III. Homogeneous Cosmological Models

Readings

The material in this section is covered in chapters 3-7 of Ryden, and in chapter 3 of Peacock.

I very much like the discussion in Jim Gunn's (1978) article *The Friedmann Models and Optical Observations in Cosmology*, from the SAAS-FEE Proceedings *Observational Cosmology*, edited by A. Maeder et al. This article is hard to track down, and I will distribute the first 40 pages or so (which are the most useful).

For distance measures, I recommend the compact summary by David Hogg, *Distance Measures in Cosmology*, available as astro-ph/9905116. It generally does not give derivations, but it summarizes the standard results in admirably clear form.

Notation differs from one treatment to another. I will try to mostly follow Ryden's notation.

The Einstein cosmological model

Cosmological Considerations on the General Theory of Relativity (Einstein, 1917, reprinted in the *Principle of Relativity* book)

Einstein constructs the first modern cosmological model, drawing on new concepts of relativity.

Gives arguments, not particularly persuasive, against an infinite universe.

Introduces hypothesis: universe is homogeneous on large scales.

How can universe be finite and homogeneous? What about boundary?

GR allows a solution: space is positively curved, like the surface of a sphere. Finite, but no boundary.

Seeks static solution with constant ρ, positive curvature.

Proves that no such solution exists.

In response, Einstein *modifies* the field equation from

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad \text{to} \qquad G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

adding the "cosmological term."

Of the "principles" that lead to the field equation, the one he drops is the requirement that spacetime is flat when $T_{\mu\nu} = 0$ everywhere.

For a specific relation between the cosmological constant Λ and the matter density ρ, this allows a static solution.

The cosmological models subsequently developed by Friedmann and LeMaître retain some ideas from the Einstein cosmology — GR, large scale homogeneity, large scale curvature — but they drop the assumption that the universe is static. They therefore do not *require* a cosmological constant.

Einstein abandoned the cosmological term for good when the cosmic expansion was discovered (in 1929). He is reputed to have called it "the greatest blunder of my life."

The Cosmological Constant

The cosmological constant idea has never completely gone away. It has been especially prominent in the last 20 years, but it is now viewed as part of $T_{\mu\nu}$.

Instead of

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad G_{\mu\nu} = 8\pi(T_{\mu\nu}^{\rm matter} + T_{\mu\nu}^{\rm VAC}), \quad \text{where}$$
$$T_{\mu\nu}^{\rm VAC} \equiv \Lambda g_{\mu\nu}/8\pi.$$

$T_{\mu\nu}^{\rm VAC}$ is the stress-energy tensor of a "false vacuum" or "scalar field" with equation of state $p = -\rho$.

The basic effect can be seen from our Newtonian limit result:

$$\nabla^2\Phi = 4\pi(\rho + 3p).$$

For $p < -\rho/3$, gravity pushes instead of pulls.

With the right choice of Λ, can have static model in which push of vacuum energy balances pull of matter (but unstable).

With larger Λ, get acceleration.

The Friedmann-Robertson-Walker metric

The *spatial* metric of the Einstein cosmological model is that of a 3-sphere:

$$dl^2 = dr^2 + R^2 \sin^2(r/R) d\gamma^2,$$

where $d\gamma^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ is the angular separation.

Here R is the curvature radius of the 3-dimensional space, and r is distance from the origin.

In the coordinate frame of a freely falling observer, time is just proper time as measured by the observer, and the spacetime metric is

$$ds^2 = -c^2 dt^2 + dl^2.$$

A natural generalization of the Einstein model is to allow the curvature radius $R(t)$ to be a function of time.

The universe is still homogeneous and isotropic on a surface of constant t, but it is no longer static.

In the 1930s, Robertson and Walker (independently) showed that there are only three possible spacetime metrics for a universe that is homogeneous and isotropic.

They can be written

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\gamma^2\right],$$

where

$$\begin{aligned} S_k(r) &= R_0 \sin(r/R_0), \qquad k = +1, \\ &= r, \qquad k = 0, \\ &= R_0 \sinh(r/R_0), \qquad k = -1. \end{aligned}$$

In this notation, $a(t)$ is dimensionless.

It is defined so that $a(t_0) = 1$ at the time t_0 (usually taken to be the present) when the curvature radius is R_0. At other times the curvature radius is $a(t)R_0$.

The radial coordinate r and the radius of curvature R_0 have units of length (e.g., Mpc).

I have followed Ryden's notation in giving $S_k(r)$ units of length. In Gunn's notation, $S_k(r)$ is dimensionless, and $a(t)$ is replaced by $R(\tau)$, where $R(\tau)$ has units of length.

For $k = +1$, the space geometry at constant time is that of a 3-sphere.

For $k = 0$, the space geometry at constant time is Euclidean, a.k.a. "flat space."

For $k = -1$, the space geometry at constant time is that of a negatively curved, 3-dimensional "pseudo-sphere."

Friedmann and LeMaître used this metric in their cosmological models of the 1920s. Robertson and Walker proved that they are the only forms consistent with the Cosmological Principle (homogeneity and isotropy).

It is commonly called the Friedmann-Robertson-Walker metric, or sometimes the Robertson-Walker metric.

The FRW metric, space curvature, and spacetime curvature

$k = +1 \Longrightarrow$positive curvature, spherical geometry, finite space

$k = 0 \Longrightarrow$no curvature, Euclidean geometry, infinite space

$k = -1 \Longrightarrow$negative curvature, pseudo-sphere geometry, infinite space

Note: these are descriptions of *space* at constant t.

For many forms of $a(t)$, *spacetime* is positively curved even if *space* is not, (this is always the case unless a cosmological constant or some other form of energy with negative pressure is important). In the special relativistic, Milne cosmology, spacetime is flat, but surfaces of constant time are negatively curved.

Positive curvature $\Longrightarrow$geodesics "accelerate" (in 2nd derivative sense) towards each other. Initially "parallel" geodesics converge.

Example: great circles on a sphere.

Zero curvature $\Longrightarrow$no geodesic "acceleration." Initially parallel geodesics stay parallel. Euclidean geometry.

Example: straight lines on a plane.

Negative curvature $\Longrightarrow$geodesics "accelerate" away from each other. Initially parallel geodesics diverge.

Example: geodesics on a saddle.

The substitution $x = S_k(r)$ allows the FRW metric to be written in another frequently used form:

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1 - kx^2/R_0^2} + x^2d\gamma^2\right].$$

Demonstration is left as a (simple) exercise for the reader.

With r as radial coordinate, radial distances are "Euclidean" but angular distances are not (unless $k = 0$). With x as radial coordinate, the reverse is true.

Comoving Observers

The metric depends on the coordinate frame of the observer.

Even a homogeneous and isotropic universe only appears so to a special set of freely falling observers, called "Fundamental Observers (FOs)" or "Comoving Observers."

These observers are "going with the flow" of the expanding universe, and the proper distance between them increases in proportion to $a(t)$.

In the coordinate frame of these observers, the FRW metric applies, and the time coordinate of the FRW metric is just proper time as measured by these observers.

Comoving spatial coordinates track the positions of these FOs, i.e., the comoving separation between any pair of FOs remains constant in time.

An observer moving relative to the local FOs has a "peculiar velocity," where peculiar is used in the sense of "specific to itself" rather than "odd."

An observer with a non-zero peculiar velocity does not see an isotropic universe – e.g., dipole anisotropy of the cosmic microwave background caused by reflex of the peculiar velocity.

Examples of application of metric

(1) Any freely falling particle follows geodesics in spacetime, whose solution in comoving coordinates could be found from the geodesic equation. For comoving particles (FO's), solution is trivial: $r,\ \theta,\ \phi =$ constant.

(2) Light rays travel along null geodesics: $ds^2 = 0$. Therefore, along a radial ray ($d\gamma = 0$), $dr = c\,dt/a(t) \implies r_o - r_e = \int_{t_e}^{t_o} c\,dt/a(t)$.

(3) In a surface of constant t, metric distance along a radial path of constant θ, ϕ is

$$l = \int ds = \int_{r_1}^{r_2} a(t)dr = a(t)(r_2 - r_1).$$

(4) In a surface of constant t, metric distance along a path of constant r, θ between two points of different ϕ is

$$l = \int ds = \int_{\phi_1}^{\phi_2} a(t)S_k(r)d\gamma = a(t)S_k(r)\sin\theta(\phi_2 - \phi_1).$$

Note that this is not a great circle (and hence shortest) path unless $\theta = \pi/2$.

(5) In a surface of constant t, metric volume of a shell of radius r and width $\Delta r \ll r$ is

$$V = \int d^3s = \int_r^{r+\Delta r} a(t)dr \int_{4\pi} a^2(t)S_k^2(r)d\gamma^2 = 4\pi S_k^2(r)a^3(t)\Delta r.$$

For $k = 0$ this is just $4\pi r^2 \Delta r \times a^3(t)$.

Note that the "metric" distances and volumes in (3)-(5) are "proper," physical distance measures and that r is the *comoving* radial coordinate.

Redshift of photons

A photon emitted by a nearby comoving source at distance d is Doppler shifted:

$$\frac{d\nu}{\nu} = \frac{-v}{c} = \frac{-Hd}{c} = -H dt,$$

where H is the Hubble parameter and the last equality follows because $d = c\,dt$.

The Hubble parameter is $H = \dot{d}/d = (\dot{ar})/(ar) = \dot{a}/a$. Thus,

$$\frac{d\nu}{\nu} = -\frac{\dot{a}\,dt}{a} = -\frac{da}{a} \implies d\ln\nu = -d\ln a.$$

Let the photon be emitted with frequency ν_e at time t_e and observed with frequency ν_o at time t_o.

Integrate to get

$$\frac{\nu_e}{\nu_o} = \frac{a_o}{a_e} = \frac{\lambda_o}{\lambda_e} \equiv (1+z), \qquad z = \text{redshift}.$$

Constant of integration fixed by demanding $\nu_o \longrightarrow \nu_e$ as $a_o \longrightarrow a_e$.

Photon wavelength proportional to $a(t)$.

Frequency shift $\Longrightarrow$time dilation. Real effect observed in, e.g., supernova light curves.

Could also derive by (a) considering successive crests traveling on null geodesics, or (b) using equation for evolution of 4-momentum along null geodesic.

Kinematic redshift

Give a particle a "peculiar" velocity with respect to the comoving frame.

The peculiar velocity decays as it catches up with receding particles.

This is a purely kinematic effect, though it looks like a fictitious "friction."

In the non-relativistic limit, a particle with speed v goes distance $v\,dt$ and changes its peculiar velocity by $dv = -H(v\,dt)$

$$\Longrightarrow \quad \frac{dv}{v} = -\frac{\dot{a}\,dt}{a} = -\frac{da}{a} \quad \Longrightarrow \quad \frac{p_e}{p_o} = \frac{mv_e}{mv_o} = \frac{a_o}{a_e}.$$

Momentum redshifts like the frequency of a photon.

Gunn shows that this continues to hold in the relativistic case (dv large).

One can also show that this implies that the de Broglie wavelength of a particle redshifts just like photon wavelengths.

Kinematic redshift profoundly affects the dynamics of instabilities: in an expanding universe (or any expanding medium), undriven disturbances decay instead of coast.

Flux, diameter, and surface brightness vs. redshift

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\gamma^2].$$

From metric application (2) above, we have the comoving distance to an object that emitted light at time t_e as

$$D_c = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}.$$

From $a \equiv (1+z)^{-1}$ we have $da = -dz(1+z)^{-2} = -a^2dz$, and from $H \equiv \dot{a}/a$ we have $dt = da/(aH) \Longrightarrow dt/a(t) = da/(a^2H) = -dz/H$.

Putting these results together yields

$$D_c = \int_0^z \frac{c\,dz'}{H(z')} \;=\; \frac{c}{H_0}\int_0^z dz' \frac{H_0}{H(z')}$$

for the comoving distance to an object at redshift z.

Using the relation for $H_0/H(z')$ that we will derive later from the Friedmann equation then reproduces equation (15) of Hogg (1999).

An object of angular size $d\gamma$ at time t_e has a transverse physical size

$$dl = a(t_e)S_k(r)d\gamma = (1+z)^{-1}S_k(r)d\gamma.$$

The angular diameter distance is

$$D_A = \frac{dl}{d\gamma} = (1+z)^{-1}S_k(r),$$

where r is the comoving distance D_c as given above.

We will later find from the Friedmann equation that the curvature radius is

$$R_0 = \frac{c}{H_0}|\Omega_k|^{-1/2},$$

where $\Omega_{\rm tot} = 1-\Omega_k$ is the ratio of the total energy density of the universe (mass, radiation, dark energy, ...) to the critical density. For $\Omega_k \to 0$, $R_0 \to \infty$ and the curvature $1/R_0^2 \to 0$.

Together with the definition $D_H = c/H_0$, this yields the result of equations (16) and (18) of Hogg (1999):

$$\begin{aligned} D_A &= (1+z)^{-1}\Omega_k^{-1/2}\sinh(\Omega_k^{1/2}D_c/D_H) \qquad k=-1, \\ &= (1+z)^{-1}D_c \qquad k=0, \\ &= (1+z)^{-1}|\Omega_k|^{-1/2}\sin(|\Omega_k|^{1/2}D_c/D_H) \qquad k=+1. \end{aligned}$$

The photons from a source at redshift z are distributed over an area $4\pi S_k^2(r)$ at the present day, since $a(t_0) \equiv 1$.

Photons emitted in a time dt_e are received over an interval $dt_0 = (1+z)dt_e$, and they are shifted downward in energy by $(1+z)$. The bolometric flux F is therefore reduced by an additional factor $(1+z)^2$:

$$F = \frac{L}{4\pi S_k^2(r)(1+z)^2} \equiv \frac{L}{4\pi D_L^2},$$

where L is the source bolometric luminosity and $D_L = (1+z)^2 D_A$ (Hogg 1999, eq. 21).

In c.g.s. units,

$$[F] = \text{erg s}^{-1}\text{ cm}^{-2}, \qquad [L] = \text{erg s}^{-1}.$$

The solid angle subtended by a source of projected area A is $\Omega = A/D_A^2$, making the surface brightness

$$I_0 \equiv \frac{F}{\Omega} = \frac{L}{4\pi D_L^2}\frac{D_A^2}{A} = \frac{L}{4\pi A}\frac{1}{(1+z)^4} = \frac{I_e}{(1+z)^4}.$$

This is the famous $(1+z)^4$ surface brightness dimming of cosmological sources, which can make high redshift galaxies very difficult to detect.

Note, however, that these relations for flux and surface brightness are bolometric, integrated over all wavelengths.

The relation for monochromatic fluxes can be written

$$\nu_0 S_{\nu_0} = \frac{\nu_e L_{\nu_e}}{4\pi D_L^2}.$$

The monochromatic or passband flux of an astronomical object is affected by the redshifting of the bandpass from ν_0 to $\nu_e = \nu_0(1+z)$.

This effect is referred to as the K-correction (see Hogg et al. 2002, astro-ph/0210394).

Most important fact about redshift:

If we measure the redshift of a source (e.g., from the frequency of a spectral line), we know a_o/a_e.

Given a model of $a(t)$, we also know the radial distance. For angular diameter and luminosity distances, we also need to know the space curvature (R_0 and k, or Ω_k).

Nearby (i.e., to first order in z),

$$cz = c\frac{\lambda_o - \lambda_e}{\lambda_e} = H_0 d, \qquad H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)},$$

independent of other cosmological parameters.

Alternatively, if we can infer distance from observed flux (of a source of known luminosity) or observed angular size (of a source of known physical size), we can reconstruct $a(t)$ from observations, and constrain cosmological parameters.

Dynamics: the Friedmann equation

Gravity hasn't entered the picture yet. But to go any further, we need $a(t)$.

Assume GR is correct. We could get equations for $a(t)$ by plugging the FRW metric into the field equation. This yields two non-trivial equations, one of which is the integral of the other.

Instead of following this derivation, we'll use the Newtonian limit, $\nabla^2\Phi = 4\pi G(\rho + 3p)$, which will get us almost all the way.

We appeal to Birkhoff's theorem, which implies that we can think about a small spherical volume in isolation, ignoring the gravitational effects of the rest of the universe (which cancel out in spherical symmetry).

Consider a shell of physical radius R comoving with the Hubble flow:

$$\ddot{R} = -\frac{4\pi}{3}G(\rho + 3p)R^3 \times \frac{1}{R^2}.$$

But $R = ar$ with r constant, so $\ddot{R} = \ddot{a}r$. Thus,

$$\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a = -\frac{4\pi}{3}G\left[3(\rho + p)a - 2\rho a\right].$$

This is an "acceleration" equation for the cosmic expansion. We see already that $\ddot{a} < 0$ if $\rho + 3p > 0$, gravity slows expansion.

We would like to have an "energy" equation for $\dot{a}$, which we can get by integrating if we know how ρ and p change with a.

Use the first law of thermodynamics (energy conservation), assuming that the expansion is adiabatic:

$$\begin{aligned} -pdV &= dU = d(\rho V) = \rho dV + V d\rho \\ \Longrightarrow d\rho &= -(\rho + p)\frac{dV}{V} = -3(\rho + p)\frac{da}{a} \\ \Longrightarrow \dot{a} &= \frac{-a}{3(\rho + p)}\dot{\rho}. \end{aligned}$$

(The adiabatic assumption is valid during most of the cosmic expansion, but it is violated at some special epochs when the number of particles changes substantially.)

Multiply both sides of the acceleration equation by $\dot{a}$ to get

$$\begin{aligned} \dot{a}\ddot{a} &= -\frac{4\pi}{3}G\left[-a^2\dot{\rho} - 2\rho a\dot{a}\right] \\ &= \frac{4\pi}{3}G\left[a^2\dot{\rho} + \rho\frac{d(a^2)}{d\tau}\right]. \end{aligned}$$

Recognize that $\dot{a}\ddot{a} = d(\dot{a}^2/2)/dt$ and that the term in [ ] is $d(a^2\rho)/dt$. Integrate with respect to t to get

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = \text{constant}.$$

Unfortunately, deriving the integration constant really does require the GR field equation.

We can guess that if the density ρ is high, space will be positively curved, and the universe will be gravitationally bound, making the constant (which plays the role of a potential energy) negative.

Conversely, if ρ is low, space will be negatively curved, and the universe will be unbound, with a positive constant.

GR leads to the conclusion that the integration constant is $-kc^2/R_0^2$.

In dimensionally correct form, the Friedmann equation can be written

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2 R_0^2} = 0.$$

We will sometimes refer to the first term as the "kinetic" term, the second as the "gravitational" term, and the third as the "curvature" term.

The density parameter

Note that $\dot{a}/a = H$, so if $k = 0$ the Friedmann equation $\Longrightarrow \rho = 3H^2/(8\pi G)$. Define the "critical density"

$$\rho_c \;=\; \frac{3H^2}{8\pi G} \;=\; \text{density of a } k = 0 \text{ Friedmann universe.}$$

We can define a dimensionless "cosmological density parameter"

$$\Omega = \frac{\rho}{\rho_c} \;\Longrightarrow\; \rho = \Omega\frac{3H^2}{8\pi G}.$$

The Friedmann equation can also be written

$$H^2(1-\Omega) = \frac{-kc^2}{a^2 R_0^2}.$$

Matching signs implies

$$\begin{aligned}
\Omega > 1 &\;\longrightarrow\; k = +1, \text{ closed universe} \\
\Omega = 1 &\;\longrightarrow\; k = 0, \text{ flat universe} \\
\Omega < 1 &\;\longrightarrow\; k = -1, \text{ open universe}
\end{aligned}$$

Note that $\Omega = \Omega(t)$, but because k doesn't change, Ω always remains within whichever of these 3 regimes it starts in.

If we define $\Omega_k = 1 - \Omega$, where Ω is the sum of all other energy densities relative to ρ_c, then the above equation implies

$$R_0 = \frac{c}{H_0}|\Omega_k|^{-1/2}.$$

Evolution of energy density

Consider an energy component with equation of state $p = w\rho$ ($c = 1$ units).

How does its energy density ρ change with expansion factor $a(t)$?

$$\begin{aligned} dU &= -pdV \\ d(\rho V) &= \rho dV + V d\rho = -w\rho dV \\ V d\rho &= -(1+w)\rho dV \\ d\ln\rho &= -(1+w)d\ln V = 3(1+w)d\ln a \qquad (V \propto a^3). \end{aligned}$$

Integrating yields

$$\rho \propto a^{-3(1+w)}.$$

Pressureless matter: $w = 0$, $\rho \propto a^{-3}$ (dilution)

Radiation: $w = 1/3$, $\rho \propto a^{-4}$ (dilution plus redshift)

Cosmological constant: $w = -1$, $\rho =$const.

If these are the energy components in the universe, then the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\left[\rho_{m,0}\left(\frac{a_0}{a}\right)^3 + \rho_{r,0}\left(\frac{a_0}{a}\right)^4 + \rho_{\Lambda,0}\right] = -\frac{kc^2}{a^2 R_0^2}.$$

Here the subscript 0 can represent any fiducial time t_0.

If it represents the present day, then $a_0/a = (1+z)$.

Note that even if the curvature term is comparable to the gravitational term today, it will be negligible at sufficiently high redshift because the ρ_m and ρ_r terms grow more rapidly with $(1+z)$.

Thus, flat universe ($k = 0$) solutions are always accurate at high z.

Solutions of the Friedmann equation: single component universe

Empty universe: $\rho = 0$, $k = -1$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{c^2}{a^2 R_0^2}.$$

Solution $a = ct/R_0$, $R_0 = ct_0$.

Metric and expansion rate of the Milne cosmology.

Flat universe: $k = 0$, $\rho = \rho_0 \left(\frac{a_0}{a}\right)^n$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^n .$$

Solution $a \propto t^{2/n}$.

Pressureless matter: $n = 3$, $a = a_0(t/t_0)^{2/3}$.

Radiation: $n = 4$, $a = a_0(t/t_0)^{1/2}$.

(Our standard notation has $a_0 = 1$.)

Λ-*dominated flat universe*: $k = 0$, $\rho = \rho_\Lambda$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_\Lambda .$$

Since $\dot{a} \propto a$, solution is exponential growth:

$$a = a_0 e^{t/t_H}, \qquad t_H = \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{-1/2} .$$

Solutions of the Friedmann equation: two component universe

Matter + curvature

We have previously written the Friedmann equation in the forms

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2 R_0^2} = 0$$

and

$$H^2(1-\Omega) = \frac{-kc^2}{a^2 R_0^2} .$$

Evaluating the second equation at $a = a_0 = 1$ gives

$$(1-\Omega_0) = \frac{-kc^2}{R_0^2 H_0^2} ,$$

which we can use to change the R_0 dependence in the first form to an Ω_0 dependence.

For a matter-dominated universe (with or without curvature),

$$\rho = \rho_0 a^{-3} = \Omega_0 \frac{3H_0^2}{8\pi G} a^{-3},$$

allowing the Friedmann equation to be written

$$H^2 - \Omega_0 H_0^2 a^{-3} + H_0^2(\Omega_0 - 1)a^{-2} = 0,$$

or

$$\frac{H^2}{H_0^2} = \Omega_0 a^{-3} + (1 - \Omega_0)a^{-2}.$$

We see that for $\Omega_0 > 1$, H becomes zero at the "turnaround" epoch

$$a_{\rm max} = \frac{\Omega_0}{\Omega_0 - 1}.$$

For $\Omega_0 < 1$, H does not reach zero, so it cannot change sign; an expanding sub-critical universe expands forever.

For $\Omega_0 > 1$, the solution (which can be verified by direct substitution and a bit of algebra) can be written in the parametric form

$$\begin{aligned} a(\theta) &= \frac{a_{\rm max}}{2}(1 - \cos\theta), \\ t(\theta) &= \frac{a_{\rm max}}{2}\frac{1}{H_0(\Omega_0 - 1)^{1/2}}(\theta - \sin\theta) \;=\; \frac{R_0}{c} \times \frac{a_{\rm max}}{2}(\theta - \sin\theta)\ . \end{aligned}$$

Maximum expansion is reached at $\theta = \pi$, and the universe collapses in a "big crunch" at $\theta = 2\pi$.

For $\Omega_0 < 1$, define

$$a_* = \frac{\Omega_0}{1 - \Omega_0},$$

and the parametric solution is

$$\begin{aligned} a(\eta) &= \frac{a_*}{2}(\cosh\eta - 1), \\ t(\eta) &= \frac{a_*}{2}\frac{1}{H_0(1 - \Omega_0)^{1/2}}(\sinh\eta - \eta) \;=\; \frac{R_0}{c} \times \frac{a_*}{2}(\sinh\eta - \eta)\ . \end{aligned}$$

where η runs from zero to infinity.

At late times ($\eta \gg 1$), the solution approaches $a \propto t$ as universe enters "free expansion."

At early times ($\eta \ll 1$, $\theta \ll 1$), both solutions approach $a \propto t^{2/3}$, as for $k = 0$.

Matter + Λ

For a flat universe with a cosmological constant, $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, and the Friedmann equation can be written

$$\frac{H^2}{H_0^2} = \Omega_{m,0} a^{-3} + (1 - \Omega_{m,0}).$$

For $\Omega_{m,0} < 1$, $\Omega_{\Lambda,0} > 0$, and the matter density and cosmological constant are equal at an expansion factor

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}.$$

The relation between time and expansion factor can be written in the cumbersome but explicit form

$$H_0 t = \frac{2}{3(1-\Omega_{m,0})^{1/2}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \left(1 + \left(\frac{a}{a_{m\Lambda}}\right)^3\right)^{1/2}\right].$$

At early times

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{2/3},$$

like a flat, matter-dominated universe, while at late times

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{\Omega_{\Lambda,0}} H_0 t),$$

giving the exponentially expanding solution for a Λ-dominated universe.

Curvature, Destiny, Topology

As the above solution shows, a matter dominated $k = +1$ universe eventually collapses, while a matter dominated $k = 0$ or $k = -1$ universe expands forever.

This equation of closed geometry with a bound universe and flat/open geometry with an unbound universe continues to hold if radiation is added.

But vacuum energy can change the picture.

Following the arguments in Problem Set 2, the Friedmann equation can be written in the form

$$H^2(a) = H_0^2 \left[\Omega_{\phi,0} \frac{\rho_\phi(a)}{\rho_{\phi,0}} + \Omega_{k,0} \left(\frac{a_0}{a}\right)^2 + \Omega_{m,0} \left(\frac{a_0}{a}\right)^3 + \Omega_{r,0} \left(\frac{a_0}{a}\right)^4\right],$$

where $\rho_\phi(a)$ is the vacuum energy density at expansion factor a and

$$\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{\phi,0}.$$

Note that $k = +1$ corresponds to negative Ω_k and vice versa.

For the universe to recollapse, we must have $H(a) = 0$ at some time in the future ($a > a_0$).

For $\Omega_{\phi,0} = 0$, this

must happen if $\Omega_k < 0$

cannot happen if $\Omega_k > 0$.

For $\Omega_{\phi,0} > 0$, recollapse can be avoided if $\rho_\phi(a)/\rho_{\phi,0}$ falls slower than a^{-2}.

Best guess current parameters are $\Omega_{\phi,0} \sim 0.7$, $|\Omega_{k,0}| \ll 1$, $\rho_\phi(a) \sim$ const., implying that the universe could be open, flat, or closed, but that expansion forever is likely.

Future recollapse is possible if $\Omega_{k,0} < 0$ and vacuum energy changes its equation of state and starts to fall faster than a^{-2} in the future.

If $\Omega_{\phi,0} < 0$ (a negative vacuum energy is not favored by observations, but it is not obviously impossible in principle), then one could have an $\Omega_k > 1$ (open) universe that recollapses.

GR does not prohibit the universe from having a complex topology, e.g. a toroidal topology in which heading off in one direction eventually brings you back to where you started.

Thus, in principle, the universe could be negatively curved or flat and still be spatially finite.

There have been some (unconvincing) claims for periodic redshifts that could be interpreted as evidence for complex topology.

People are seriously searching for signs of complex topology in the pattern of CMB anisotropies.