$

Finally, recalling the definition of x^* and using formula (47)

$$E\left(X_{t+n} \mid D_{t-1}\right) = \left(1 - b^n\right) x^* + b^n E\left(X_t \mid D_{t-1}\right) \tag{52}$$

which expresses long term rational expectation as a weighed average, with weights depending on the length of the time horizon n, of the short term expectation and the steady state value x^*. Since $b < 1$, the formula also shows that, while expectations for the immediate future are what they are depending on recent experience, long term rational expectations tend to approach x^*, the unconditional mean of the process, as $n \to \infty$.

The property expressed by equation (50) suggests the possibility of defining rational expectations in an alternative way, a less demanding definition than the strict definition given above in this section. Assuming the representative agent to be endowed with all the knowledge and technical expertise necessary to compute the conditional means $E\left(\cdot \mid \cdot\right)$ of the model seems hardly credible. In most cases it may be sufficient to define rational expectations as simply the capacity of forming expectations that, taken in long series, turn out to be correct on average and not to neglect systematic causal factors. As we have seen from equation (50), the strict definition entails this capacity as a consequence of excessively strong epistemic assumptions, but the reverse implication does not generally hold: agents may come to acquire this capacity, for example through protracted professional practice, without need to assume that they know everything and are able to run complex computations.

It must be noticed, however, that this 'weak' definition of rational expectations supports many of the properties that can be deducted from the strict definition, but not all of them. In particular, one cannot prove properties (51) and (52) without assuming that subjective conditional expectations coincide with the objective $E\left(\cdot \mid \cdot\right)$ of the model. In this sense, one may say that weakly defined rational expectations are not Bayesian in the same sense as strict rational expectations.

According to the weak definition, 'rational expectation' is any prediction x_t^e such that the difference

$$x_t - x_t^e \equiv \eta_t$$

behaves like WN and is uncorrelated with ε_t in the sense that

$$Cov\left(\varepsilon_t, \eta_{t+n}\right) = E\left(\varepsilon_t \eta_{t+n}\right) = 0$$

for all $n = 0, 1, ...$

By inserting this definition into (43) and by utilizing (44) and (47) we get

$$\begin{aligned} x_t^e &= \alpha x_t^e - \eta_t + \beta z_t \\ &= \frac{-\eta_t}{1-\alpha} + \frac{\beta}{1-\alpha}\left(a + b z_{t-1} + \varepsilon_t\right) \\ &= \frac{\beta}{1-\alpha}\left(a + b z_{t-1}\right) + \frac{\beta\varepsilon_t - \eta_t}{1-\alpha} \\ &= E\left(X_t \mid D_{t-1}\right) + \frac{\beta\varepsilon_t - \eta_t}{1-\alpha} \end{aligned} \tag{53}$$

from which we see that the only difference between 'strict' and 'weak' rational expectations is given by the last term on the right-hand side of the last equation. From the solution of the next

Exercise 17 *Show that, if η_t is WN uncorrelated with ε_t, also $\beta\varepsilon_t - \eta_t$ is WN.*

it turns out that the difference between strictly and weakly rational expectations is entirely made of WN.

Also the actual dynamics of x_t maintains the characteristics of the dynamics generated by strict rational expectations with the only difference of a WN. To prove this, insert the weak definition into (43) and develop

$$\begin{aligned} x_t &= \alpha\left(x_t - \eta_t\right) + \beta z_t \\ &= -\frac{\alpha\eta_t}{1-\alpha} + \frac{\beta}{1-\alpha}\left(a + b z_{t-1} + \varepsilon_t\right) \\ &= \frac{\beta}{1-\alpha}\left(a + b z_{t-1}\right) + \frac{\beta\varepsilon_t - \alpha\eta_t}{1-\alpha} \\ &= E\left(X_t \mid D_{t-1}\right) + \frac{\beta\varepsilon_t - \alpha\eta_t}{1-\alpha} \end{aligned}$$

Now indicate with x_t^o the actual dynamics when expectations are strictly rational and conform to (50): in the last formula we have

$$\begin{aligned} x_t &= x_t^o - \beta\varepsilon_t + \frac{\beta\varepsilon_t - \alpha\eta_t}{1-\alpha} \\ &= x_t^o + \alpha\frac{\beta\varepsilon_t - \eta_t}{1-\alpha} \end{aligned}$$

from which (see (53) above)

$$x_t - x_t^o = \alpha \left[x_t^e - E\left(X_t \mid D_{t-1} \right) \right] = \alpha \frac{\beta \varepsilon_t - \eta_t}{1 - \alpha}$$

Thus, the difference between the two dynamics, like the difference between the expected values, is only a matter of WN.

Exercise 18 *Show that x_t and x_t^o have the same unconditional mean x^* but x_t has greater unconditional variance and autocovariances than x_t^o.*

If the differences pointed out in the solution to this exercise play no role in the economic model one is using, then from the point of view of that model it appears that adopting a strict or a weak definition of rational expectations does not make much difference. As mentioned above, however, the weak definition does not guarantee that second order expectations and long term expectations behave in accordance with formulas (51) and (52). If these properties do play a role in the model, then the definition of rational expectations must be the strict one.

5.3 Adaptive expectations

If the formation of expectations conforms to a model different from rational expectations, then also the dynamics of the system will develop according to generally different rules, although the structure of the economy remains the same described by the equations (43) and (44) of the previous section. We examine this issue by means of a non-rational, non-Bayesian, non-probabilistic model of expectation formation called 'adaptive' or 'extrapolative' expectations. This is still used in macroeconomic literature as a radical alternative to rational expectations.

Expectations about an economic variable are said to be adaptive when they are entirely based on knowledge of the past history of the variable without taking the characteristics of the stochastic process that drives it into account – either because these characteristic are unknown, or because agents are supposed to be unable to utilize them in a technically adequate way. Of course, even the simple extrapolation of past observations onto future realizations implicitly utilizes indirect knowledge of the underlying stochastic process, but this is done unaware and not by means of statistical methods, contrary to what happens with rational expectations.

An example of adaptive expectations on the variable x in the model formed by equations (43) and (44) is represented by the following rule

$$x_t^e - x_{t-1}^e = \phi \left(x_{t-1} - x_{t-1}^e \right)$$

with ϕ, a number included in the $(0, 1)$ interval, playing the role of the factor by which the last observed error $\left(x_{t-1} - x_{t-1}^e\right)$ is used to correct the current prediction. Correction will be faster or slower depending on ϕ being nearer to 1 or to 0. This is an exclusively "backward looking" rule, and also a naive rule in that it does not use any statistical estimator, nor even the notion of probability. It allows for updating with respect to new information, but of course not updating of the Bayesian kind.

A change in the structure of the system, for example a change in the parameters a or b of equation (44), would be absorbed in a rational expectations world as soon as agents become aware of it, as is shown by equation (47). With adaptive expectations, instead, the change would be absorbed only gradually and indirectly, through the cumulated observations of the time series generated by the modified process. In the case of changes due to new operative rules or new targets being followed by policy-makers such as Central Bank or Government, assuming rational rather than adaptive expectations means assuming two different kinds of response of the public to changes in the criteria regulating economic policy. With rational expectations the public will react as soon as the new policy is perceived and will embody it entirely in the formation of expectations and consequent behavior. The response of an adaptive public will instead be delayed and gradual.

The differences just mentioned are important enough to justify different theories concerning the effects of macroeconomic policies on the basis of different assumptions on expectations. But if we look at the working of the system inside a given policy regime, i.e. with all the parameters controlled by policy-makers remaining constant, the consequences of choosing the one or the other assumption on expectations turn out to be less dramatic than one might expect. The dynamics in the two cases is of course different but, as we are going to see, the differences are not extreme.

From the adaptive rule we derive the following first order difference equation in the x^e variable

$$x_t^e = (1 - \phi)\, x_{t-1}^e + \phi x_{t-1}$$

Solving the equation by means of the usual methods we get

$$x_t^e = \phi \sum_{h=0}^{\infty} (1 - \phi)^h\, x_{t-1-h}$$

From this we see that current expectation is determined by a moving average of all past observations, the weights tending to zero as the observations go back further and further into the past. Notice that all past observations refer to x, not to the realizations of the shock ε as in equation (48).

Now the actual dynamics of x_t induced by adaptive expectations. Starting from the process (46) that governs z_t, and using the definitions of steady state values z^* and x^*, we find

$$\begin{aligned} x_t &= \alpha\left(\phi\sum_{h=0}^{\infty}(1-\phi)^h x_{t-1-h}\right)+\beta\left(z^*+\sum_{h=0}^{\infty}b^h\varepsilon_{t-h}\right) \\ &= (1-\alpha)x^*+\alpha\phi\sum_{h=0}^{\infty}(1-\phi)^h x_{t-1-h}+\beta\sum_{h=0}^{\infty}b^h\varepsilon_{t-h} \end{aligned} \tag{54}$$

Notice that in the last equation we can write

$$\sum_{h=0}^{\infty}(1-\phi)^h x_{t-1-h}=(1-(1-\phi)L)^{-1}x_{t-1}$$

and therefore, multiplying both sides of equation (54) by $(1-(1-\phi)L)$

$$(1-(1-\phi)L)x_t=(1-(1-\phi)L)\left[(1-\alpha)x^*+\beta\sum_{h=0}^{\infty}b^h\varepsilon_{t-h}\right]+\alpha\phi x_{t-1} \tag{55}$$

where

$$\begin{aligned} (1-(1-\phi)L)x_t &= x_t-(1-\phi)x_{t-1} \\ (1-(1-\phi)L)(1-\alpha)x^* &= \phi(1-\alpha)x^* \\ (1-(1-\phi)L)\beta\sum_{h=0}^{\infty}b^h\varepsilon_{t-h} &= \beta\sum_{h=0}^{\infty}b^h(\varepsilon_{t-h}-(1-\phi)\varepsilon_{t-h-1}) \\ &= \beta\sum_{h=0}^{\infty}b^h(1-(1-\phi)L)\varepsilon_{t-h} \end{aligned}$$

By replacing the last four lines into (55) we get

$$x_t=\phi(1-\alpha)x^*+(1-\phi(1-\alpha))x_{t-1}+\beta\sum_{h=0}^{\infty}b^h(1-(1-\phi)L)\varepsilon_{t-h}$$

or, more simply, using the abbreviation $\theta\equiv 1-\phi(1-\alpha)$

$$x_t=(1-\theta)x^*+\theta x_{t-1}+\beta\sum_{h=0}^{\infty}b^h(1-(1-\phi)L)\varepsilon_{t-h}$$

This difference equation can be solved either backwards or forward. Let us assume that $0<\theta<1$, i.e. (with $\alpha>0$) $\alpha<1$, and solve backwards in successive steps. After n steps we have

$$\begin{aligned} x_t=\theta^n x_{t-n}+\left(1+\theta+...+\theta^{n-1}\right)(1-\theta)x^*+ \\ +\beta\sum_{h=0}^{\infty}b^h\sum_{j=0}^{n-1}\theta^j(1-(1-\phi)L)\varepsilon_{t-h-j} \end{aligned}$$

As the number of steps n becomes larger and larger, $\theta^n \to 0$ and $\left(1+\theta+...+\theta^{n-1}\right) \to \frac{1}{1-\theta}$; the sequence x being by hypothesis bounded, the solution is

$$x_t = x^* + \beta \sum_{h=0}^{\infty} b^h \sum_{j=0}^{\infty} \theta^j \left(1-(1-\phi)L\right) \varepsilon_{t-h-j}$$

Using L, and rearranging the terms under summation, the last formula becomes

$$\begin{aligned} x_t &= x^* + \beta \sum_{h=0}^{\infty} b^h \left[\varepsilon_{t-h} + (\theta - (1-\phi)) \sum_{j=0}^{\infty} \theta^j \varepsilon_{t-h-(j+1)} \right] \\ &= x^* + \beta \sum_{h=0}^{\infty} b^h \left[\varepsilon_{t-h} + \alpha\phi \sum_{j=0}^{\infty} \theta^j \varepsilon_{t-h-(j+1)} \right] \\ &= x^* + \beta \sum_{h=0}^{\infty} b^h A_h \varepsilon_{t-h} \end{aligned} \tag{56}$$

where

$$\begin{aligned} A_0 &= 1 \\ A_1 &= 1 + \alpha\phi b^{-1} \\ A_2 &= 1 + \alpha\phi b^{-1} + \alpha\phi b^{-2}\theta \\ &\quad ... \qquad ... \qquad ... \\ A_h &= 1 + \alpha\phi \sum_{j=1}^{h} b^{-j}\theta^{j-1} \end{aligned}$$

If we compare the dynamics described by (56) with the rational expectations dynamics (x_t^o) of equation (49) we find

$$\begin{aligned} x_t - x_t^o &= \beta \sum_{h=1}^{\infty} b^h \left(A_h - \frac{1}{1-\alpha} \right) \varepsilon_{t-h} \\ &= \alpha\beta \sum_{h=1}^{\infty} b^h \left(\phi \sum_{j=1}^{h} b^{-j}\theta^{j-1} - \frac{1}{1-\alpha} \right) \varepsilon_{t-h} \end{aligned}$$

Summing up: the two processess have the same trend and unconditional mean x^*. Both are weakly stationary and asymptotically independent. The only differences between them concern variances and autocovariances, as may be checked with some calculus.

An interesting comparison is provided by the 'impulsive response coefficients', the coefficients that, by measuring the impact of a shock ε_{t-n} on x_t,

indicate the 'memory' of the process at a distance n. With rational expectations, as is seen from (49), the impact is given by

$$\frac{\beta b^n}{1-\alpha}$$

and converges towards zero as $n \to \infty$. With adaptive expectations the same impact is measured by (see (56))

$$\beta b^n A_n = \beta b^n \left[1 + \alpha\phi \sum_{j=1}^{n} b^{-j}\theta^{j-1}\right]$$

which also converges to zero for $n \to \infty$ but at different speed. In the difference

$$\frac{\beta b^n}{1-\alpha} - \beta b^n A_n = \alpha\beta b^n \left[\frac{1}{1-\alpha} - \phi \sum_{j=1}^{n} b^{-j}\theta^{j-1}\right]$$

the expression in square brackets may have any sign for small values of n but it surely declines as n increases. It is not difficult to check that, even if it is positive for small n, the expression in square brackets must become negative from some value of n on (just show this for $b > \theta$). Thus, in the long run the 'memory' of adaptive expectations turns out to be more persistent than in the case of rational expectations. This explains why the autocorrelation of the actual dynamic process is higher with the former than with the latter.

Finally, as to prediction error with adaptive expectations: taking $x_t^e = \frac{x_t - \beta z_t}{\alpha}$ from equation (43) and using (46) and (56) in sequence

$$\begin{aligned}
x_t^e - x_t &= \frac{1-\alpha}{\alpha} x_t - \frac{\beta}{\alpha} z_t \\
&= \frac{1-\alpha}{\alpha} x_t - \frac{\beta}{\alpha}\left(z^* + \sum_{h=0}^{\infty} b^h \varepsilon_{t-h}\right) \\
&= \frac{1-\alpha}{\alpha}\left(x_t - x^*\right) - \frac{\beta}{\alpha}\sum_{h=0}^{\infty} b^h \varepsilon_{t-h} \\
&= \frac{\beta}{\alpha}\sum_{h=0}^{\infty}\left[\left(1-\alpha\right)A_h - 1\right]\ b^h \varepsilon_{t-h}
\end{aligned}$$

The last equation shows that, differently from rational expectations, prediction errors with adaptive expectations are not WN but a MA process, although its unconditional mean remains zero. Differently from the case of rational expectations, therefore, autocovariance between errors at t and at $t+n$ is never zero no matter how great the value of n. It remains true,

however, that autocovariance approaches zero as $n \to \infty$ (asymptotic independence). As one might have expected intuitively, adaptive predictions may fail not only because of the presence of the unpredictable chance factor ε, but also because they do not account for the systematic autocorrelation embodied in the structure of the model underlying the dynamics of the system.

6 Optimization under uncertainty, with applications

6.1 Optimal control in a stochastic environment

Consider a generic optimal control problem along the lines seen in sections 3.4 and 3.5 above. The setup of the problem is as usual: x_s and y_s indicate respectively the state and control variable of the problem, the objective is the recursive function

$$F = \sum_{h=0}^{\infty} \beta^h \varphi(y_h) \tag{57}$$

As in 3.5, the instantaneous benefit φ is made to depend on control alone. The only difference with respect to the deterministic case is that the law of motion is now of the 'Markov' or 'stochastic recursive' kind (see section 4.3 above), like the following

$$x_{s+1} = g(x_s, y_s, \varepsilon_{s+1}) \tag{58}$$

with $\varepsilon \sim$WN a stochastic shock independent of the decision maker's actions. At the moment of choosing a plan, $s = 0$, x_o is taken as given. The decision maker chooses the sequence of controls $\{y_o, y_1, y_2...\}$ but has no way of knowing the resulting sequence $\{x_1, x_2...\}$. This fact has two implications:

(i) Apart from the action (level of control) chosen at $s = 0$, all successive actions at $s = 1, 2...$ will be conditional on the actual states $x_1, x_2...$ occurring at the successive dates (and supposedly known once a date is reached). Thus, a plan is not simply a sequence of actions ('at s, do y_s') but a sequence of actions conditional on information available at the moment of carrying them out ('at s, do y'_s if you observe x'_s; y''_s if x''_s; etc...'). Differently said, a plan is a sequence of decision functions, one for each date, mapping information available at that date into actions. This is usually called a 'contingent plan' or a 'policy rule'. One may notice that this notion of policy corresponds to the game-theoretic notion of 'strategy'. In making the prescription at s conditional on the sole value x_s, and not on the whole sequence of values from 0 to s, we are in fact restricting the range of policies/strategies admitted but, as we shall see, in the recursive approach to optimal control this restriction is inessential.

(ii) As a consequence of (i), both x_s and y_s for $s \geq 1$ become random variables, X_s and Y_s, with probability distributions depending on the probability distribution of ε and on the chosen plan itself (the distribution of X_1 depends on the distribution of ε and on the chosen y_o; and so on...). Consequently,

also the instantaneous benefit at each $s \geq 1$ becomes a random variable, $\Phi_s = \varphi(Y_s)$, and so also the objective $\mathsf{F} = \sum \beta^h \Phi_h$.

This framework describes the objective structure of a situation of choice under uncertainty. The standard approach of economic theory to this type of situations consists in assuming that the decision-maker is aware of the uncertainty, forms expectations on the random variables involved, and chooses a policy that maximizes the expected value of the objective F, expectations being based on information available at the moment of choosing, i.e. at $s = 0$. This approach raises a first fundamental issue, the so-called time-consistency of the plan.

Call σ a policy, consisting of a sequence of decision functions $\{\sigma_o, \sigma_1, \sigma_2, ...\}$, each σ_s mapping the values of X_s into Y_s. For each sequence of random shocks $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\}$ the policy σ determines (through (58)) a unique sequence of realizations $\{x_s, y_s = \sigma_s(x_s) : s = 0, 1, 2...\}$, hence (through (57)) a value $\mathsf{F} = F_{\sigma,\varepsilon}$, the value that the objective function takes on if the agent applies policy σ and if ε is the sequence of shocks that actually occur. Consequently, having subjective expectations of some sort on ε, at $s = 0$, with information limited to x_o, the agent can attach an expected value to σ, call it $E(\mathsf{F} \mid \sigma, x_o)$, and look for the σ that maximizes it.

Now imagine $\hat{\sigma}$ to be the optimal policy as evaluated at $s = 0$. At a successive date $s > 0$ the agent knows x_s and according to his policy should perform the action $\hat{y}_s = \hat{\sigma}_s(x_s)$. Since at s the agent knows more than at 0, his expectations at s may in general be different from those he had at 0. Will 'do $\hat{y}_s$ if x_s' be still an optimal choice on the basis of these generally different expectations? In other words, will action $\hat{y}_s$ maximize the expected value $E(\mathsf{F} \mid \hat{\sigma}, x_s)$ if the policy is re-assessed on the basis of the new state of information? If, as it seems to be generally possible, the answer happens to be 'no', then the policy $\hat{\sigma}$ is time-inconsistent. As soon as the agent realizes this, he will no longer act according to it. Hence, $\hat{\sigma}$ will not characterize the agent's behavior over time. Its validity will in general be limited to $s = 0$, losing all intertemporal reliability.

For a dynamic theory of behavior based on optimal planning to make sense, therefore, it is necessary that policies are both optimal at the time of conception and time-consistent, i.e. optimal at all times from conception to end. We now show that a sufficient condition for this is that expectations are Bayesian in the sense explained in section 5.1 above: under this condition, optimality and time-consistency come to be one and the same thing in the sense that a plan is optimal if and only if it is consistent.

Start from the property of second order Bayesian expectations proved in

the solution to exercise 15

$$E(\cdot\,|D_o) = E(E(\cdot\,|\mathcal{D}_s)\,|D_o) \tag{59}$$

where D_o is information at $s = 0$ and $\mathcal{D}_s$ information available at $s > 0$, a random variable if looked at from date 0. Now decompose the value of the objective function, given $\hat{\sigma}$ and a sequence of shocks ε, into value from 0 to $s-1$ plus value from s to ∞ as follows

$$\begin{aligned} F_{\hat{\sigma},\varepsilon} &= \sum_{h=0}^{s-1} \beta^h \varphi(\hat{\sigma}_h(x_h)) + \beta^s \sum_{n=0}^{\infty} \beta^n \varphi(\hat{\sigma}_{s+n}(x_{s+n})) \\ &= F_{\hat{\sigma},\varepsilon}|_0^{s-1} + \beta^s\, F_{\hat{\sigma},\varepsilon}|_s^{\infty} \end{aligned}$$

Its expected value can be decomposed accordingly

$$E(\mathsf{F} \mid \hat{\sigma}, x_o) = E\left(\mathsf{F}_0^{s-1} \mid \hat{\sigma}, x_o\right) + \beta^s E(\mathsf{F}_s^{\infty} \mid \hat{\sigma}, x_o) \tag{60}$$

At date s the agent's information includes the time series of the realizations of X from 0 to s. Thus, the first term of the sum on the right-hand side of (60) is by now a known quantity, call it f_{s-1}. The second term remains an expected value but, since x_s is known, it can be updated to the available information, so that we can write

$$E(\mathsf{F} \mid \hat{\sigma}, x_s) = f_{s-1} + \beta^s E(\mathsf{F}_s^{\infty} \mid \hat{\sigma}, x_s) \tag{61}$$

If expectations are bayesian we can apply (59) to (60), and by using equation (61) get

$$\begin{aligned} E(\mathsf{F} \mid \hat{\sigma}, x_o) &= E(E(\mathsf{F} \mid \hat{\sigma}, X_s)\,|\hat{\sigma}, x_o) \\ &= E\left(\mathsf{F}_0^{s-1}|\hat{\sigma}, x_o\right) + \beta^s E(E(\mathsf{F}_s^{\infty} \mid \hat{\sigma}, X_s)\,|\hat{\sigma}, x_o) \end{aligned}$$

Now go back to the problem of time-consistency. If the action $\hat{y}_s = \hat{\sigma}_s(x_s)$ is part of the optimal policy as assessed at $s = 0$, and if expectations are bayesian, it cannot be the case that $\hat{y}_s$ is no longer optimal if re-assessed at $s > 0$ on the basis of information x_s. For, if this were the case, then by choosing some action $y_s \neq \hat{y}_s$, hence a policy $\sigma \neq \hat{\sigma}$ from date s onwards, the agent would realize an expected value greater than $E(\mathsf{F}_s^{\infty} \mid \hat{\sigma}, x_s)$ in (61); and since this would not affect f_{s-1}, this different policy would lead to

$$E(\mathsf{F} \mid \sigma, x_s) = f_{s-1} + \beta^s E(\mathsf{F}_s^{\infty} \mid \sigma, x_s) > E(\mathsf{F} \mid \hat{\sigma}, x_s)$$

The last formula shows that this different choice would also increase the expected value at 0 resulting in (from (60))

$$E(\mathsf{F} \mid \sigma, x_o) = E\left(\mathsf{F}_0^{s-1} \mid \hat{\sigma}, x_o\right) + \beta^s E(\mathsf{F}_s^{\infty} \mid \sigma, x_o) > E(\mathsf{F} \mid \hat{\sigma}, x_o)$$

contradicting the hypothesis that $\hat{\sigma}$ is an optimal policy at $s = 0$.

Exercise 19 *The last step in the previous argument assumes implicitly that the probability of the event $X_s = x_s$ conditional on the adoption of policy $\hat{\sigma}$ is positive (with the usual caveat if X_s is continuous, see 4.2): explain why.*

Exercise 20 *The previous argument shows that, with bayesian expectations, time-consistency is a necessary condition of optimality. Show that it is also a sufficient condition.*

Bayesian expectations thus provide the appropriate environment for a theory of optimal planning under uncertainty based on the maximization of expected value. First of all, the perfect overlapping between optimality and time consistency serves to extend the recursive property from the objective function to the value function of the optimal control problem, in a way similar to what we saw in section 3.5 with reference to deterministic problems. A further consequence is the possibility of analyzing solutions by means of Bellman methods of dynamic programming.

Assuming $\hat{\sigma}$ to be an optimal policy, we define the value function as in 3.5 as $V(x_o) \equiv E(\mathsf{F} \mid \hat{\sigma}, x_o)$. Take any date s, and operate the decomposition

$$E(\mathsf{F}_s^\infty \mid \hat{\sigma}, x_s) = \varphi(\hat{\sigma}_s(x_s)) + \beta E\left(\mathsf{F}_{s+1}^\infty \mid \hat{\sigma}, X_{s+1}\right)$$

From the property of time consistency of $\hat{\sigma}$ it follows that the last formula can be rewritten as (in the next formulas we omit conditionalization bars)

$$\begin{aligned} V(x_s) &= \varphi(\hat{\sigma}_s(x_s)) + \beta EV(g(x_s, \hat{\sigma}_s(x_s), \varepsilon_{s+1})) \\ &\geq \varphi(y_s) + \beta EV(g(x_s, y_s, \varepsilon_{s+1})) \end{aligned} \tag{62}$$

for any y_s. This shows that the recursive property of the objective function is extended to the expected value of the solution to the optimal control problem.

If the function $\varphi + \beta EV$ is differentiable and concave with respect to y, we can also characterize the last inequality by means of Bellman equation in the form

$$0 = \varphi'(\hat{\sigma}_s(x_s)) + \beta E\left[\frac{\partial g(x_s, \hat{\sigma}_s(x_s), \varepsilon_{s+1})}{\partial y_s} V'\right]$$

where V' is the random variable 'derivative of the value function with respect to the random variable X_{s+1}'.

For expository reasons, we find it convenient to introduce a further restriction on expectations: not only will they be assumed to be bayesian, but also to be based on full knowledge of the structure of the model, i.e. of the motion equation (58) and of the stochastic process that rules ε. This is equivalent to setting the optimal control problem in an environment characterized by rational expectations. The reason for this restriction is because in such

an environment the properties of the optimal plan come out in the simplest form.

A first simplification consists in the fact that we can limit the search for optimal policies to stationary policies, 'stationary' meaning that a policy $\sigma = (\sigma_o, \sigma_1, \sigma_2, ...)$ is composed of the same decision function at all dates, $\sigma_o = \sigma_1 = \sigma_2 = ...$ Why that? The argument starts from the fact that the ε_s are i.i.d. (independent and identically distributed) variables, so that the probability distribution (both conditional and unconditional) of sequences $(\varepsilon_s, \varepsilon_{s+1}, ...)$ from s to ∞ is independent of s. If the subjective probabilities are formed in accordance with the objective stochastic process, this fact implies that, given a policy σ and an initial state $x_s = x$, also the expected value $E(\mathsf{F}_s^\infty \mid \sigma, x)$ associated with following policy σ from s to ∞ is independent of s. Hence, provided the initial state is the same $x_t = x_s = x$, following the same policy from, say, $t > s$ to ∞, guarantees the same expected value $E(\mathsf{F}_t^\infty \mid \sigma, x) = E(\mathsf{F}_s^\infty \mid \sigma, x)$. As a consequence, if the action $y = \sigma_s(x)$ is optimal at s, the same action must be optimal at t. Thus, nothing prevents an optimal policy from including the same decision function at the two dates s and t, $\sigma_s = \sigma_t$. As this argument can be replicated for all pairs of dates, the conclusion is that the decision function that is optimal for one date can also be optimal for all dates, making up a stationary optimal policy. In this context, the terms 'policy' and 'decision function' almost coincide in the sense that a policy is simply a repeated decision function.

Following from this simplification, the recursive property (62) of the value function can be rewritten dropping some of the date subscripts

$$V(x_s) = \varphi(\hat{\sigma}(x_s)) + \beta E V(g(x_s, \hat{\sigma}(x_s), \varepsilon_{s+1}))$$

and the same in Bellman equation

$$0 = \varphi'(\hat{\sigma}(x_s)) + \beta E\left[\frac{\partial g(x_s, \hat{\sigma}(x_s), \varepsilon_{s+1})}{\partial y_s} V'\right]$$

Again as a consequence of assuming rational expectations, notice that the expected value of a policy applied from date s to ∞ depends only on the state of the system at s, and not on the previous history of the state variable x_t with $t < s$: here x_s represents all that is necessary to know in order to form rational expectations. This justifies the comment made above at the end of point (i) about restricting the admitted policies.

One last important point concerns the issue of the existence of solutions to an optimal control problem of the kind outlined in this section. Up to now we have proceeded as if a solution were always available without inquiring into the conditions that guarantee its existence and, in the case of existence,

whether the solution is unique or not. Existence theorems in this (however simplified) context involve a considerable amount of mathematics and cannot be considered here. For those interested, see below an intuitive outline of the main mathematical argument and a reference for further reading.

[*Optional*] Briefly, the argument for existence can be outlined thus. Given a generic function $v : \mathbb{R} \to \mathbb{R}$, and assuming $\varphi(y) + \beta E v(g(x, y, \varepsilon))$ to be bounded, construct a policy σ (this too an $\mathbb{R} \to \mathbb{R}$ function) associated with v by means of the definition

$$\sigma(x) := \arg\max_y [\varphi(y) + \beta E v(g(x, y, \varepsilon))]$$

(assuming that a max exists for all x; otherwise use $\arg\sup_y$). Here Ev is computed on the basis of the stationary distribution of ε. Call T the operator on the space of $\mathbb{R} \to \mathbb{R}$ functions defined by

$$Tv(x) := \varphi(\sigma(x)) + \beta E v(g(x, \sigma(x), \varepsilon))$$

Suppose that the functional operator T has a fixed point, i.e. that a function v^* exists such that $v^* = Tv^*$. We then have

$$v^*(x) = \varphi(\sigma(x)) + \beta E v^*(g(x, \sigma(x), \varepsilon))$$

for some policy σ such that

$$\sigma(x) = \arg\max_y [\varphi(y) + \beta E v^*(g(x, y, \varepsilon))]$$

As a consequence

$$v^*(x) \geq \varphi(y) + \beta E v^*(g(x, y, \varepsilon))$$

As is clear from the last three formulas, together with a fixed point v^* we have found a value function, v^* itself, and an optimal (stationary) policy σ at one stroke. It is easily seen that the reverse is also true, an optimal policy together with its corresponding value function clearly defines a fixed point of the T operator. The mathematical problem of defining conditions for the existence and uniqueness of the solution to an optimal control problem is therefore the same as defining conditions for the existence and uniqueness of fixed points of an operator in a function space. If T has a unique fixed point, the value function is also unique and at least one (but not necessarily unique) optimal policy exists. An exposition of most of the theorems relevant in developing the argument sketched here can be found in Stachurski, *Economic Dynamics,* cit., chapter 10. According to a fixed point theorem by Banach, the proof of the existence and uniqueness relies on T being a uniform contraction in the space of bounded $\mathbb{R} \to \mathbb{R}$ functions. [*End of the optional part*]

6.2 A model of optimal portfolio choice

We now apply these notions to the portfolio choice model of Wickens chapter 10 (11 in the 2nd edition).

In the DGE model of Wickens chapter 4 there is only one financial asset available as an instrument for accumulating wealth, and its return is known in advance with certainty. Now we assume that financial markets trade a variety of assets with different characteristics and uncertain future returns. In trying to maximize total discounted utility the representative household must not only distribute its income between consumption and savings each period, but also choose how to distribute its savings among the existing assets knowing the prices at which they are currently traded but ignoring their future prices and returns. This is an example of optimal control problem in a stochastic environment.

Assume that there are $N+1$ financial assets named with numbers from 0 (conventionally referred to the risk-free asset) to N. At date s the price of each asset h, p_{hs}, is known. The 'payoff' of h, i.e. the yield from keeping one unit of h in portfolio from s to $s+1$, is indicated by $x_{h,s+1}$ and becomes known only at $s+1$. The payoff is determined by the market price at $s+1$, $p_{h,s+1}$, plus the cash flows (if any) to which the owner may be entitled, such as dividends, interest payments and other instalments the maturity of which expires at $s+1$. The reasons why payoffs may not be known in advance are obvious. Both the asset prices and the actual cash flows paid are assumed to be regulated by an unpredictable random process about which agents may only have expectations. Thus, at s, $x_{h,s+1}$ is a random variable indicated with $X_{h,s+1}$. And the (gross) rate of return from keeping one unit of h between s and $s+1$, defined by

$$1 + r_{h,s+1} \equiv \frac{x_{h,s+1}}{p_{hs}}$$

is also a random variable, indicated with $R_{h,s+1}$. Of course $X_{h,s+1} \equiv p_{hs} R_{h,s+1}$. In this presentation, we assume that all the hypotheses of the previous section concerning the underlying stochastic process and expectations still hold.

The household's portfolio at date s is the $N+1$ components vector $a_s = (a_{os}, a_{1s}, ...a_{Ns})$, with each component $a_{hs} \geq 0$ representing the number of units of asset h held from s to $s+1$. For the sake of simplicity we assume that returns from assets are the only source of income of the household. Thus, c_s being consumption expenditure at s, the budget constraint

$$\sum_h p_{hs} a_{h,s+1} = \sum_h x_{hs} a_{hs} - c_s \tag{63}$$

(income available at s is either spent in consumption or saved in the form of assets purchased at s and held in portfolio until $s+1$) represents the equation

of motion of the state vector a_s as regulated by the control variable c_s. Notice that in equation (63) prices and payoffs are measured in units of consumption goods. The model we are considering is 'real'; the corresponding 'nominal' model will be examined later.

As usual, we define the objective function of the representative household as

$$\sum_{s=0}^{\infty} \beta^s u(c_s)$$

(consistently with the budget constraint, households provide no labour).

At $s = 0$ the initial portfolio a_o is given. According to what we have seen in the previous section, the optimal policy is a stationary decision function that maps information available at each date s, i.e. a_s, current payoffs x_s and prices p_s, into consumption c_s and the composition of the portfolio a_{s+1} transferred from s to the next date. Let us mark with a 'hat' $\hat{\cdot}$ the values of variables determined in accordance with the optimal policy. In this particular case, the general recursive condition (62) that characterizes the value function takes the special form

$$V(\hat{a}_s) = u(\hat{c}_s) + \beta E V(\hat{a}_{s+1}) \tag{64}$$

In the formula, $\hat{a}_s$ has a hat because it is the result of past optimal choices up to s; $\hat{c}_s$ and $\hat{a}_{s+1}$ indicate the current optimal choice at s, given information $\hat{a}_s, x_s, p_s$. This choice must of course be consistent also with the budget constraint (63). Notice that, although $\hat{a}_{s+1}$ is known at s, $V(\hat{a}_{s+1})$ is uncertain because future consumption $\hat{c}_{s+1}$ depends on future payoffs and prices.

Since $\hat{c}_s$ maximizes $V(\hat{a}_s)$, by differentiating (64) with respect to c_s we get Bellman equation

$$\begin{aligned} 0 &= u'(\hat{c}_s) + \beta \frac{dEV(\hat{a}_{s+1})}{dc_s} \\ &= u'(\hat{c}_s) + \beta E \frac{dV(\hat{a}_{s+1})}{dc_s} \\ &= u'(\hat{c}_s) + \beta E \left(\sum_{h=0}^{N} \frac{\partial V(\hat{a}_{s+1})}{\partial a_{h,s+1}} \frac{da_{h,s+1}}{dc_s} \right) \end{aligned} \tag{65}$$

At the first step we have used the linearity of the E operator ($dEV = EdV$); at the second step we have used the formula of the total differential for $dV(\hat{a}_{s+1})$. Now we take advantage of the fact that, according to the budget constraint,

$$-dc_s = \sum_{h=0}^{N} p_{hs} da_{h,s+1} \tag{66}$$

or, equivalently

$$-1 = \sum_{h=0}^{N} p_{hs} \frac{da_{h,s+1}}{dc_s}$$

This allows us to rewrite the form (65) of Bellman equation as follows

$$\begin{aligned} 0 &= -u'(\hat{c}_s) \sum_{h=0}^{N} p_{hs} \frac{da_{h,s+1}}{dc_s} + \beta E \left(\sum_{h=0}^{N} \frac{\partial V(\hat{a}_{s+1})}{\partial a_{h,s+1}} \frac{da_{h,s+1}}{dc_s} \right) \\ &= \sum_{h=0}^{N} \left(-u'(\hat{c}_s) p_{hs} + \beta E \frac{\partial V(\hat{a}_{s+1})}{\partial a_{h,s+1}} \right) \frac{da_{h,s+1}}{dc_s} \end{aligned}$$

Notice that this modified Bellman equation must hold for all the admissible vectors of $N+1$ ratios $\frac{da_{h,s+1}}{dc_s}$, i.e. independently of the way in which the variation in consumption dc_s is offset by variations in the amounts of assets held in portfolio provided equation (66) is satisfied. It is easy to check that this is equivalent to imposing the condition that for each asset $h = 0, 1, ...N$ the following equation holds

$$-u'(\hat{c}_s) p_{hs} + \beta E \frac{\partial V(\hat{a}_{s+1})}{\partial a_{h,s+1}} = 0 \tag{67}$$

Exercise 21 *Prove that condition (67) for $h = 0, 1...N$ is necessary (sufficiency is obvious) for Bellman equation to hold for all the admissible plans of investment (or disinvestment) of the positive (or negative) saving due to a variation $-dc_s$.*

The intuition behind formula (67) is the following: along the optimal plan the discounted marginal contribution of each asset h to the expected future value of the plan must equal the variation in present utility due to the purchase or sale of one unit of h at its current price (remember that in this model prices are measured in units of consumption goods).

As a last step, in order to arrive at a characterization of the optimal policy in terms of Euler equation, we compute the partial derivatives $\frac{\partial V(\hat{a}_{s+1})}{\partial a_{h,s+1}}$ in (67) following the Euler procedure that was described in section 3.5. This procedure requires that, after a variation dc_s, the subsequent variation $da_{h,s+1}$ in the amount held of each asset h is offset by a variation dc_{s+1} such that at $s+2$ the portfolio composition is brought back to the composition $\hat{a}_{s+2}$ that we would have had if no variation of plans had occurred at s and $s+1$. This means that, whatever payoff $x_{h,s+1}$ realizes at $s+1$, consumption must absorb the differential income $x_{h,s+1} da_{h,s+1}$ so that

$$da_{h,s+2} = x_{h,s+1} da_{h,s+1} - dc_{s+1} = 0$$

This implies

$$\frac{dc_{s+1}}{da_{h,s+1}} = x_{h,s+1} = p_{hs}\left(1 + r_{h,s+1}\right) \tag{68}$$

From the recursive condition (64) applied at $s+1$

$$V\left(\hat{a}_{s+1}\right) = u\left(\hat{c}_{s+1}\right) + \beta EV\left(\hat{a}_{s+2}\right)$$

we have also

$$\frac{\partial V\left(\hat{a}_{s+1}\right)}{\partial a_{h,s+1}} = u'\left(\hat{c}_{s+1}\right)\frac{dc_{s+1}}{da_{h,s+1}}$$

This is because, according to Euler's procedure, $EV\left(\hat{a}_{s+2}\right)$ is constant with respect to $a_{h,s+1}$. By replacing from (68) into the last equation we get

$$\frac{\partial V\left(\hat{a}_{s+1}\right)}{\partial a_{h,s+1}} = u'\left(\hat{c}_{s+1}\right)\left(1 + r_{h,s+1}\right)p_{hs}$$

Now we can go back to Bellman equations (67) and, using the last equation (and dividing by p_{hs}), transform them into the following

$$-u'\left(\hat{c}_s\right) + \beta E\left(U'_{s+1}R_{h,s+1}\right) = 0 \tag{69}$$

where both $u'\left(\hat{c}_{s+1}\right)$ and $\left(1 + r_{h,s+1}\right)$, if looked at from the standpoint of information available at s, are random variables indicated respectively as U'_{s+1} and $R_{h,s+1}$. This is the form taken by Euler condition when returns to financial investment are uncertain..

Recall from section 3.5 that Euler conditions are not equivalent to Bellman's: they are necessary but in general not sufficient for dynamic maximization. In order to make these two sets of conditions equivalent each Euler condition (for each asset h) should be supplemented with a transversality condition

Exercise 22 *Check that the transversality condition in this problem of optimal portfolio choice is, for each asset h,*

$$\lim_{n\to\infty}\beta^n E\left(u'_n p_{hn} a_{hn+1}\right) = 0$$

6.3 CAPM utility-based and consumption-based

From Euler condition (69) the optimal portfolio choice model can be developed in two ways, leading to two parallel versions of the classical Capital Asset Pricing Model (CAPM) of the theory of financial markets. These two developments are called CAPM 'utility-based' (UBCAPM) and 'consumption-based' (CBCAPM).

We examine UBCAPM first. Go back to Euler equations (69) and rewrite them, using the definition of covariance between the two random variables U' and R_h, as

$$0 = -u'(\hat{c}_s) + \beta \left[cov\left(U'_{s+1}, R_{h,s+1}\right) + EU'_{s+1} ER_{h,s+1} \right] \tag{70}$$

Remember that we have assumed that a risk-free asset exists, the one indicated by the name $h = 0$. The risk-free asset pays a fixed amount of consumption goods (our units of account for prices) to its owner with absolute certainty. Therefore, its future payoff $x_{o,s+1} = (1 + r_{o,s+1}) p_{o,s}$ is known with certainty or, which is the same, its rate of return $R_{o,s+1}$ is a 'degenerate' random variable with $ER_{o,s+1} = (1 + r_{o,s+1})$ and $varR_{o,s+1} = 0$. Thus, in the case of $h = 0$ equation (70) takes the form

$$0 = -u'(\hat{c}_s) + \beta (1 + r_{o,s+1}) EU'_{s+1}$$

Use the last equation to replace the term $u'(\hat{c}_s)$ in all the Euler equations (70) for $h \neq 0$. After some rearrangement of terms we have

$$ER_{h,s+1} - (1 + r_{o,s+1}) = -\frac{cov\left(U'_{s+1}, R_{h,s+1}\right)}{EU'_{s+1}} \tag{71}$$

The difference on the left-hand side of the equation is recognizable as the risk premium provided by asset h, call it

$$\rho_{h,s+1} \equiv ER_{h,s+1} - (1 + r_{o,s+1})$$

(notice that ρ is an expected value). The equation implies that h is held in the optimal portfolio from date s to $s+1$ provided $\rho_{h,s+1}$ equals the negative of the relative (relative to expected marginal utility) covariance between its return and marginal utility.

This condition becomes clearer if we consider the expected return of the whole optimal portfolio. This is given by

$$\begin{aligned} ER_{s+1} &= \frac{E \sum_{h=0}^{N} x_{h,s+1} a_{h,s+1}}{\sum_{i=0}^{N} p_{is} a_{i,s+1}} \\ &= \frac{\sum_{h=0}^{N} \left(p_{hs} ER_{h,s+1}\right) a_{h,s+1}}{\sum_{i=0}^{N} p_{is} a_{i,s+1}} \\ &\equiv \sum_{h=0}^{N} \omega_{hs+1} ER_{h,s+1} \end{aligned} \tag{72}$$

where the symbol ω_{hs+1} stands for the share of the value of the entire portfolio occupied by asset h evaluated at the prices current at s

$$\omega_{hs+1} \equiv \frac{p_{hs}a_{h,s+1}}{\sum_{i=0}^{N} p_{is}a_{i,s+1}}$$

The next step is left as an exercise.

Exercise 23 *Demonstrate that*

$$E\left(R_{s+1}\right) - \left(1 + r_{o,s+1}\right) = -\frac{cov\left(U'_{s+1}, R_{s+1}\right)}{EU'_{s+1}} \tag{73}$$

[Hint: start from (72); replace $ER_{h,s+1}$ from (71); develop using the linearity property of covariance

$$\sum_h \omega_h cov\left(X, Y_h\right) = cov\left(X, \sum_h \omega_h Y_h\right)$$

As in equation (71), the risk premium of the entire portfolio, $\rho_{s+1} \equiv ER_{s+1} - (1 + r_{o,s+1})$, is expressed in (73) as a function of the negative of the relative covariance between portfolio return and marginal utility.

From (73) we get

$$EU'_{s+1} = -\frac{cov\left(U'_{s+1}, R_{s+1}\right)}{\rho_{s+1}}$$

which can be used to replace EU'_{s+1} in (71) with the result that

$$\rho_{h,s+1} = \frac{cov\left(U'_{s+1}, R_{h,s+1}\right)}{cov\left(U'_{s+1}, R_{s+1}\right)}\rho_{s+1} \tag{74}$$

In this formula the equilibrium risk premium of each asset h is shown as a proportion of the portfolio risk premium, with a proportionality coefficient which depends on the ratio between two covariances between returns (to h, and to portfolio) and marginal utility. In the language of CAPM, this ratio is called the 'beta' of asset h with respect to portfolio.

Formula [74] represents the fundamental equation of UBCAPM. Notice that according to this approach to portfolio choices the risk associated with the presence of an asset in portfolio, and with the portfolio as a whole, is indicated by the negative of the covariance between the rate of return (of the asset, or the portfolio) and marginal utility (see formulas (71) and (73)).

Since marginal utility is inversely related to the level of consumption, the sign of this covariance indicates whether returns tend to vary in the same direction or in the opposite direction with respect to the consumption level. In general, a positive covariance implies that the asset (portfolio) acts as a stabilizer of the time profile of consumption, in the sense that its yield tend to be greater when this is more needed, i.e. in situations of low consumption, and smaller when less needed, as in situations of high consumption. Indeed, such an asset (or portfolio), as shown by formulas (71) and (73), commands a negative risk premium. With negative covariance, instead, yields tend to be greater (smaller) when less (more) needed; this is taken as a the essential feature of risky assets (portfolios), assets that act as destabilizers of the consumption time profile.

In the case of a risky portfolio, i.e. a portfolio with $\rho_{s+1} > 0$, or equivalently with $cov\left(U'_{s+1}, R_{s+1}\right) < 0$, a 'safe' asset is simply an asset whose 'beta' is less than one. This implies

$$cov\left(U'_{s+1}, R_{h,s+1}\right) > cov\left(U'_{s+1}, R_{s+1}\right)$$

and includes cases of both stabilizing assets (with $cov\left(U'_{s+1}, R_{h,s+1}\right) > 0$ or, equivalently, $\rho_{h,s+1} < 0$) and destabilizing assets ($cov\left(U'_{s+1}, R_{h,s+1}\right) < 0$) which, however, are less risky than the whole portfolio.

CBCAPM provides an alternative formulation of the same model, based on variations in consumption instead of marginal utility. The relationship between the two can be obtained starting from the linear approximation

$$u'\left(c_{s+1}\right) \cong u'\left(c_s\right) + u''\left(c_s\right)\left(c_{s+1} - c_s\right)$$

This can be rewritten as

$$\begin{aligned} u'\left(c_{s+1}\right) &\cong u'\left(c_s\right)\left[1 + \frac{u''\left(c_s\right)}{u'\left(c_s\right)}\left(c_{s+1} - c_s\right)\right] \\ &= u'\left(c_s\right)\left[1 + c_s \frac{u''\left(c_s\right)}{u'\left(c_s\right)} g_{s+1}\right] \end{aligned}$$

where g_{s+1} stands for the rate of growth of consumpt between s and $s+1$

$$g_{s+1} \equiv \frac{c_{s+1} - c_s}{c_s}$$

We know from section2.7 above that the term

$$\sigma_s \equiv -c_s \frac{u''\left(c_s\right)}{u'\left(c_s\right)}$$

a measure of the elasticity of marginal utility, is also used as an index of 'Relative Risk Aversion' (RRA). If for the sake of simplicity we assume the utility function to be a CRRA ('constant RRA') function, with $\sigma_s = \sigma$ ranging in the semi-closed interval $(0, 1]$ (0 corresponds to linear utility, equivalent to indifference to risk; 1 corresponds to logarithmic utility, which we take to represent the maximum level of risk aversion), then we can write (still within the limits of linear approximation)

$$u'(c_{s+1}) = u'(c_s)(1 - \sigma g_{s+1})$$

From the last formula it is clear that, given the level of information available at s, the random variable U'_{s+1} depends on two known parameters, $u'(c_s)$ and σ, and the random variable G_{s+1}, the rate of growth of consumption, unknown at s, between s and the next date

$$U'_{s+1} = u'(c_s)(1 - \sigma G_{s+1})$$

From this, by exploiting the properties of mean value and covariance, it is easy to get

$$EU'_{s+1} = u'(c_s)(1 - \sigma E G_{s+1})$$

and

$$cov\left(U'_{s+1}, R_{h,s+1}\right) = -\sigma u'(c_s)\, cov\left(G_{s+1}, R_{h,s+1}\right)$$

Now we are able to replace all the occurrences of U'_{s+1} in formulas (71), (73) and (74) with the appropriate expression in terms of G_{s+1}. In particular, we get

$$ER_{h,s+1} - (1 + r_{o,s+1}) = \frac{\sigma cov\left(G_{s+1}, R_{h,s+1}\right)}{1 - \sigma E G_{s+1}}$$

for the risk premium $\rho_{h,s+1}$, and

$$E\left(R_{s+1}\right) - (1 + r_{o,s+1}) = \frac{\sigma cov\left(G_{s+1}, R_{s+1}\right)}{1 - \sigma E G_{s+1}}$$

for the portfolio risk premium ρ_{s+1}. From these two formulas we get the CAPM formula in terms of the 'beta' of each asset h

$$\rho_{h,s+1} = \frac{cov\left(G_{s+1}, R_{h,s+1}\right)}{cov\left(G_{s+1}, R_{s+1}\right)} \rho_{s+1}$$

The interpretation of the CBCAPM formulas confirms the interpretation of risk and security, based on the sign of the covariance between return and consumption, given above for the UBCAPM formulas. Notice in particular the following

Exercise 24 *Check that, while the 'beta' of an asset is independent of the degree of RRA as measured by the parameter σ, the risk premium of any asset increases or decreases with σ depending on whether its return covaries positively or negatively with the rate of growth of consumption.*

6.4 Nominal CAPM

In the model considered so far all values (prices, payoffs and rates of return) are 'real' values in the sense that they are referred to units of consumption goods. The same model can be expressed in terms of nominal (monetary) values by using a nominal consumption price index p_{cs}. The definitions of the nominal variables corresponding to the real variables that we have called $p_{hs}, x_{h,s+1}$ and $(1 + r_{h,s+1})$ are the following:

$$\begin{aligned} \overset{n}{p}_{hs} &\equiv p_{cs} p_{hs} \\ \overset{n}{x}_{h,s+1} &\equiv p_{c,s+1} x_{h,s+1} \equiv (1 + \pi_{s+1})\, p_{cs} x_{h,s+1} \\ 1 + \overset{n}{r}_{h,s+1} &\equiv \frac{\overset{n}{x}_{h,s+1}}{\overset{n}{p}_{hs}} \equiv \frac{(1 + \pi_{s+1})\, x_{h,s+1}}{p_{hs}} \equiv (1 + \pi_{s+1})\,(1 + r_{h,s+1}) \end{aligned}$$

In the definitions, π_{s+1} is the rate of inflation between s and $s+1$ as measured by the consumption price index

$$1 + \pi_{s+1} \equiv \frac{p_{c,s+1}}{p_{cs}}$$

The rest of the model can be developed according to these definitions. Notice that, given all the information available at s, 'nominal' uncertainty depends on all the factors that contribute to 'real' uncertainty plus the additional uncertainty concerning future inflation π_{s+1} (at s, future inflation is a random variable). A consequence of this fact is that the risk-free asset (by definition, the asset called '0' in the model) cannot be the same asset in the real and the nominal model. A nominal risk-free asset must be one which pays a fixed nominal payoff $\overset{n}{x}_{s+1}$ known in advance with certainty, while a real risk-free asset must be one whose nominal payoff varies in the same proportion as the nominal consumption price index $p_{c,s+1}$. Thus, in the formulas of nominal CAPM, nominal risk premia are referred to a nominal risk-free return which is generally different from the one used in the real CAPM, both UB and CB.

The development of the formulas for equilibrium risk premia and the 'betas' is left as an exercise. Hint: for the UBCAPM start from Euler equation

(69); call Π_{s+1} the random variable $(1+\pi_{s+1})$ and rewrite (69) as

$$\begin{aligned} 0 &= -u'(\hat{c}_s) + \beta E\left(\frac{U'_{s+1}\overset{n}{R}_{h,s+1}}{\Pi_{s+1}}\right) \\ &= -u'(\hat{c}_s) + \beta cov\left(\frac{U'_{s+1}}{\Pi_{s+1}}, \overset{n}{R}_{h,s+1}\right) + \beta E\left(\frac{U'_{s+1}}{\Pi_{s+1}}\right) E\left(\overset{n}{R}_{h,s+1}\right) \end{aligned}$$

Apply this to the (nominal) risk-free asset $h=0$ and replace

$$u'(\hat{c}_s) = \beta\left(1+\overset{n}{r}_{o,s+1}\right) E\left(\frac{U'_{s+1}}{\Pi_{s+1}}\right)$$

In the end, in the UB formulation the nominal risk premium of asset h turns out to be

$$\overset{n}{\rho}_{h,s+1} = -\frac{cov\left(\frac{U'_{s+1}}{\Pi_{s+1}}, \overset{n}{R}_{h,s+1}\right)}{E\left(\frac{U'_{s+1}}{\Pi_{s+1}}\right)}$$

and, for the entire portfolio

$$\overset{n}{\rho}_{s+1} = -\frac{cov\left(\frac{U'_{s+1}}{\Pi_{s+1}}, \overset{n}{R}_{s+1}\right)}{E\left(\frac{U'_{s+1}}{\Pi_{s+1}}\right)}$$

6.5 Contingent claims

According to a different approach to portfolio choices known as the 'contingent claims approach', trading in securities in order to obtain an equilibrium portfolio composition can be thought of as trading in money (or any other asset that can be exchanged for final commodities) conditional on possible 'states of the world' in order to obtain an equilibrium profile of one's own wealth conditional on the same states. The underlying idea is that uncertainty means that there are features of the surrounding world that are (at least temporarily) unknown, so that from the point of view of the available information the 'true' world that a generic individual inhabits is one of many possible worlds, as many as the possible realizations of the unknown features. When the unknown features are related to the value of the assets that constitute the individual's wealth, we have the typical case of economic uncertainty. The implications of this way of looking at uncertainty for portfolio choices can be shown by a simple example.

Example 25 *Suppose that, on the basis of today's information, tomorrow's payoff (in monetary units) of security i may be either $x_i = 30$ or $x_i = 10$, while the payoff of a different security j may be $x_j = 0$ or $x_j = 40$. We can describe this situation as being in a world which may take on 4 possible forms or 'states' corresponding to the 4 pairs of payoffs $(30, 0)$, $(30, 40)$, $(10, 0)$ and $(10, 40)$. Call the 4 states (respectively) α, β, γ and δ. Each security can be considered as a bunch of claims on future money, each claim being conditional ('contingent') on the occurring of one state out of the 4 possible states of the world. Hence the expression 'contingent claims'. Choosing to keep given amounts of i and j in portfolio is equivalent to choosing to keep the resulting bunch of contingent claims.*
The example shows that exchanging one security for the other at the ruling market prices is the same as exchanging value across the 4 states at rates of exchange that are determined by the market prices of the securities. For example, with prices p_i and p_j such that $p_i = \frac{3}{4}p_j$, 3 units of j can be sold in exchange for 4 units of i. This transaction is equivalent to selling 80 units of value conditional on the occurring of state δ in exchange for 40 units conditional on γ plus 120 conditional on α. Looked at in this way, a financial transaction is similar to an insurance policy (buying money conditional on the occurring of a specified state by paying a premium in all the other states) or more generally to a wager. The risk profile of the portfolio is modified accordingly. At the prices of the example, and assuming that securities can be traded in fractions, the risk profile of a portfolio containing 3 units of j can be transformed into any risk profile of the type

State	α	β	γ	δ
Value of portfolio	$120\,(1-y)$	120	$40\,(1-y)$	$40+80y$

with y ranging from 0 (only i in portfolio) to 1 (only j).

This representation of trading in uncertain securities as trading across states of the world reveals a 'hidden side' of financial transactions and provides an insight into aspects that might otherwise escape notice. One such aspect is the possibility of arbitraging activities that are not immediately obvious from the point of view of the exchange of securities. In the previous example we have seen the possibility of transferring value from δ to α and γ, or vice versa, at a ratio of exchange that depends on the prices of the two securities involved in the operation. May be the same possibility is afforded by exchanging a security h for a security k, with the prices p_h and p_k determining a different rate of exchange. In such a case an arbitrageur might devise a transaction scheme involving all the four securities so as to transfer value from δ to α and γ and the reverse, and realize a net arbitrage gain

in the process thanks to the difference in the rates of exchange. It is clear that security prices that allow the carrying out of a transfer of value between states simultaneously at different rates of exchange cannot be equilibrium prices. A number of arbitrageurs would embark on schemes of purchasing and selling that would inevitably end by affecting the existing prices.

This argument emphasizes an implicit property of equilibrium prices in financial markets: the prices of securities should be such that, if a transfer of value across given states can be carried out in more than one way (i. e., through different securities), the implicit rates of exchange across states are the same whatever the chosen way may be.

A second aspect emphasized by the contingent claims approach concerns the 'degree of completeness' of the financial structure. In the above example there are inter-state exchanges that, with whatever prices p_i and p_j, cannot be carried out if i and j are the only two securities traded in the financial market. Think, for example, of a transfer of value from α to δ or viceversa, with the amount of value available at β and γ remaining unchanged. While value at β may not be affected by a transaction involving i and j provided the rate of exchange is $\frac{p_i}{p_j} = \frac{3}{4}$, value at γ is inevitably modified whatever the prices of the two securities may be. In a case like this we can say that the financial structure is incomplete because there are inter-state transfers of value that the existing assets do not allow. To see what a complete financial structure may look like, imagine that, with the same two securities, the set of possible states of the world were reduced to 2, for example by assuming that the returns on i and j are known to be positively (negatively) correlated, so that states α and δ (β and γ) are excluded a priori. In such a case, all transactions in securities would coincide with transfers of value between the residual two states.

Exercise 26 *Check that, in the case of a positive correlation between returns, the ratio of exchange between value in β and value in γ (how many euros in β are worth one euro in γ) equals*

$$3 - 4\frac{p_i}{p_j}$$

Argue that a price ratio $\frac{p_i}{p_j} \geq \frac{3}{4}$ could not be an equilibrium price ratio because of arbitrage. Compute the ratio of exchange between α and δ, and the no arbitrage limit to the equilibrium $\frac{p_i}{p_j}$ ratio, in the case of a negative correlation.

Both issues, uniqueness of inter-state rates of exchange and completeness of the financial structure, can be analyzed in more general terms under the simplifying assumption that the number of possible states of the world is

finite (quite a strong simplification with respect to the analysis of the previous sections, in which we dealt with continuous random variables, implying a continuum of possible states of the world). The analysis utilizes a convenient device called 'Arrow securities' (from Nobel laureate Ken Arrow who introduced it in the 1950s), i.e. fictional securities that pay a positive amount of money if one and only one state occurs, zero in all the other cases.

Let $N+1$ be the number of financial assets, as usual, and W the number of possible states of the world. Each asset i is associated with a list of possible payoffs x_{iw}, one for each state $w = 1, ...W$, i.e. with a payoff vector x_i with W components. Uncertainty about payoffs is represented by the variance of the components of x_i. The risk-free asset ($i = 0$ by convention) is the only one characterized by x_{ow} constant with respect to w.

By aligning vertically the payoff row vectors of all the $N + 1$ assets we get a payoff matrix, call it X, of dimension $(N + 1) \times W$. A transaction in securities is represented by an $N + 1$ (row) vector, call it δ, such that each component δ_i indicates the amount of i bought (if $\delta_i > 0$) or sold (if $\delta_i < 0$), with $\delta_i = 0$ indicating that i is not involved in the transaction. The W-component (row) vector δX is the vector of transfers of value across states consequent on the transaction, i.e. the change in the portfolio risk profile following from the purchases and sales of securities indicated by δ. If we call p the $N + 1$ component (column) vector of the current security prices, the product δp represents the cost of the operation.

Let us define a notional Arrow security relative to state w as a security having payoff 1 in state w and 0 in all the other states. The unit vector $u^{(w)}$ (1 at place w and all zeros elsewhere) is its payoff vector. If a complete array of Arrow securities existed, one for each state w and each one with its own price, call it q_w, then by buying and selling Arrow securities it would be possible to carry out any desired transfer of value between states at well-defined prices. In such a case the financial structure would obviously be complete.

In general, Arrow securities do not actually exist, or only some of them exist but in very special circumstances, so that this definition of completeness may sound completely abstract. Actual existence is not that relevant, however, because an Arrow security may be 'replicated' by a transaction in the actually existing securities provided there are enough securities in the market and the payoff matrix X has a suitable structure. As is easily seen, the notional Arrow security w can be replicated by the existing securities if the system of W equations in the $N + 1$ unknowns δ_i

$$\delta X = u^{(w)} \tag{75}$$

has at least one solution. In the end, it all depends on the structure of the payoff matrix. If a solution to the system exists, call it $\delta^{(w)}$, then agents can

buy or sell one euro conditional on state w by carrying out the transaction $\delta^{(w)}$ or $-\delta^{(w)}$ at cost $\delta^{(w)}p$ or with proceeds $-\delta^{(w)}p$, as if the Arrow security w with its price q_w actually existed.

A definition of complete financial structure follows naturally from this discussion: a financial market that works as if a complete array of Arrow securities were available (whether or not they exist in fact), enabling agents to transfer value across states in all conceivable ways at well-defined prices, is what we mean by a 'complete financial structure'. The mathematical condition implied by this definition is that the system (75) must have at least one solution $\delta^{(w)}$ for each $w = 1, ...W$. Selecting W solutions and arranging the corresponding δ vectors in a single $W \times (N+1)$-dimensional matrix, call it D, we must therefore have

$$DX = I \tag{76}$$

with I the W-dimensional identity matrix. As is clear from the last formula, the issue of the existence of a matrix like D is strictly related to the existence of the inverse of matrix X, or the inverse of a submatrix of the latter.

Let r indicate the rank of X. Clearly, $r \leq \min\{N+1, W\}$, but $r < W$ is easily seen to exclude completeness.

Exercise 27 *Show that, with $r < W$, no matrix of transactions D can satisfy the system (76) [Hint: consider that the hypothesis implies that at least one column of X is a linear combination of other columns; check that it is impossible to find a matrix of transactions D such that all the products $Dx^{(w)}$ (where the $x^{(w)}$ are columns of X) are unit vectors.]*

Completeness therefore requires $r = W \leq N + 1$, i.e. (i) a number of securities not smaller than the number of states, and (ii) a minimum degree of diversification among securities, in the sense that at least W out of the $N + 1$ securities must have linearly independent payoff vectors.

In the case $r = W = N + 1$, X is a square nonsingular matrix and the system (76) has one and only one solution $D = X^{-1}$. In this case, an Arrow security w is replicated by the vector of transactions

$$\delta^{(w)} = u^{(w)}X^{-1} = \left(X^{-1}\right)^{(w)}$$

where $\left(X^{-1}\right)^{(w)}$ stands for the w-th row of X^{-1}. The price of one euro conditional on w is given by the notional price of the corresponding Arrow security

$$q_w = \delta^{(w)}p = \left(X^{-1}\right)^{(w)} p$$

and the whole vector of Arrow prices is

$$q = Dp = X^{-1}p \tag{77}$$

Transfers of value between any two states w and v may be carried out combining the two transaction schemes $\delta^{(w)}$ and $\delta^{(v)}$, at a unique rate of exchange determined by the ratio between q_w and q_v.

When $r = W < N + 1$ we still have completeness but in a more complex situation, one that may be open to arbitrage. With X having more rows than columns, or in other words a 'redundancy' of securities, there is more than one nonsingular square submatrix of dimension W that can be extracted out of X. This is the same as saying that there is more than one set of W securities that can provide a replica of a complete array of Arrow securities.

Consider for example one such case with two groups of securities, A and B, corresponding to the two submatrices X_A and X_B, both nonsingular square matrices of dimension W. Correspondingly, we have two transaction matrices $D_A = X_A^{-1}$ and $D_B = X_B^{-1}$, and two vectors of Arrow prices,

$$q_A = X_A^{-1}p_A \quad \text{and} \quad q_B = X_B^{-1}p_B$$

where p_A and p_B are the price vectors of the securities belonging respectively to group A and group B.

Equilibrium requires security prices to leave no room for arbitrage. An implication of the no-arbitrage condition is therefore that there is no state w such that $q_{Aw} \neq q_{Bw}$, because in such a case arbitrageurs would gain from the difference in Arrow prices by carrying out transactions $\delta_A^{(w)}$ and $\delta_B^{(w)}$ in opposite direction at the same time. Thus, $q_A = q_B$, which means, in terms of security prices, $D_A p_A = D_B p_B$, or indifferently one of the following

$$\begin{aligned} p_A &= X_A X_B^{-1} p_B \\ p_B &= X_B X_A^{-1} p_A \end{aligned}$$

In other words, the prices of securities in the two groups must be linked to each other by the linear transformation $X_A X_B^{-1}$ or its inverse $X_B X_A^{-1}$.

One last implication follows from the last condition. Consider any security h with its W-component payoff vector x_h. Form a group A of W securities with linearly independent payoff vectors taking care to include h in the group (we know from elementary linear algebra that this is always possible). Then, as we have just seen, in an equilibrium of a complete financial market we must have

$$p_A = X_A q$$

with the h component of the p_A vector given by

$$p_h = x_h q$$

The value of security h must equal the value of its contingent claims evaluated at Arrow prices. As this must be true of any security, in a complete market equilibrium security prices and Arrow prices must satisfy the system

$$p = Xq \tag{78}$$

6.6 A contingent claims interpretation of optimal portfolio choice

In the hypothesis of a complete financial structure, the contingent claims approach to portfolio choice provides a slightly different view of the conditions that characterize the equilibrium choice.

Although Arrow securities may not exist as such, a complete market provides perfect replicas in the form of transaction schemes $\delta^{(w)}$ with well-defined prices $q_w = \delta^{(w)} p$ for all states $w = 1, ...W$. Let us then define the expected rate of return of an Arrow security w (or of its replica) as

$$ER_w = \sum_{v=1}^{W} \pi_v \frac{x_{wv}}{q_w} = \frac{\pi_w}{q_w}$$

In the formula we have indicated the probability of each state v with π_v, and taken account of the fact that the payoff vector x_w has $x_{wv} = 1$ for $v = w$ and $x_{wv} = 0$ for all $v \neq w$.

Following Wickens, ch 10 p 235 (ch 11 in the second edition), we now introduce the definition of the 'stochastic discount factor' of an Arrow security w as the inverse of its expected return

$$m_w \equiv \frac{1}{ER_w} = \frac{q_w}{\pi_w}$$

from which of course we have $q_w = \pi_w m_w$.

The stochastic discount factor m_w can be interpreted as the current 'price' of the probability of two simultaneous events, the occurring of state w *and* getting 1 euro. While in general a stochastic discount factor will be strictly less than 1 (why should anybody pay 1 euro or more for an uncertain euro?), its higher or lower value reflects the implicit utility of money in the state to which it is linked. If, of two states v and w, it turns out for example that $m_v > m_w$, this means that the market evaluates the probability of 1 euro in v

more highly than the same probability in w. This may be taken as evidence of the fact that, for reasons that remain unspecified, money is commonly held to be more useful in v rather than in w.

Equipped with these definitions we go back to the equilibrium condition (78) and rewrite it for each security h as

$$\begin{aligned} p_h &= \sum_w \pi_w x_{hw} m_w \\ &= p_h \sum_w \pi_w \frac{x_{hw}}{p_h} m_w \\ &= p_h \sum_w \pi_w \left(1 + r_{hw}\right) m_w \end{aligned}$$

Indicating with R_h and M respectively the random variables $(1 + r_h)$ and m – they are random variables at the current state of information because which state w will occur is still unknown – we rewrite the last formula as

$$\begin{aligned} 1 &= E\left(R_h M\right) \\ &= cov\left(R_h, M\right) + E\left(R_h\right) E\left(M\right) \end{aligned}$$

The last equation comes from the security prices equilibrium condition and applies to all securities h from $h = 0$ to $h = N$. But in the case of $h = 0$ – conventionally, the risk-free security with payoff (rate of return) constant in all states – the equation reduces to

$$1 = R_o E\left(M\right)$$

and allows us to replace the term $E\left(M\right)$, equal to the inverse of the risk-free rate of return R_o, in all the equations with $h \neq 0$. After some rearranging we thus get

$$E\left(R_h\right) - R_o = -R_o \, cov\left(R_h, M\right) \tag{79}$$

Formula (79) expresses the equilibrium risk premium, ρ_h, of a generic asset h as determined by the negative of the covariance between the asset's return and the stochastic discount factor. A risky asset is one the return to which covaries inversely with m, i.e. pays more (less) in states in which m is lower (higher); according to the interpretation given above, this means that on average the asset pays more/less in cases in which, for any reason, money is commonly considered to be less/more useful. This somehow confirms the interpretation of risk premia that we reached with utility based and consumption based CAPM (see above), but is more general in that it does not require the utility of money to depend on the level of consumption.

If an equilibrium condition like (79) holds for all assets h (including $h = 0$), then it is easy to check (go back to section 6.3, exercise 23) that the

following condition holds for portfolio as a whole (with R for the return to portfolio)

$$E(R) - R_o = -R_o \, cov(R, M)$$

From (79) and the latter formula we obtain

$$E(R_h) - R_o = \frac{cov(R_h, M)}{cov(R, M)} \; (E(R) - R_o)$$

which emphasizes the CAPM 'beta' of asset h as determined by the covariance between R_h and M relative to the covariance between the portfolio return and M.