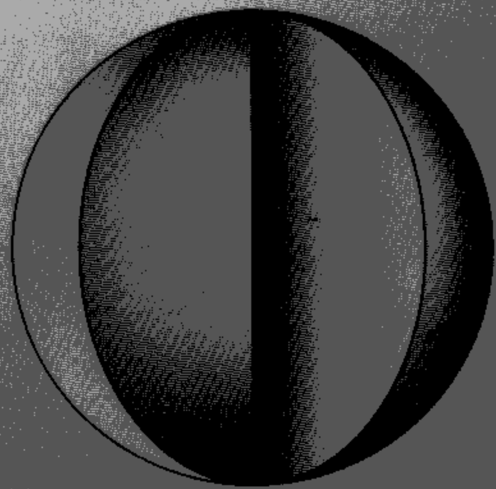
ELEMENTARY MATHEMATICS
from an Advanced Standpoint

ARITHMETIC
ALGEBRA
ANALYSIS



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converge toward the single-valued function e^w, which is spread over the smooth sphere, while the sequences of values on the other sheets have, in general, no limit whatever. We thus have a complete explanation of the right complicated and wonderful passage to the limit from the many valued power to the single-valued exponential function.

As a general moral of these last considerations we might say that a complete understanding of such problems is possible only when they are taken into the field of complex numbers. Is this, then, not a sufficient reason for teaching complex function theory in the schools? Max Simon, for one, has in fact supported similar demands. I hardly believe, however, that the average pupils, even in the highest class, can be carried so far, and I think, therefore, that we should abandon those aspects of method as to algebraic analysis in the schools which incline toward such considerations, in favor of the simple and natural way which we have developed above. I am, to be sure, all the more desirous that the teacher shall be in full possession of all the function-theoretic connections that come up here; for the teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them.

After these detailed discussions we can now be briefer in the corresponding consideration of the goniometric functions.

II. The Goniometric Functions

Let me say, before beginning, that the name *goniometric functions* seems preferable to the customary name *trigonometric functions*, since trigonometry is but a particular application of these functions, which are of the greatest importance for mathematics as a whole. Their inverse functions are analogous to the logarithm, while they themselves are analogous to the exponential function. We shall call these inverse functions the *cyclometric functions*.

1. Theory of the Goniometric Functions

As a starting point for our theoretical considerations let me suggest the question as to the most appropriate way of introducing the goniometric functions in the schools. I think that here also it would be best to make use of our general principle of quadrature. The customary procedure, which begins with the measurement of the circular arc, does not seem to me to be so very obvious, and it lacks, above all, the advantage of affording a simple and coherent control both of elementary and advanced fields.

Again I shall make immediate use of analytic geometry. Let us start with the unit circle

$$x^2 + y^2 = 1$$

and consider the sector formed by the radii to the points A ($x = 1$, $y = 0$) and P (x, y) (see Fig. 63). In order to be in agreement with the usual notation, I shall denote the area of this sector by $\varphi/2$. (Then the arc in the customary notation will be φ.)

I shall define the goniometric functions sine and cosine of φ as the lengths of the coordinates x and y of the limiting point P of the sector $\varphi/2$:

$$x = \cos\varphi, \qquad y = \sin\varphi.$$

The origin of this notation is not clear. The word "sinus" probably arose through an erroneous translation of an Arabic word into Latin. Since we did not start from the arc we cannot well designate the inverse functions, i. e., the double sector, as, a function of the coordinates, by using the customary terms arc sine and arc cosine, but it is natural by analogy to call $\varphi/2$ the "area" of the sine (or cosine) and to write

$$\varphi = 2 \text{ area } \sin y = \text{arc } \sin y,$$
$$\varphi = 2 \text{ area } \cos x = \text{arc } \cos x.$$

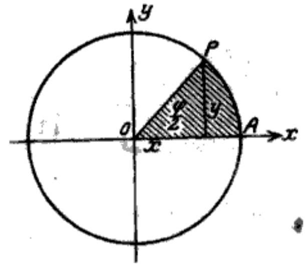

Fig. 63.

The following notation, used in England and in America is also quite appropriate:

$$\varphi = \cos^{-1} x, \qquad \varphi = \sin^{-1} y.$$

The further goniometric functions:

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi}, \qquad \text{ctn}\,\varphi = \frac{\cos\varphi}{\sin\varphi}$$

(in the older trigonometry also secant and cosecant) are defined as simple rational combinations of the two fundamental functions. They are introduced only with a view to brevity in practical calculation and have for us no theoretical significance.

If we follow the coordinates of P with increasing φ we can at once obtain qualitatively a representation of the cosine and sine curves in a rectangular coordinate system. They are the well known wave lines with a certain period 2π (see Fig. 64), where π is defined as the area of the entire unit circle, instead of as usual, the length of the semi-circle.

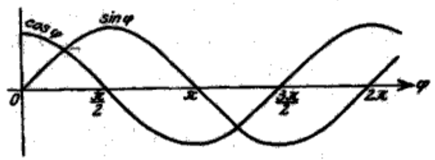

Fig. 64.

Let us now compare once more our introduction of the logarithm and the exponential function with these definitions. You will recall that our point of departure was a rectangular hyperbola referred to its asymptotes as axes.

$$\xi \cdot \eta = 1 .$$

The semi axis of this hyperbola is $OA = \sqrt{2}$ (see Fig. 65), whereas the circle had the radius 1. Let us now consider the area of the strip between the fixed ordinate AA' ($\xi = 1$) and the variable ordinate PP'. If this is called Φ, we may put $\Phi = \log \xi$, and the coordinates of P are expressed in terms of Φ in the form

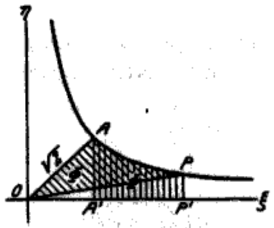

Fig. 65.

$$\xi = e^{\Phi}, \qquad \eta = e^{-\Phi}.$$

You notice a certain analogy with the preceding discussion, but that the analogy fails in two respects. In the first place, Φ is not a sector as it was before, and furthermore the two coordinates are now expressed rationally in terms of one function e^{Φ}, whereas, in the case of the circle, we had to introduce two functions, sine and cosine, to secure rational expressions. We shall see however that this divergence can be easily resolved.

Notice, in the first place, that the area of the triangle $OP'P$, namely $\frac{1}{2}\xi \cdot \eta = \frac{1}{2}$, is independent of the position of P. In particular, then, it is the same as that of $OA'A$. Therefore, if we add the latter triangle to Φ and then subtract the former triangle from this sum, we see that Φ can be defined as the area of a hyperbolic sector lying between a radius vector to the vertex A and one to a variable point P, just as in the case of the circle. There is still a difference in sign. Before, the arc AP, looked at from O, was counterclockwise, whereas now it is clockwise. We can remove this difference by reflecting the hyperbola in OA, i.e., by interchanging ξ and η. We get then as coordinates of P

$$\xi = e^{-\Phi}, \qquad \eta = e^{\Phi}.$$

Finally let us introduce the principal axes in place of the asymptotes as axes of reference, by turning Fig. 65 through 45° (after reflection in OA). If we call the new coordinates (X, Y), the equations of this transformation are

$$X = \frac{\xi + \eta}{\sqrt{2}}, \qquad Y = \frac{-\xi + \eta}{\sqrt{2}}.$$

The equation of the hyperbola then becomes

$$X^2 - Y^2 = 2,$$

and the sector Φ now has precisely the same position that sector $\Phi/2$ had in the circle. The new coordinates of P as functions of Φ may be written in the form

$$X = \frac{e^{\Phi} + e^{-\Phi}}{\sqrt{2}}, \qquad Y = \frac{e^{\Phi} - e^{-\Phi}}{\sqrt{2}}.$$

It remains only to reduce the entire figure in the ratio $1 : \sqrt{2}$ in order to make the semi axis of the hyperbola 1 instead of the $\sqrt{2}$, as it was in the case of the circle. Then the sector in question has the area $\varphi/2$, in complete accord with the preceding. If we call the new coordinates (x, y) again, they will be the following functions of Φ

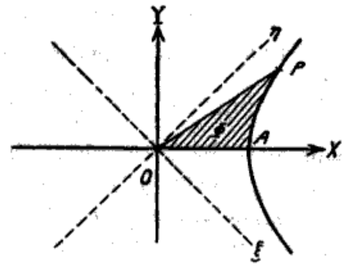

Fig. 66.

$$x = \frac{e^{\Phi} + e^{-\Phi}}{2}, \qquad y = \frac{e^{\Phi} - e^{-\Phi}}{2},$$

which satisfy the relation

$$x^2 - y^2 = 1,$$

which is the equation of a hyperbola. These functions are called hyperbolic cosine and sine and are written in the form

$$x = \cosh \Phi = \frac{e^{\Phi} + e^{-\Phi}}{2}, \qquad y = \sinh \Phi = \frac{e^{\Phi} - e^{\Phi}}{2}.$$

The final result, then, is that if we treat the circle and the rectangular hyperbola, each with semiaxis one, in literally the same way we obtain on the one hand the ordinary goniometric functions, on the other the hyperbolic functions, so that these functions correspond fully to one another.

You know that these functions cosh and sinh can be used to advantage in many cases. Nevertheless we have really taken a step backward here, so far as the treatment of the hyperbola is concerned. Whereas at first, the coordinates (ξ, η) could be rationally expressed in terms of a single function e^{Φ}, it now requires two functions, which are connected by an algebraic relation (the equation of the hyperbola). It is natural, therefore to attempt a converse treatment for the goniometric functions, analogous to the original developments for the hyperbola. This is, in fact, quite easy if one does not object to the use of complex quantities, and it leads to the setting up of a single fundamental function in terms of which $\cos \varphi$ and $\sin \varphi$ can be expressed rationally, just as $\cosh \Phi$ and $\sinh \Phi$ are in terms of e^{Φ}, and which is therefore entitled to play the chief rôle in the theory of the goniometric functions.

To this end we introduce into the equation of the circle $x^2 + y^2 = 1$ (where $x = \cos \varphi$, $y = \sin \varphi$) the new coordinates

$$x - iy = \xi, \qquad x + iy = \eta,$$

which gives

$$\xi \cdot \eta = 1.$$