Chapter 2
Special Sequences

This chapter is devoted to a study of some special sequences that arise often in analysis. We begin with two familiar sequences which converge to e, the base of natural logarithms. As a byproduct of the analysis, the number e is found to be irrational. This prompts an elementary proof that π is irrational as well. Further topics include Euler's constant γ and the infinite product formulas of Vieta and Wallis. The chapter concludes with a crown jewel of classical analysis, Stirling's approximation to the factorial function.

2.1. The number e

The number $e = 2.71828182845904\ldots$, known as the base of natural logarithms, arises through the limit

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and is found to have the form

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Our aim is to reconcile the two expressions and to show in particular that the first limit exists. We will also show that e is an irrational number.

First of all, let us prove that the infinite series converges, so that we can legitimately define the number e as its sum. Let

$$s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

be the nth partial sum, and observe that $s_n < s_{n+1}$, since all terms are positive. On the other hand, the inequality $k! \geq 2^{k-1}$ shows that

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3$$

for all n. Thus the sequence $\{s_n\}$ is monotonic and bounded, and is therefore convergent. We denote its limit, or the sum of the infinite series, by e.

Now consider the expressions

$$x_n = \left(1 + \frac{1}{n}\right)^n , \qquad n = 1, 2, 3, \ldots .$$

In order to study the behavior of the sequence $\{x_n\}$, we can use a hand calculator to compute

$$x_1 = 2\,,\ x_2 = 2.25\,,\ x_3 = 2.370\ldots\,,\ x_4 = 2.441\ldots\,,\ x_5 = 2.488\ldots\,, \text{etc.}$$

The data suggest that $\{x_n\}$ is an increasing sequence. Our next step will be to show that this is actually true.

According to the binomial theorem, we can write

$$\begin{aligned} x_n = \left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\ &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 \\ &\quad + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}\left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ &\quad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) . \end{aligned}$$

A similar expansion of x_{n+1} gives

$$x_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{1}{n+1}\right)^k .$$

Observe now that for each fixed k in the range $2 \leq k \leq n$, the term

$$\begin{aligned} \binom{n+1}{k}\left(\frac{1}{n+1}\right)^k &= \frac{1}{k!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right) \\ &\geq \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) = \binom{n}{k}\left(\frac{1}{n}\right)^k . \end{aligned}$$

Furthermore, the expansion for x_{n+1} contains an extra term $\left(\frac{1}{n+1}\right)^{n+1} > 0$, corresponding to $k = n+1$. These two inequalities combine to show that $x_{n+1} > x_n$ for $n = 1, 2, 3, \ldots$.

The above expansion for x_n also shows that

$$x_n \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = s_n < 3\,.$$

Therefore, $x_1 < x_2 < x_3 < \cdots < 3$, and so the sequence $\{x_n\}$ is monotonic and bounded, hence convergent. We are going to prove that

$$\lim_{n\to\infty} x_n = e\,,$$

the limit of the sequence $\{s_n\}$. Since $x_n \leq s_n$ for all n, it follows that

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} s_n = e\,.$$

On the other hand, for each fixed $m \leq n$ the sum in the previous calculation can be truncated to give

$$x_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

Now hold m fixed and let $n \to \infty$ to infer that

$$\lim_{n\to\infty} x_n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} = s_m$$

for each index m. Letting $m \to \infty$, we see from this that

$$\lim_{n\to\infty} x_n \geq \lim_{m\to\infty} s_m = e\,.$$

Combining this with the earlier inequality $\lim_{n\to\infty} x_n \leq e$, we conclude that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} s_n = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\,.$$

The infinite series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges quite rapidly. For instance $s_7 = 2.71825396\ldots$ already gives the correct value of e to four decimal places. To estimate the rate of convergence, we can write

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+1} + \left(\frac{1}{n+1}\right)^2 + \cdots\right\} = \frac{1}{n!\,n}, \end{aligned}$$

after summing the geometric series. Therefore,

$$0 < e - s_n < \frac{1}{n!\,n} \qquad \text{for} \quad n = 1, 2, 3, \ldots .$$

It is now easy to show that e is irrational. Suppose on the contrary that e is rational, so that $e = \frac{m}{n}$ for some positive integers m and n. By the estimate just given,

$$0 < n!(e - s_n) < \frac{1}{n} \le 1 .$$

But

$$n!s_n = n!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)$$

is an integer, and $n!e$ is also an integer under the assumption that $e = \frac{m}{n}$. Hence that assumption has led to the conclusion that $n!(e - s_n)$ is an integer between 0 and 1, which is impossible. The contradiction shows that e is irrational.

In fact, it is known that e is a transcendental number. Recall that a real number x is said to be *algebraic* if it satisfies some polynomial equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

with integer coefficients $a_0, a_1, \ldots, a_n$. All rational numbers are algebraic, and so are many irrational numbers such as $\sqrt{2}$. A real number that is not algebraic is said to be *transcendental.* Both e and π are known to be transcendental, but those assertions are not easy to prove. The transcendence of e was proved by Charles Hermite [**2**] in 1873. Then in 1882 Ferdinand von Lindemann [**4**] adapted Hermite's method to establish the transcendence of π. A simpler version of the Hermite–Lindemann proof can be found in the book by Ivan Niven [**5**]. In the next section we present Niven's proof of the more elementary fact that π is irrational.

2.2. Irrationality of π

We now digress from the theme of this chapter to prove that the number π is irrational. This fact lies intrinsically deeper than the irrationality of e, and was proved by more sophisticated methods before Ivan Niven [**5**] found the remarkably elementary proof that will be presented here. In fact, the proof yields the stronger result that π^2 is irrational.

Consider the polynomial

$$f(x) = \frac{1}{n!}\, x^n (1-x)^n\,,$$

where n is a positive integer to be specified later. This function satisfies

$$0 < f(x) < \frac{1}{n!} \qquad \text{for} \ \ 0 < x < 1\,.$$

It is easy to see that each of the derivatives $f^{(k)}(0)$ is an integer. Indeed, $f^{(k)}(0) = 0$ for $0 \le k < n$, and for $k \ge n$ a calculation shows that the kth derivative of $x^n(1-x)^n$ at the origin is an integer divisible by $n!$. (This remains true if the factor $(1-x)^n$ is replaced by any other polynomial with integer coefficients.) By the symmetry relation $f(1-x) = f(x)$, it follows that every derivative $f^{(k)}(1)$ is also an integer. Observe finally that $f^{(k)}(x) \equiv 0$ for all $k > 2n$, since f is a polynomial of degree $2n$.

Now suppose, for purpose of contradiction, that π^2 is rational, so that $\pi^2 = p/q$ for some positive integers p and q. Define the polynomial

$$g(x) = q^n \left[\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)\right]\,,$$

and note that both $g(0)$ and $g(1)$ are integers under the supposition that $\pi^2 = p/q$. Because of the structure of g, we see that

$$g''(x) + \pi^2 g(x) = q^n \pi^{2n+2} f(x)\,.$$

In view of this relation, a simple calculation gives

$$\begin{aligned} \frac{d}{dx}\{g'(x)\sin \pi x - \pi g(x) \cos \pi x\} &= [g''(x) + \pi^2 g(x)] \sin \pi x \\ &= q^n \pi^{2n+2} f(x) \sin \pi x = \pi^2 p^n f(x) \sin \pi x\,. \end{aligned}$$

Consequently,

$$\pi p^n \int_0^1 f(x) \sin \pi x \, dx = \left[\frac{1}{\pi}\, g'(x) \sin \pi x - g(x) \cos \pi x\right]_0^1 = g(1) + g(0)\,,$$

which is an integer. On the other hand, since $0 < f(x) < 1/n!$ for $0 < x < 1$, we find that

$$0 < \pi p^n \int_0^1 f(x) \sin \pi x \, dx < \frac{\pi p^n}{n!} < 1$$

if n is chosen sufficiently large. But this is impossible, because there is no integer between 0 and 1. Thus the assumption that π^2 is rational has led to a contradiction, and so we conclude that π^2 is irrational, which implies that π is irrational.

The transcendence of π has an interesting application to classical geometry. It settles once and for all the ancient problem of "squaring the circle" with straight edge and compass. Given an arbitrary circle, the problem is to construct a square of the same area. Since the circle has area πr^2, this amounts to starting with a line segment of unit length and constructing a segment of length $\sqrt{\pi}$. A segment of length π could then be constructed, since from any segment of length ℓ it is possible to construct a segment of length ℓ^2. But it can be shown that the length of every segment constructible with straight edge and compass, starting with a segment of unit length, is an algebraic number. (A good reference for this fact is the book by Courant and Robbins [**1**].) Therefore, if it were possible to square the circle with straight edge and compass, the number π would have to be algebraic. But π is transcendental, so it is impossible to square the circle.

2.3. Euler's constant

Euler's constant is

$$\gamma = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} - \log n \right\} .$$

It is named for Leonhard Euler, who first discussed it in 1734. The number γ is an important constant that occurs frequently in mathematical formulas. The existence of the limit is not obvious. Our aim is to prove that the limit exists and to determine its approximate numerical value.

Consider the curve $y = 1/x$ for $1 \le x \le n$, where $n = 2, 3, \ldots$. The area under the curve is given by

$$A_n = \int_1^n \frac{1}{x} \, dx = \log n .$$

Now construct rectangular boxes of heights $1/k$ over the intervals $[k, k+1]$, as shown in Figure 1.

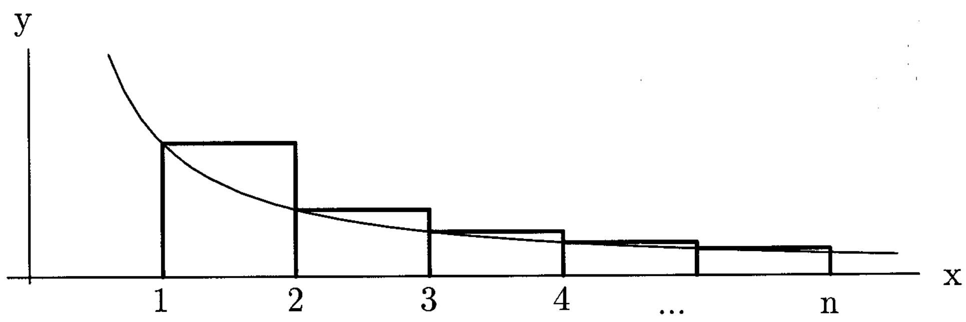

Figure 1. The curve $y = 1/x$ and rectangular boxes.

Since

$$\frac{1}{k+1} \le \frac{1}{x} \le \frac{1}{k} \qquad \text{for } \ k \le x \le k+1,$$

it follows that

$$\frac{1}{k+1} = \int_k^{k+1} \frac{1}{k+1}\,dx \le \int_k^{k+1} \frac{1}{x}\,dx \le \int_k^{k+1} \frac{1}{k}\,dx = \frac{1}{k}$$

for $k = 1, 2, \ldots$. Adding these inequalities over $k = 1, 2, \ldots, n-1$, we have

$$\sum_{k=1}^{n-1} \frac{1}{k+1} \le \int_1^n \frac{1}{x}\,dx \le \sum_{k=1}^{n-1} \frac{1}{k}\,.$$

With the notation

$$S_n = \sum_{k=1}^{n} \frac{1}{k}\,,$$

this says that $S_n - 1 \le A_n \le S_{n-1}$. The two inequalities can be rearranged to give

$$0 \le S_{n-1} - A_n \le 1 - S_n + S_{n-1} = 1 - \frac{1}{n}\,.$$

This shows that the sequence $\{S_{n-1} - A_n\}$ is positive and is bounded above by 1.

Geometrically, the quantity $S_{n-1} - A_n$ is the sum of areas of those portions of the boxes that lie above the curve $y = 1/x$ from $x = 1$ to n. In order to estimate this total area, imagine that all of these boxes are slid to the left until they lie inside the first box, as shown in Figure 2, where the shaded regions have total area $S_{n-1} - A_n$. Since the regions are nonoverlapping and lie inside a square of area 1, this conceptual exercise gives a geometric interpretation of the inequality $S_{n-1} - A_n \le 1$.