The Blancmange Function Continuous Everywhere but Differentiable Nowhere
Author(s): David Tall
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The blancmange function
Continuous everywhere but differentiable nowhere

DAVID TALL

One of the problems of the first introduction to the calculus and the subsequent mental imagery developed by the student is that the functions involved are usually given by simple formulae such as $f(x) = x^n$ and the derivatives calculated by formulae crunching: $f'(x) = nx^{n-1}$. The fundamental ideas of the calculus and any relational understanding recede into the background. Getting out of the strait-jacket and considering more general functions at some stage is rarely considered. When it is, it is usually performed in the context of university analysis where pictures are banned because they are claimed to mislead the intuition.

Such an attitude is utterly destructive. What we must do is to retain our intuition so that the theorems of analysis became natural, giving us a more coherent view of the theory. With the coming of the microchip and high resolution graphics, the drawing of much more general functions will become a reality in the classroom in the next twenty years. Now is the time to begin to reorient our understanding of the calculus to take advantage of the new facilities and a broader understanding of the concepts.

My aim here is to give a refined conceptual explanation of continuity and differentiation which are formally correct and have a suitable pictorial interpretation. As a particular example I shall introduce the *blancmange function* (whose name was first coined by my colleague John Mills). The graph of the function is illustrated in Fig. 1.

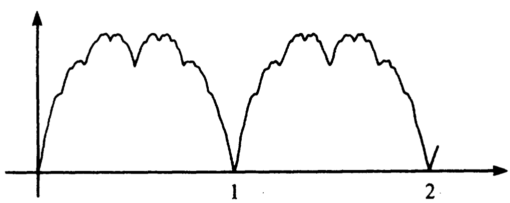

FIGURE 1.

The ideas presented here should be easily within the grasp of students armed only with graph paper, a ruler, a pencil and a lively mind. As we shall see, the blancmange function turns out to be continuous everywhere but differentiable nowhere. We shall learn how to view a graph highly magnified to see how this comes about.

The blancmange recipe

The first ingredient in the blancmange is a simple saw-tooth graph which has the value $f_1(x) = x$ for $0 \leq x \leq \frac{1}{2}$ and $f_1(x) = 1 - x$ for $\frac{1}{2} < x < 1$,

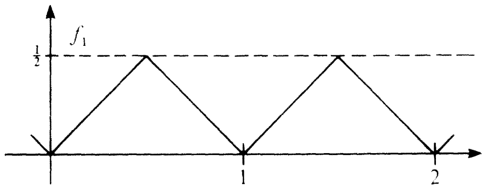

FIGURE 2.

then repeats its values over each succeeding unit interval, as illustrated in Fig. 2. It may be described alternatively as the distance from x to the nearest integer. If $x = k + d$ where k is a whole number and d is the decimal expansion of the rest of x, $(0 \leq d < 1)$, then

$$f_1(k + d) = d \quad \text{when } 0 \leq d \leq \tfrac{1}{2}$$

and

$$f_1(k + d) = 1 - d \quad \text{when } \tfrac{1}{2} < d < 1.$$

For instance, $f(2\frac{1}{4}) = \frac{1}{4}$, $f(3\frac{3}{4}) = \frac{1}{4}$, $f(\sqrt{2}) = (\sqrt{2}) - 1$, $f(e) = 3 - e$ (because e is between $2\frac{1}{2}$ and 3).

The next ingredient is another saw-tooth graph $y = f_2(x)$ where

$$f_2(x) = \tfrac{1}{2}f_1(2x).$$

When $0 \leq 2x \leq \frac{1}{2}$, then $f_1(2x) = 2x$, so

$$f_2(x) = \tfrac{1}{2}f_1(x) = x \quad (0 \leq x \leq \tfrac{1}{4}).$$

Likewise $\frac{1}{2} < 2x < 1$ gives $f_1(2x) = 1 - 2x$, so

$$f_2(x) = \tfrac{1}{2}f_1(x) = \tfrac{1}{2} - x \quad (\tfrac{1}{4} < x < \tfrac{1}{2}).$$

For $x = \frac{1}{2} + a$, we have

$$f_2(\tfrac{1}{2} + a) = \tfrac{1}{2}f_1(1 + 2a) = \tfrac{1}{2}f_1(2a) = f_2(a).$$

Just as $f_1(x)$ repeats its values when x is increased by 1, so $f_2(x)$ repeats its values when x is increased by $\frac{1}{2}$, as shown in Fig. 3. An alternative

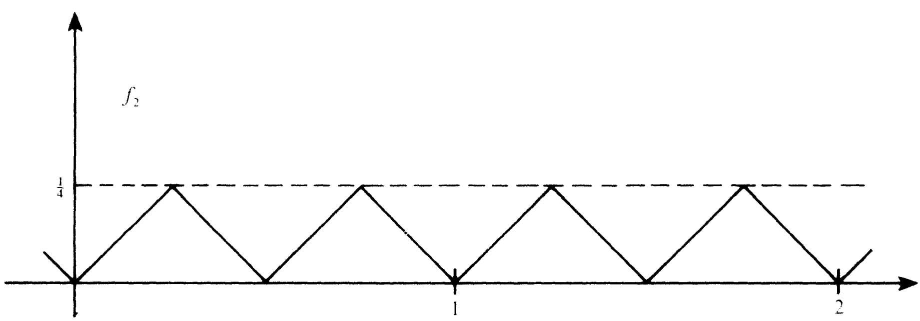

FIGURE 3.

way to see the same thing is to note that as x increases, the graph of $y=\frac{1}{2}f_1(2x)$ goes up and down twice as fast and to half the height of $y=f_1(x)$.

Now the idea is repeated with

$$f_3(x)=\tfrac{1}{4}f_1(4x)$$

and successively we draw

$$f_4(x)=\tfrac{1}{8}f_1(8x),\ldots,f_n(x)=(\tfrac{1}{2})^{n-1}f_1(2^{n-1}x).$$

At each stage the graph is half the size of the previous one (Fig. 4).

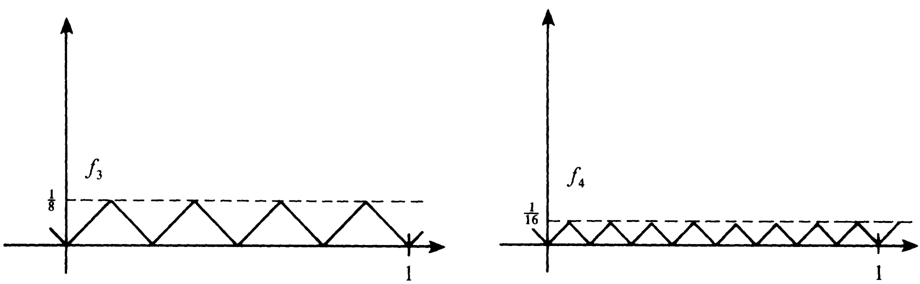

FIGURE 4.

To get the blancmange function we add these graphs together. It may be done in stages. First we let $b_1(x)=f_1(x)$, then we draw

$$b_2(x)=f_1(x)+f_2(x),$$
$$b_3(x)=f_1(x)+f_2(x)+f_3(x),$$

and, in general,

$$b_n(x)=f_1(x)+f_2(x)+\cdots+f_n(x)=b_{n-1}(x)+f_n(x).$$

The interesting thing is that after a few graphs, round about b_6, b_7 or b_8, depending on the scale, the new additions become so small, they don't significantly alter the status quo. The reader should draw the graphs to any appropriate scale. The physical act of drawing, as opposed to passively looking at the final static product, will indelibly imprint this fact in the memory (Fig. 5).

The *blancmange function* b is the limit of the sequence of b_n's:

$$b(x)=\lim_{n\to\infty} b_n(x) \quad \text{(for every real number } x\text{)}.$$

From the inequalities on $f_1(x), f_2(x), \ldots, f_n(x)$, we clearly have

$$0\le b_{n-1}(x)\le b_n(x)\le \tfrac{1}{2}+(\tfrac{1}{2})^2+\cdots+(\tfrac{1}{2})^n.$$

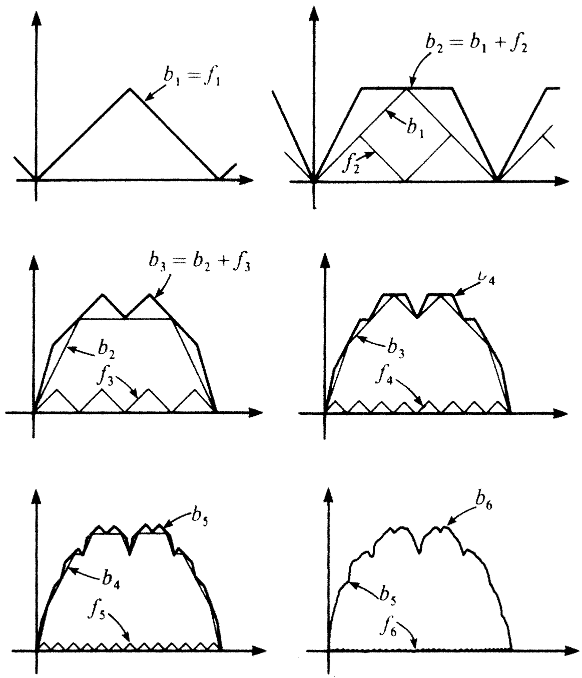

FIGURE 5.

But

$$\tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^n = 1 - (\tfrac{1}{2})^n,$$

so for all n we have

$$0 \le b_{n-1}(x) \le b_n(x) \le 1.$$

Thus for any fixed value of x, the sequence $b_1(x)$, $b_2(x)$, ..., $b_n(x)$, ... is increasing and bounded above by 1. It therefore tends to a limit $b(x)$ which is not more than 1.

For practical purposes we do not have to compute many terms to get a good approximation to $b(x)$. For instance, for $n > 20$ we have

$$\begin{aligned} 0 \le b_n(x) - b_{20}(x) = f_{21}(x) + \cdots + f_n(x) &\le (\tfrac{1}{2})^{21} + \cdots + (\tfrac{1}{2})^n \\ &= (\tfrac{1}{2})^{21}(1 - (\tfrac{1}{2})^{n-20})/(1 - \tfrac{1}{2}) \\ &< (\tfrac{1}{2})^{20} \\ &< 0{\cdot}000001, \end{aligned}$$

so

$$b_{20}(x) \le b_n(x) \le b_{20}(x) + 0{\cdot}000001 \quad (n > 20).$$

Allowing n to increase so that $b_n(x)$ tends to $b(x)$, we get

$$b_{20}(x) \le b(x) \le b_{20}(x) + 0{\cdot}000001.$$

Using 1 metre as a unit length and a fine drawing pen giving a line of width 0·1 millimetres, the graphs of the blancmange function b and of b_{20} are quite indistinguishable.

However, there is an important *theoretical* difference between b and b_{20}. The graph of b_{20} is made up of very short straight line segments but, as we shall see, the graph of the blancmange function wobbles all the time. This can be seen by imagining what happens were we to start adding the saw-teeth starting not at f_1 but at some later tooth, say

$$f_{n+1}(x) + f_{n+2}(x) + \cdots.$$

The saw-tooth f_{n+1} is a $(\frac{1}{2})^n$-scale version of f_1, then f_{n+2} is scaled down by a further factor $\frac{1}{2}$ and so on. Thus the above sum is just a $(\frac{1}{2})^n$-scale blancmange. To be precise, we have

$$\begin{aligned} & f_{n+1}(x) + \cdots + f_{n+r}(x) + \cdots \\ &= (\tfrac{1}{2})^n f_1(2^n x) + \cdots + (\tfrac{1}{2})^{n+r-1} f_1(2^{n+r-1}) + \cdots \\ &= (\tfrac{1}{2})^n f_1(2^n x) + \cdots + (\tfrac{1}{2})^{r-1} f_1(2^{r-1}(2^n x)) + \cdots \\ &= (\tfrac{1}{2})^n f_1(2^n x) + \cdots + f_r(2^n x) + \cdots \\ &= (\tfrac{1}{2})^n b(2^n x). \end{aligned}$$

This gives

$$\begin{aligned} b(x) &= f_1(x) + \cdots + f_n(x) + f_{n+1}(x) + \cdots + f_{n+r}(x) + \cdots \\ &= b_n(x) + (\tfrac{1}{2})^n b(2^n x). \end{aligned}$$

Thus the graph of the blancmange function $b(x)$ is obtained by adding together the graph of $b_n(x)$ and the $(\frac{1}{2})^n$-sized blancmange $(\frac{1}{2})^n b(2^n x)$. As an example, for $n = 1$, the identity

$$b(x) = b_1(x) + \tfrac{1}{2}b(2x)$$

tells us that if we add the graph of $y = b_1(x)$ to a half size blancmange $y = \frac{1}{2}b(2x)$, then we get the full-size blancmange $y = b(x)$ (Fig. 6).

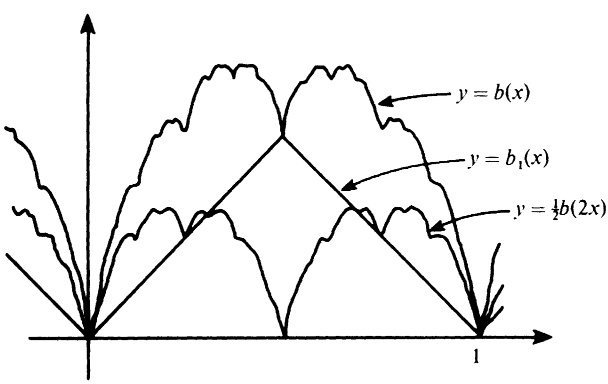

FIGURE 6.

More generally, if the graphs of $b(x)$ and $b_n(x)$ are drawn in the same picture, it will reveal the fact that the blancmange function has tiny sheared mini-blancmanges growing everywhere. For instance Fig. 7 shows the 1/16-size blancmanges growing on $b_4(x)$. Where b_4 has flat portions the mini-blancmanges are perfect scaled down versions, elsewhere they are sheared.

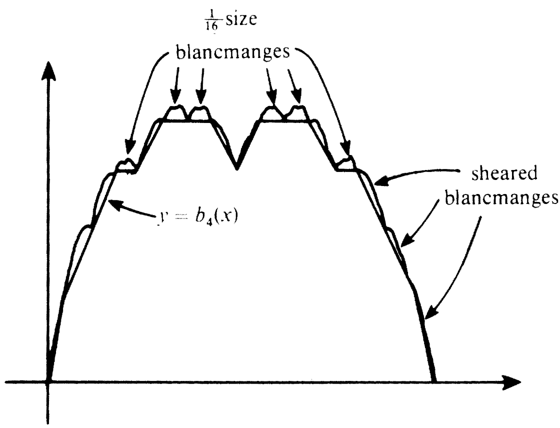

FIGURE 7.

Pictorial continuity: the blancmange function is continuous

A central property of a continuous function $f: D \to \mathbb{R}$ (where D is a subset of the real numbers $\mathbb{R}$) is that it can be drawn over any closed interval $[a, b]$ in D without taking the pencil off the paper. In short, a continuous function can be drawn 'continuously' in the colloquial sense. Regrettably, the experiences sixth-formers get of the notion of continuity are often at variance with the formal definition. Some believe that a 'continuous function' is one that 'has its graph in one piece', others that 'it is given by a single formula', and yet others erroneously link it with differentiation, for instance 'it has a smoothly turning tangent'.

A function $f: D \to \mathbb{R}$ is continuous at $a \in D$ if

$$\lim_{x \to a} f(x) = f(a).$$

In this sense the function $f(x) = 1/x$ is continuous at all points in

$$D = \{x \in \mathbb{R} : x \neq 0\}$$

because

$$\lim_{x \to a} 1/x = 1/a \quad (a \neq 0)$$

but the graph of $y = 1/x$ is not in one piece. The problem is that the domain D has a *gap* in it. We can only hope to draw the graph 'in one piece' over parts of the domain which have no gap, so we attempt to draw it over an interval. There is an additional hazard; if we try to draw the graph over an interval with one or more endpoints omitted, for instance

$$I = \{x \in \mathbb{R} : 0 < x \leq 1\},$$

then it may happen that the function is unbounded near the missing endpoint (as $1/x$ is unbounded near zero) and so the graph cannot be captured on a finite piece of paper.

With this in mind, let us define a function $f: D \to \mathbb{R}$ to be *pictorially continuous* if over any interval $[a, b]$ in D, given $k > 0$ we can find $c > 0$ such that $s, t \in [a, b]$ and $|s - t| < c$ implies $|f(s) - f(t)| < k$ (see Fig. 8).

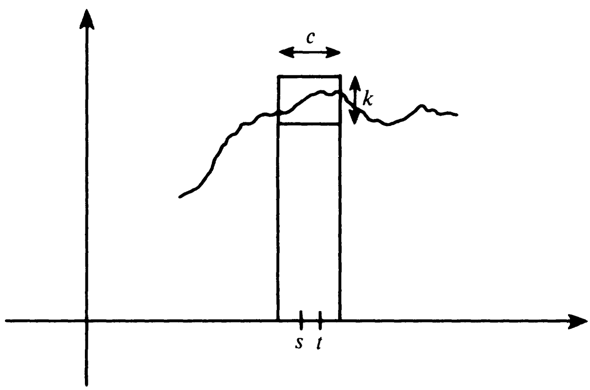

FIGURE 8.

Given $k > 0$, if s, t differ by less than the corresponding c, then the values $f(s), f(t)$ differ by less than k

(Provided that domains like $\mathbb{Q}$ are avoided, where inside points and outside points are inextricably mixed, this is equivalent to continuity.)

Suppose that we wish to draw a pictorially continuous function over an interval $[a, b]$. Given $k > 0$ cover the interval $[a, b]$ with successive small intervals $[a, a + d]$, $[a + d, a + 2d], \ldots$ where d is less than the c corresponding to k. Then the graph is captured in a succession of boxes width d, height k (Fig. 9(a)). For a given size pencil point we just choose k, d small enough to make the rectangle fit in the mark made on paper by the pencil. Then we move over the rectangles in succession to draw a pencil line which captures the graph inside it. Fig. 9(b) is a caricature of this process: 9(c) is more realistic for in practice a pencil line captures the very small rectangles within it.

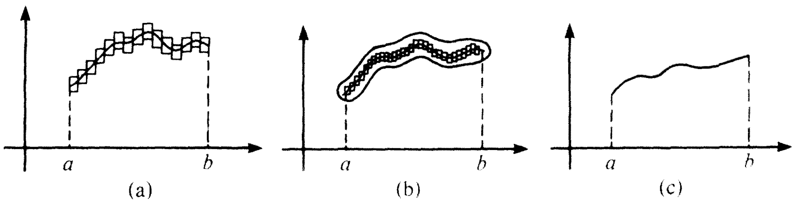

FIGURE 9.

We can now establish the (pictorial) continuity of the blancmange function. Let $k > 0$ be given. We shall find a $c > 0$ such that, throughout $\mathbb{R}$,

$$|s-t| < c \quad \text{implies} \quad |b(s)-b(t)| < k.$$

We do this in two stages. We have already seen that we can take n large enough to make $b(x)-b_n(x)$ as small as we like. So choose n such that $0 \leq b(x)-b_n(x) < k/2$ for all real x. Then, in particular, for any s, t in $\mathbb{R}$,

$$-\frac{k}{2} < (b(s)-b_n(s)) + (b_n(t)-b(t)) < \frac{k}{2}$$

Next, note that the graph of b_n is made up of straight line segments by adding n saw teeth together. But each saw-tooth has line segments each of gradient -1 or $+1$, so the gradients of the line segments of b_n lie between $-n$ and $+n$. If s, t happen to lie under the same segment of b_n we have

$$|b_n(s)-b_n(t)| \leq n|s-t|.$$

But if they lie under different segments then this inequality still holds (even more so!). So

$$|s-t| < \frac{k}{2n} \quad \text{implies} \quad -\frac{k}{2} < b_n(s)-b_n(t) < \frac{k}{2}.$$

Putting these facts together, we have found a c (namely $k/2n$) so that $|s-t| < c$ implies

$$|b(s)-b(t)| = |(b(s)-b_n(s)) + (b_n(s)-b_n(t)) + (b_n(t)) - b(t))| < k$$

as required: the continuity of b is established.

Differentiable functions: the blancmange function is differentiable nowhere

The difference between a merely continuous function and a differentiable function is easily seen by magnifying the graph and looking at it closely. If a differentiable function is highly magnified, its graph looks like a straight line. A continuous function which is not differentiable will not level out in the same way. For instance the graph of $y = x^2$ magnified near $x = 1$, $y = 1$ looks like a straight line of gradient 2 (as shown in Fig. 10).

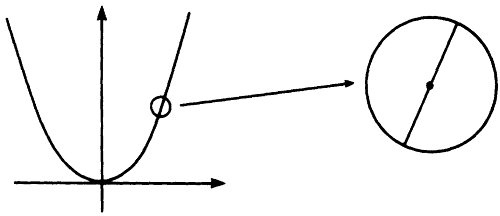

FIGURE 10.

In a practical sense, this may be done by plotting values over a small interval, say [0·999, 1·001], to a large scale of magnification, either by straightforward calculations, or by high resolution graphics on a micro-computer.

Theoretically, suppose that a function $f: D \to \mathbb{R}$ is differentiable at $a \in D$ and that the interval $[a-r, a+r]$ lies in D for some $r > 0$. Then

$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0$$

and so, given $k > 0$, there exists $c > 0$ such that

$$0 < |x - a| < c \quad \text{implies} \quad \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < k,$$

whence

$$|f(x) - f(a) - f'(a)(x - a)| < k|x - a|.$$

Now take any λ with $0 < \lambda < c$. Then for $0 < |x - a| \leq \lambda$, we have

$$|f(x) - f(a) - f'(a)(x - a)| < k\lambda.$$

By substitution, this inequality also holds for $x = a$; whence for $x \in [a - \lambda, a + \lambda]$, we have

$$-k\lambda < f(x) - f(a) - f'(a)(x - a) < k\lambda.$$

This may be rewritten as

$$f(a) + f'(a)(x - a) - k\lambda < f(x) < f(a) + f'(a)(x - a) + k\lambda,$$

which means that the graph of $y = f(x)$ lies between the two parallel straight lines

$$y = f(a) + (x - a)f'(a) - k\lambda \quad \text{and} \quad y = f(a) + (x - a)f'(a) + k\lambda$$

(which are themselves parallel to the tangent to $y = f(x)$ at the point $(a, f(a))$). These two lines are a *vertical* distance $2k\lambda$ apart (as in Fig. 11).

Suppose that we wish to draw a picture of the graph, scaling up the interval $[a - \lambda, a + \lambda]$ to occupy a width w; then we shall need to multiply

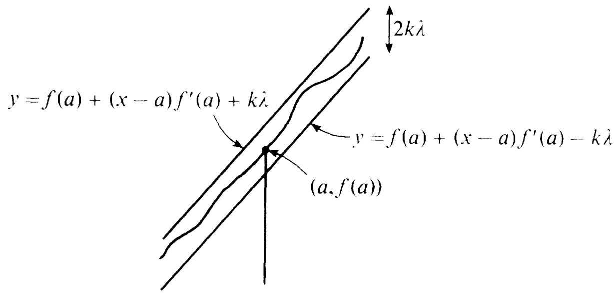

FIGURE 11.

by a scaling factor $w/2\lambda$. Scaling the whole picture by this factor, the vertical height between the two parallel lines is scaled to a length

$$2k\lambda . w/2\lambda = kw.$$

Now we can choose k to be *any* positive number we like and this then gives us an appropriate value of λ. For instance taking $k = h/w$ gives the following theorem:

Theorem Suppose that f is defined on a domain including an interval $[a-d, a+d]$ and f is differentiable at $x = a$. Then for any positive numbers w and h, there exists an interval $[a-\lambda, a+\lambda]$ such that the graph of $y = f(x)$ over this interval scaled up by a factor $w/2\lambda$ lies between two parallel straight lines of gradient $f'(a)$ which are a vertical distance h apart on the scaled drawing.

As a practical application, let us take $w = 20$ centimetres and $h = 0{\cdot}01$ centimetres, (these values being chosen to represent a decent paper-width w and the depth of the line drawn by a fine drawing pencil). Then there exists some $\lambda > 0$ such that the graph of $y = f(x)$ over the interval $[a-\lambda, a+\lambda]$ scales up to lie inside a fine straight pen-line of width $0{\cdot}01$ centimetres over an interval of width 20 centimetres (Fig. 12). The pen-line

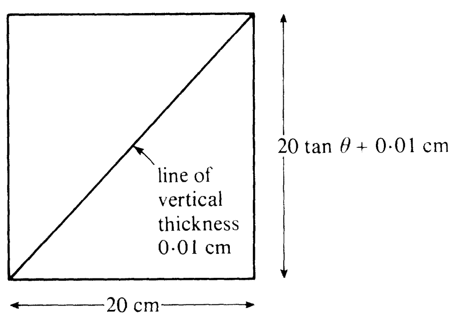

FIGURE 12.

has gradient $\tan\theta = f'(a)$ and the height of the picture is $20\tan\theta + 0{\cdot}01$ centimetres. (If the graph is steep this may be quite high!)

This reasoning may be used to show that the blancmange function does not have a derivative anywhere. If it happened that it *were* differentiable at $x = a$, say, then over some interval $[a - \lambda, a + \lambda]$, the graph scaled to a horizontal width 20 centimetres lies between two parallel lines which are a vertical distance $0{\cdot}01$ centimetres apart.

Choose integers m, n $(n \geq 1)$ such that $[m/2^n, (m + 1)/2^n]$ is the largest such interval lying inside $[a - \lambda, a + \lambda]$, which means that

$$\frac{m-1}{2^n} < a - \lambda \leq \frac{m}{2^n} < \frac{m+1}{2^n} \leq a + \lambda < \frac{m+2}{2^n}.$$

Then

$$2\lambda = (a + \lambda) - (a - \lambda) < \frac{m+2}{2^n} - \frac{m-1}{2^n} = \frac{3}{2^n}.$$

which gives $1/2^n > 2\lambda/3$. This means that when $[a - \lambda, a + \lambda]$ is scaled up to a length 20 centimetres, then $[m/2^n, (m + 1)/2^n]$ is scaled up to a length of at least $20/3 = 6\frac{2}{3}$ centimetres. But, from the first section, the graph of the blancmange function over the interval $[m/2^n, (m + 1)/2^n]$ is a sheared mini-blancmange. Since the blancmange function on $[0, 1]$ rises to a height of more than $\frac{1}{2}$, then scaled up to a base length of more than $6\frac{2}{3}$ centimetres, a blancmange rises to a height of more than $\frac{1}{2} \times 6\frac{2}{3} = 3\frac{1}{3}$ centimetres. When sheared, there is no way that it can lie between two parallel lines of vertical distance $0{\cdot}01$ centimetres apart! The reason why the blancmange function is differentiable nowhere now becomes manifestly obvious—it wobbles too much!

Consequences

We have seen that the ideas of continuity and differentiability can be given obvious pictorial interpretations: a continuous function can be drawn without taking the pencil off the paper over any closed interval in the domain, and a differentiable function when highly magnified looks very much like a straight line. These ideas are quite familiar to practical mathematicians, physicists and engineers, though the notion of continuity may be encrusted with other personal interpretations that cloud its true meaning. Many formalistic mathematicians, on the other hand, deny these very helpful picture images. Some would find it difficult to imagine an everywhere continuous, nowhere differentiable function; yet the blancmange function has a recipe for drawing that gives a very clear idea of why it has these seemingly unusual properties.

The fact is that pictures, *correctly interpreted*, can play a very important role in giving insight to ideas in mathematical analysis. It is salutary to note that, by using pictorial ideas, it is possible to show that the blancmange function is nowhere differentiable, a fact that is considered 'too difficult' to explain in most undergraduate mathematics courses. What is more important is that these practical ideas translate into a correct formal proof, now invested with geometric insight sadly lacking in so much formal mathematics.

"Intuition" is not a low-level phenomenon to be excluded from higher mathematics, it is a highly personal mental activity produced by experience. If we give the right experiences and enhance intuition then it can result in a much more profound understanding.

DAVID TALL

Mathematics Education Research Centre, University of Warwick, Coventry CV4 7AL

Sums of powers of integers: a little of the history

A. W. F. EDWARDS

The lack of any obvious pattern amongst the Bernoulli numbers $(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, \dots)$ is one of the shocks of analysis which subsequent familiarity with the many beautiful and simple means of deriving them does not altogether assuage. Historically, they first arose in connection with the sums of the rth powers of the first n integers

$$\sum_{i=1}^{n} i^r = 1^r + 2^r + 3^r + \cdots + n^r \tag{1}$$

which it is convenient to write as $\sum n^r$. The Greeks, Hindus, and Arabs all had rules amounting to

$$\left.\begin{aligned} \textstyle\sum n &= \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n^2 + \tfrac{1}{2}n \\ \textstyle\sum n^2 &= \tfrac{1}{6}n(n+1)(2n+1) = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n \\ \textstyle\sum n^3 &= [\tfrac{1}{2}n(n+1)]^2 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 \end{aligned}\right\} \tag{2}$$

whilst a fifteenth-century Arab rule for the fourth powers was equivalent to

$$\begin{aligned} \textstyle\sum n^4 &= \tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \\ &= \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n, \end{aligned} \tag{3}$$

there being no n^2 term in the second form.