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Concept definition, concept image and the notion of function

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A simple model for cognitive processes will be constructed using the notions of concept image and concept definition. The model will be used to analyse some phenomena in the process of the learning of the function concept in grades 10 and 11. Educational conclusions, similar to those in [3] which were based on historical arguments, will be drawn.

1. Mental pictures and images of concepts

We would like to explain first what we mean by the mental picture of a concept. Let C denote a concept and let P denote a certain person. Then P's mental picture of C is the set of all pictures that have ever been associated with C in P's mind. (This definition was first used in [6].)

The word 'pictures' here is used in the broadest sense of the word and it includes any visual representation of the concept (even symbols). Thus a graph of a specific function and the symbols '$y=f(x)$' might be included (together with many other things) in someone's mental picture of the concept of function.

Besides the mental picture of a concept there might be a set of properties associated with the concept (in the mind of our person P). For instance, somebody might think that an altitude in a triangle should always fall inside the triangle. He might think that functions should always be defined by means of algebraic expressions. This set of properties together with the mental picture will be called by us the concept image. It is clear that as defined here the image of a concept depends upon the person about whom we are speaking. However, in the following discussions we will not mention it for the sake of convenience. It will be clear from the context.

2. Concept definition and concept formation

By 'concept definition' we mean here a verbal definition that accurately explains the concept in a non-circular way. For some of our concepts we also have concept definition in addition to the concept image. For many other concepts we do not. For instance, we do not have a definition for 'house', 'orange' etc., although we have very clear concept images for them. They were acquired when we were children probably by means of 'ostensive definitions'. It is true that some concepts, such as these, could be introduced to us by means of verbal definitions. The word 'forest' could be introduced to us by saying: 'many many trees together are a forest'. We were supposed then to visualize many trees together and thus to form a concept image. However we claim that (1) in order to handle concepts one needs a concept image and not a concept definition, (2) concept definitions (where the concept was introduced by means of a definition) will remain inactive or even will be forgotten. In thinking, almost always the concept image will be evoked.

This is true mainly about informal learning of concepts. In formal learning the situation might be different. Here the concept definition becomes a part of the game. (In [1, p. 181] it is claimed that 'in England the tradition is to rely on the formal definition in higher education and informal or ostensive definitions in primary or secondary schools. However, considering some textbooks, it is not clear whether this is a tendency or a practice'.) Thus, for concepts such as 'chemical reaction', 'coordinate system' and 'equilateral triangle' we do have definitions. These definitions are either taught to us or they are made up by us when we are asked to explain the concepts to somebody. The definitions made up by us are a result of our experience with the concept. They are a description of our concept image. The definitions which we are taught, on the other hand, are part of a general system (in the case of scientific or mathematical concepts), a system with which we are not necessarily familiar. Sometimes definitions are introduced to us before we have any concept image. We expect further learning to fill this gap. One can argue whether this is an effective way. This question is discussed in [4, Ch. 2]. However, if one does not require definitions for the sake of definitions, the reason for requiring them is the belief that definitions help to form the concept images and also that they are useful in carrying out some cognitive tasks.

3. Concept image and concept definition—a model

For each concept, assume the existence of two different cells in the cognitive structure (to avoid confusion, we do not mean biological cells). One cell is for the definition(s) of the concept and the second one is for the concept image. One cell or even both of them might be void. There might be an interaction between the two cells although they can be formed independently. A child might have a concept image for the notion of coordinate system as a result of seeing many graphs in various situations. According to this concept image the two axes of a coordinate system are perpendicular to each other. Later on the child's Math teacher might define a coordinate system as any two intersecting straight lines. As a result of this, three scenarios might occur: (I) The concept image will be changed to include also coordinate systems the axes of which do not form a right angle. (II) The concept image cell will remain as it is. The definition cell will contain the teacher's definition for a while but this definition will be forgotten or distorted after a while and when the child is asked to define a coordinate system he will talk about axes forming a right angle. (III) Both cells will remain as they are. The moment the child is asked to define a coordinate system he will repeat his teacher's definition, but in all other situations he will think of a coordinate system as having two perpendicular axes.

A similar process might occur when a concept is first introduced by means of a definition. Here the concept image cell is empty in the beginning. After several examples and explanations it is gradually filled. However, it does not necessarily reflect all the aspects of the concept definition. Similar scenarios to (I)–(III) above might occur at this stage too.

This is shown in figure 1.

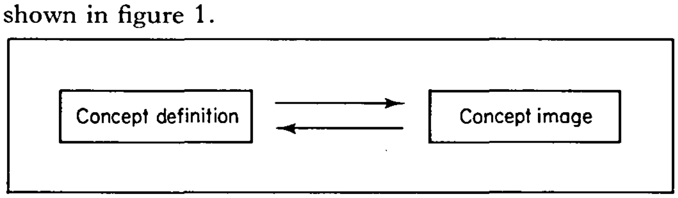


Figure 1.

This diagram refers to the stages of concept formation. It seems to us that many teachers at the secondary and the collegial levels expect at a certain stage of the concept formation a one way process as shown in figure 2.

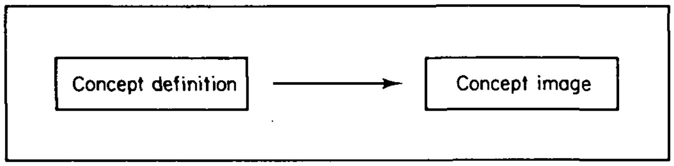


Figure 2.

Namely, the concept image is formed by means of the concept definition and under its control. We, however, consider this as wishful thinking.

In addition to the stages of the concept formation there is also the stage of performance. At this stage a cognitive task is given and the concept image and the concept definition cells are supposed to be activated. In this paper we will be concerned only with identification or construction tasks. Again, our impression is that the model, implicitly assumed by many teachers, is one of those described in figures 3 to 5.

Again, we believe that figures 3 to 5 do not reflect the practice. There is no way to force a cognitive structure to use definitions, either in order to form concept images or in order to handle a cognitive task. Some definitions are too complicated to deal with. They do not help in creating concept images in the students' mind. Hence, they

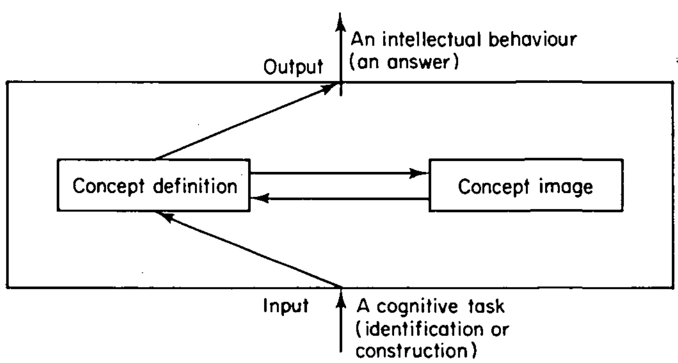


Figure 3.

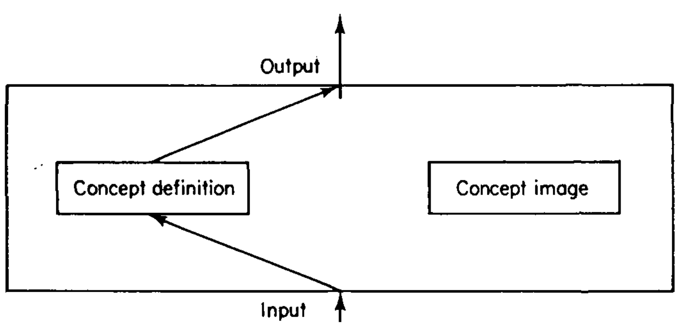


Figure 4.

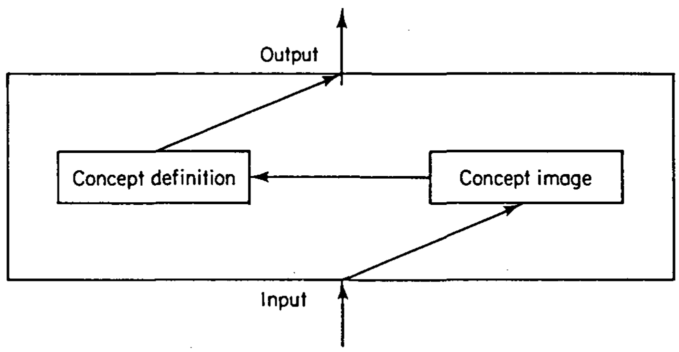
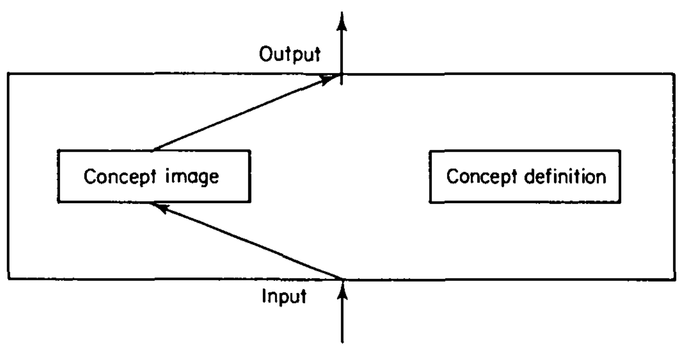


Figure 5.

are useless. On the other hand, there are some definitions that do make sense but the moment some specific examples are given by the teacher or by the book, they form the concept image and because of (1) and (2) of the previous section the definitions might become inactive or even be forgotten. Thus, the model for the above cognitive task is more similar to the following:



4. Revealing the concept image and teaching

We would like to make our point in this section by an example from geometry. Sometimes we test students in order to see whether they have the knowledge of some very simple geometrical concepts such as right angle, isosceles triangle, an altitude in a triangle and so forth. We do it by means of a multiple choice questionnaire where they are asked to identify the concepts or to construct specific examples. Very often we discover that our students do not know the above concepts and they have wrong concept images. These images might be a result of the specific set of examples given to the students. Probably there is an (implicit) assumption that the students are supposed to use the concept definitions when a cognitive task is imposed on them and therefore there is no need to give them numerous different examples, but, as we claimed before, such an assumption has no ground.

Here are three examples.

(1) In a textbook (in Hebrew) for the second grade the author defines an isosceles triangle as a triangle which has two equal sides. All the isosceles triangles which are drawn in the text have a horizontal basis. Let us now draw a set of triangles some of which are isosceles triangles but only one of these has a

horizontal basis. Will it be a surprise if only this triangle is identified as an isosceles triangle by our students?

(2) In a geometry textbook (*Geometry with Coordinates*, SMSG, part I, pp. 143–144) an angle is considered as 'the union of two concurrent rays'. A straight angle is 'the union of two opposite rays'. There is only one drawing of a straight angle in the book, of course, a horizontal one. Again, will it be a surprise if a student will identify only straight angles which are horizontal?

(3) In another book by SMSG (*Mathematics for Junior High School*, Volume 2, part I, pp. 194–195) right triangles are discussed. In all of them one side of the right angle is horizontal (there are ten right triangles altogether). It seems unnecessary to ask our question once again.

Thus, revealing the concept images of our students becomes very important for teaching; not only might it give us a better understanding of our students (knowing what caused them to act as they acted) but also it might suggest some improvements to our teaching which formed such wrong concept images.

5. The temporary concept image

There is an important point to be made here. In a specific intellectual task sometimes only parts of the concept image cell or concept definition cell are actually activated. Hence, the concept image (or the concept definition) cannot be determined by a single observation of a specific behaviour. We, therefore, must speak of that part of the cell that was activated when working on a given task. Hence, we actually deal with the concept image (or the concept definition) cell at a certain moment. We can call it the temporary concept image (or concept definition).

For instance, one can be asked to draw a quadrilateral such that by drawing a straight line it is possible to obtain 4 triangles (the reader is advised to try to solve this before he continues further). The question is taken from [2, p. 142]. When trying to solve it many people work on simple quadrilaterals (namely those which do not intersect themselves). Sticking to that type of quadrilateral they fail in the task. Some of them are really not familiar with non-simple quadrilaterals, but others simply fail to recall them in spite of knowing the concept. We can say that their *temporary* concept image did not include the non-simple quadrilaterals, although their permanent concept image does include them.

We will use the term 'temporary concept image' only when we believe that only the proper part of the concept image cell was activated; otherwise we will simply say 'concept image'. Of course, one can argue about the ground of the above belief and one can be absolutely right about it. So in the following sections when we say 'concept image' and the reader believes it is a temporary concept image (and vice versa), we will not claim that he is wrong. The controversy cannot be resolved on the basis of the information we have without interviews. But our intention in this paper was not to determine whether a certain concept image is temporary or not, but to expose either temporary or permanent concept images.

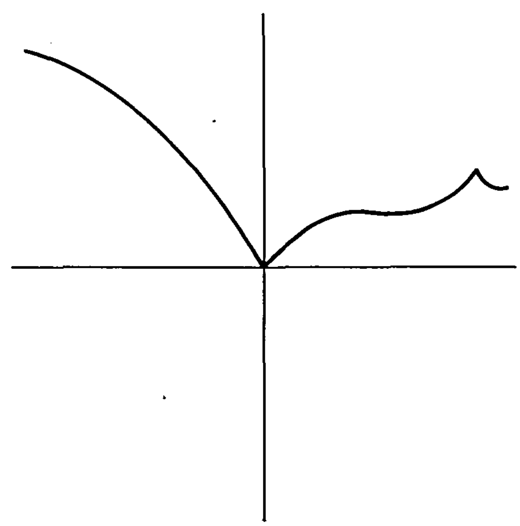
6. The concept of function and the problem

We are going to use the above model to analyse some phenomena in the learning of the concept of function in grades 10 and 11. We are dealing with students who met (in one way or another) the concept of function in the 9th grade. In the 10th grade, the concept of function was formally introduced to them as if they had never had it before. The teachers of these students were using a textbook according to which a function is any correspondence between two sets (a domain and a range) such that every element in the domain has exactly one element in the range that corresponds to it. The teachers used the textbook definition and also introduced to their students nonmathematical functions as correspondences between people and their identity card numbers, children and their mothers, etc. We asked two questions: (1) To what extent did the concept definitions of the students suit the textbook definition? (2) To what extent did the concept definition and the concept image of the students suit each other in cases where the concept definition and the textbook definition did suit each other?

7. The questionnaire and the sample

We will introduce the questionnaire first and then make some comments about it. It had five questions. In the first four questions, the student had to choose between *yes* and *no* and to *explain his choice in words*.

(1) Is there a function that corresponds to each number different from 0 its square and to 0 it corresponds -1?
(2) Is there a function that corresponds 1 to each positive number, corresponds -1 to each negative number, and corresponds 0 to 0?
(3) Is there a function that admits integral values for nonintegral numbers and admits nonintegral values for integral numbers?
(4) Is there a function the graph of which is the following?



(5) In your opinion what is a function?

It was explained to the students that the questionnaire is not a test and that it intends only to examine some of their concepts. They were not asked to write their names on the questionnaires. They were asked to cooperate, and our impression was that we received good cooperation. We also kept reminding them to explain their answers and here also there was a good reaction.

Some comments about the questionnaire: It was a strange one. The questions do not look like questions in regular tests or homework assignments. The answer to the first four questions is 'yes' (students expect some negative answers as well). The moment you understand the question and you know the concept of function as it is defined in the textbook everything becomes so simple that you start to doubt whether this was the intention of the question. By having these elements in the questionnaire, we expected to avoid the well-known conditioned behaviour of students in tests. This expectation was supported by the written explanations we received. In our analysis we have considered only questionnaires with written explanations.

The task imposed by questions 1–4 seems to be a construction task at first sight. At a second glance it looks more like an identification task, since the conditions in questions 1, 2, and 4 can be used as they are to define the required functions. Only in question 3 is a little effort required to define explicitly a function fulfilling the required conditions.

Our sample included 65 students in grade 10 and 81 students in grade 11. They studied in two academically selective high schools in Jerusalem in 1978. The 10th grade students were tested several months after they had learned the chapter about functions. The 11th grade students learned this chapter when they were in the 10th grade.

8. The main concept definitions

In this section we analyse the answers to question 5 (In your opinion what is a function?). Four main categories were distinguished. We shall define the categories and quote some answers that belong to each of them.

Category I: The textbook definition (see § 6) sometimes mixed with elements from the concept image cell

Here the textbook definition was repeated in the student's own words. The formulations might be inaccurate or even bad. In a regular test some of the formulations will not get credit points, but here they were evidence that the concept definition cell had something similar to the textbook definition (in spite of being imperfect).

(1) It is a correspondence of a number belonging to one set of numbers (the domain) to a number in another set (the range). To each number in the domain corresponds only one number in the range, but numbers in the range can have several numbers in the domain. There must not be numbers in the function. Anything can do (concrete objects, animals, etc) (Grade 11).
(2) Every point in the domain has a point in the range (Grade 10).
(3) Function in my opinion is that every x has one number or one object in y but not vice versa (Grade 10).

Category II: The function is a rule of correspondence

This eliminates the possibility of an arbitrary correspondence. A rule and an arbitrary correspondence are contradictory. In addition to the word 'rule' the students also use the words 'law', 'relation', 'dependence between variables', etc.

In this category the influence of Category I is sometimes felt. The student uses words from the textbook definition. However, the aspect of rule is dominant.

(1) It is a *relation* between two sets of numbers based on a *certain law* (Grade 10).
(2) It is a *relation* between two factors *depending* on each other. This relation is described in the graph (Grade 10).

(3) It is taking something and changing it by means of a *constant process* determined by the specific function in consideration (Grade 10).
(4) A function is a method to obtain from one set of numbers another set by means of *a certain rule*. I denied 4 (in the questionnaire) as a function because generally in everyday life applications a function has a definite rule. It is not defined for each point separately (Grade 11).
(5) A function is a *relation* in which every element in one set is related to a single element from another set according to a certain *law*. We have to discover the *law* but it always must exist (Grade 11).

The idea of this category was also expressed in answers to questions 1–4. For instance, an answer to question 3: 'No, such a function with a law does not exist. This is because there is no law in the nonintegral numbers that might be even irrational' (Grade 10).

Category III: The function is an algebraic term, a formula, an equation, an arithmetical manipulation, etc.

(1) A function is a set of numbers such that when *doing to them a certain arithmetical operation* we obtain another set of numbers which is functional to the first set (Grade 10).
(2) A function is a reflection of numbers by means of *an equation*, namely, when a number is substituted in an algebraic term we get a number which is the reflection of the first number (Grade 10).
(3) As far as I know a function is an *arithmetical operation* performed on numbers. Every element in the domain has a single element in the range and there is no element in the domain that does not have an image (Grade 10).
(4) A function is an *equation* that has a range at one side and a domain at the other side. To each number (or factor) in the domain corresponds a factor in the range (Grade 10).
(5) A function is something like an *equation*. When you put numbers instead of the unknown you get a solution (Grade 10).

Note the influence of the textbook definition in answers 3 and 4. According to our model there was an interaction between the concept image cell and the definition cell that first contained the textbook definition. After a while the concept image cell became dominant but some traces of the textbook definition were left.

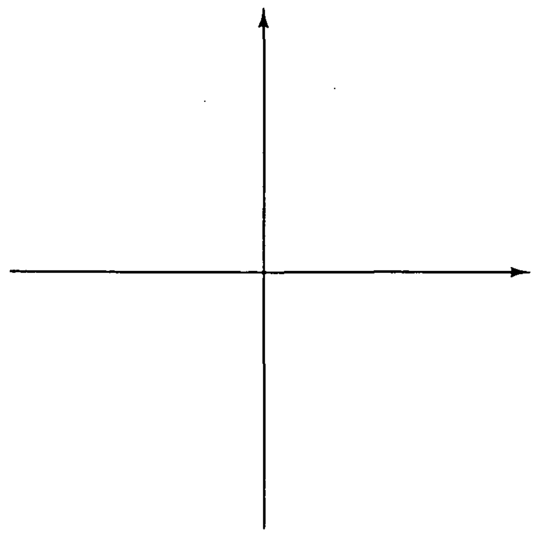
The idea of this category (as well as the idea of Category II) was also expressed in answers to questions 1–4. For instance, an answer to question 4: 'Maybe yes and maybe no. The graph might have a formula and it might have no formula. If it does not have a formula then it is not a function' (Grade 11).

Category IV: Some elements in the mental picture are taken as a definition for the concepts

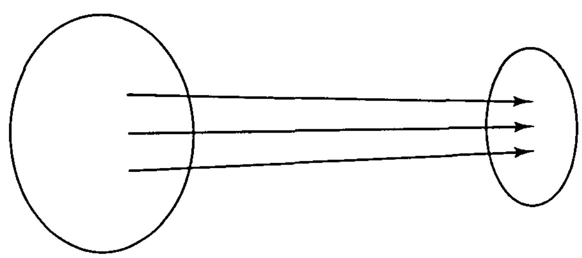
The function is identified with a graph, with the symbols '$y=f(x)$' and with the 'two potatoes with arrows' diagram.

(1) A function is a *curved line in a coordinate system* such that to every point corresponds exactly one point. There might be points to which no point corresponds and there might be points on the y-axis to which there are more than one point on x (Grade 11).

(2) Values of one graph y depending on another graph x according to $y=f(x)$. (Grade 11).



(3) A function is an expression corresponding elements from one set to another under the condition that there will not be more than one arrow for each number. (Grade 11).



Note that in answers 1 and 3 the influence of the textbook definition is felt. Hence one can argue whether it is justified to include certain answers in certain categories. This is true, but we want to emphasize that the above categories are not necessarily disjoint. In order to decide to which category a certain answer (that has several aspects) should belong we look at (what seems to us) the dominant aspect. The possibility of including one answer in more than one category is consistent with our model. Moreover, it confirms it. Of course, as a result of this the statistical treatment will be inaccurate. One should look at the statistical tables (§11) only as a rough estimate.

9. The main concept images

Analysing the answers to questions 1–4 in the questionnaire we have found some concept images that are not consistent with the textbook definition (although sometimes consistent with the student's own definition). Students have some expectations of functions, expectations that do not have any logical ground and do not logically relate to the textbook definition. These expectations are probably a result of the experience the students have had with functions. However, one has to admit that students' observations of their experience were quite selective. Their experience included all 'strange functions' by means of which the 'correct image' of the concept of function should have been formed. In spite of that, the cognitive process that formed certain students' concept image ignored this part of their experience, perhaps because it looked strange, artificial, or unnatural.

(1) A function should be given by one rule. If two rules are given for two disjoint domains we are concerned with two functions. If the correspondence between the numbers looks arbitrary to a student he might speak of infinitely many functions, as if each number has its own rule of correspondence.

The above approach is expressed in the following answers:

To question 1: No, because such a function should give also the square of 0 (Grade 10).
No, because if you square then the result is positive (Grade 11).
No, because it contradicts the concept of function (Grade 10).
No, $0^2=0$ and not -1 (Grade 11).

To question 2: No, there are three different functions. One of them gives $+1$ for all positive numbers, the second gives -1 for all negative numbers and the third one gives 0 (Grade 10).
No, because such a function is a constant function but the constant should be the same all the way through (Grade 11).

A student's comment to all questions: There does not exist one function in any of the questions, but by means of all kinds of compositions it is possible to get everything, even something as in question 4 (Grade 10).

(2) A function can be given by several rules relating to disjoint domains providing these domains are half lines or intervals. But a correspondence as in question 1 (a rule with one exception) is still not considered as a function.

This approach was expressed in the fact that many students whose reply to question 2 was positive, chose to say 'no' in question 1. It is also beautifully expressed in an explicit manner in the following two answers of the same student: (*a*) No, because there is no number the square of which is negative. (*b*) Yes, because contrary to (*a*) there are various functions such that the first one is equal to 1 for all positive numbers, the second one is equal to -1 for all negative numbers, and the third one is 0.

(3) Functions (which are not algebraic) exist only if mathematicians officially recognize them (by giving them a name or denoting them by specific symbols). This view was expressed by answers like the following:

To question 2: No, As a matter of fact perhaps there is such a function but I do not know about it (Grade 10).

Even students who replied yes to question 2 justified it by saying that this is the sign function (the denotation of which is 'sg') as if the existence of the function depends on this fact (some of the students learned about this function in class). The same thing happened with question 3. Instead of defining a function that fulfills the required conditions, students mentioned the integral part function sometimes denoted by $[x]$, and the fractional part function, sometimes denoted by $\{x\}$, and even did not bother to show how these functions are related to the question.

(4) A graph of a function should be 'reasonable'. Many students denied the graph in question 4 to be a graph of a function because it is not regular. They claimed that a graph of a function should be symmetrical, persistent, always increasing or always decreasing, reasonably increasing, etc.

Here are some answers to question 4:

> No, I do not think so. I always believed that a function is something *persistent* (Grade 10).
> No. Since a function is constructed by means of a fixed equation it is impossible for it to increase in an *unproportional* manner (Grade 10).
> No. A function *either increases or decreases* but here it is neither nor.
> No. There is *no regularity* in the graph and therefore there is no function that can describe this graph. It might be an arbitrary correspondence from x to y without any regularity (Grade 10).
> No. There is no constant relation between x and y (Grade 11).

(5) For every y in the range there is only one x in the domain that corresponds to it. This point of view is a result of *a failure to recall* the textbook definition correctly.

(6) A function is a one-to-one correspondence. This point of view is a result of a distortion of the textbook definition. This distortion might be a result of what we call an implicit requirement for symmetry. If for every x in the domain there is only one y in the range then the contrary should also be true. Thus, contrary to (5) which was caused by a memory failure only, some creativity (whether you like it or not) was involved in forming (6).

Because of space considerations, we will not quote answers reflecting (5) or (6). Essentially, these answers denied correspondences that did not fulfill the requirements in (5) or (6) to be functions.

10. The image of rule and a temporary image of numbers

It is impossible in such a paper as this to mention all interesting points that have been discovered in the answers. We shall mention only two of them. The first one concerns the concept of rule. It turned out that many students conceive rule as an instruction involving manipulations or an algebraic formula. One has to 'do something' to a number in order to get the number corresponding to it. An arithmetical operation should be involved. This was mainly expressed in answers to question 2. Students found it unsatisfactory to say: 'Correspond 1 to each positive number, -1 to each negative number, and 0 to 0.' Instead of this they suggested: "$\frac{x}{x}$ for positive numbers and $\frac{x}{-x}$ for negative numbers", or "$\frac{x}{x}$ for any number", etc. The same phenomena occurred in question 3. The students tried very hard to avoid a rule which arbitrarily corresponds an integer to nonintegers and vice versa. A rule and arbitrary correspondence seemed to them contradictory. So they tried to construct a formula that will fulfill the requirements of question 3. Here we come to the second point we wanted to mention. Several students (independently) tried: $f(x)=\frac{1}{x}$ and then substituted x with integers. Forming the substitutions, they found themselves successful. Now they had to try the above formula for nonintegers. But after having in the previous computations nonintegers of the form $\frac{1}{x}$ (x integer), they temporarily had the image of $\frac{1}{x}$ for nonintegers. Thus, they wrote $\frac{1}{x}$ instead of x in the above $f(x)$ and they got $f(\frac{1}{x}) = 1/(\frac{1}{x}) = x$ which confirmed their guess that $f(x) = \frac{1}{x}$ was a good candidate. Their temporary concept image (§ 5) of numbers included here integers and fractions of the form $\frac{1}{x}$ (x integer only). This is an example of how temporary concept images (being casual and unstable as they might be) can determine a whole line of thought which unfortunately leads to serious mistakes.

11. Some tables

Tables 1–3 are intended to answer the following questions respectively:

(1) What was the distribution of students between categories (1) to (4) (§ 8)?
(2) Among the students who gave the textbook definition, how many acted according to it? (A student who says, for instance in 4 that the given graph is not a graph of a function because it is not symmetrical is not considered as somebody whose behaviour is directed by the textbook definition.)
(3) What was the distribution of correct answers among questions 1–4?

Different tables were prepared first for the 10th graders and the 11th graders, but no significant difference was found. Hence, we combined the tables, N = 146, and here are the results:

Category I	Category II	Category III	Category IV	No Answer
57	14	14	7	8

Table 1. Percentage classification.

Percentage of students acting according to the textbook definition in the whole sample	20
Percentage of students above, out of the students having the textbook definition	34

Table 2.

Nobody acted according to the textbook definition without having some form of it as an answer to question 5.

	Question 1	Question 2	Question 3	Question 4
Correct answer (percent)	39	66	49	57

Table 3.

12. An educational comment

We already mentioned the importance of knowing the student's concept image for teaching (§ 4). At this point it has become quite clear that a definition and some examples are not sufficient to form the desirable concept image. There are numerous factors that determine the concept image, some of them beyond our control. It is important for teachers and textbook authors to know this. First, one has to know what he is supposed to expect from students. Second, it increases the understanding we have for our Math students (an understanding which they desperately need). The naive approach of 'let us write a wonderful unit and all the problems will be solved' proves again to be wrong. Also an a priori comprehensive analysis of the concept as suggested in [5] is not sufficient.

In [3] it was clearly shown that when considering the notion of function from the historical point of view one should get a totally different perspective to our problem. It is noted there that great mathematicians like Euler, D'Alembert and Lagrange had a concept image different from the one suggested by the modern textbook definition. This definition can be considered as the Dirichlet–Bourbaki definition. As a matter of fact, the modern approach to function was rejected by some great mathematicians. In [3] it is argued that because of that and because of the well known difficulties that students have with the Dirichlet–Bourbaki definition it is better to avoid it in all courses preceding Analysis, Topology and Algebra at the university level.

On the other hand, it is clear that if one does want to insist on the general definition in spite of everything, one has to provide the students with examples that form the desired concept image not only in the beginning of the chapter but throughout the whole period of learning. Elements (of the concept image) which are not constantly reinforced have a good chance of being forgotten and thus the concept image is distorted. Providing the students with such examples during the whole period of learning might not be such an easy task (the problem is to find interesting examples in the right context without giving our students the feeling that they are indoctrinated). Again, one has to decide whether one wants to go through all that. Unfortunately, the alternatives are either to do it like that or not to do it at all.

But all this, however, might lead to another conclusion. We have to admit that in applications or in regular achievement tests the concept image does not have to play a crucial role. Students can succeed in examinations even when having a wrong concept image. Hence if we do not care about the concept image we do not have to bother with it. On the other hand, if we do care about it then what we do is not enough. More time and pedagogical efforts are required. Of course, the question of priorities in Mathematics teaching should be raised at this point, but this is not a question to be resolved in this paper.

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