Anonymous and neutral majority rules*

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Abstract

In the standard arrovian framework and under the assumption that individual preferences and social outcomes are linear orders on the set of alternatives, we provide necessary and sufficient conditions for the existence of anonymous and neutral rules and for the existence of anonymous and neutral majority rules. We also determine a general method for constructing and counting these rules and we explicitly apply it to some simple cases.

Keywords: Social welfare function; anonymity; neutrality; majority; linear order; group theory.

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1 Introduction

Committees are often required to provide a strict ranking of a given family of alternatives, not only to determine which alternative is top-ranked. It is possible to design many procedures to aggregate committee members' preferences on alternatives into a strict ranking of alternatives. Among them, we are interested to analyse those satisfying certain principles usually invoked by social choice theorists. The first principle is the requirement that the identities of individuals are not used to determine the social outcome so that every individual opinion influences equally the collective decision. The second one is instead the requirement that any two alternatives are equally treated. These two principles, called anonymity and neutrality respectively, simply say that individual and alternative names are immaterial. Finally, we assume that the decision process also obeys a majority principle, that is, each time a precisely specified and large enough amount of committee members ranks an alternative over another, that ranking has to be maintained in the final decision. The paper investigates under which conditions such special collective decision procedures can be really designed.

Before starting our inquiry, we need to clarify how committee members express their preferences. Usually the format for expressing preferences are voting for one alternative or strict ranking all alternatives (being these two methods equivalent when the alternatives are two). Of course, as we decided to deal with anonymous aggregation rules, the way preferences can be expressed has to be the same for each member in the committee. Moreover, as also neutrality is required, we are forced to focus only on the strict ranking format for preferences. Indeed, when alternatives are at least three, if all members of the committee unanimously voted the same alternative, they could not strict rank the not voted alternatives without treating them impartially. In other words, neutrality is not consistent with voting an alternative only, or more generally, with each way to express preferences leaving room for indifference between two or more alternatives.

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Thanks to the observations above, the considered aggregation rules can be now easily formalized within the well known arrovian framework. We consider h individuals in a committee and n alternatives to be ranked, and we assume that individual preferences and decision outcomes (or social preferences) are linear orders on the set of alternatives, that is, indifference between any pair of distinct alternatives is not allowed. A preference profile is a list of h linear orders each of them associated with the name of a specific committee member and representing her preferences. A rule is a function from the set of preference profiles to the set of social preferences: it represents a particular decision process which determines a ranking of alternatives whatever individual preferences the committee members express. A rule is anonymous if it has the same value over any pair of preference profiles such that we can get one from the other by figuring to permute individual names. A rule is instead neutral if, for every pair of preference profiles such that we can get one from the other by figuring to permute alternative names, the social preferences associated with them coincide up to the considered permutation. Finally, given an integer ν not exceeding the number of members in the committee but exceeding half of it, a ν-majority rule is a rule ranking an alternative over another one if that alternative is preferred to the other by at least ν individuals. Of course, each anonymous and neutral majority rule cannot be independent on the irrelevant alternatives due to Arrow's impossibility theorem.

Anonymity and neutrality are principles often used in social choice literature as they are usually considered criteria able to guarantee some extent of equity and fairness. They are also two of the main properties leading to characterizations of relative and absolute majority rules. In the specific case of two alternatives and when indifference is allowed both for individual and social preferences, May (1952) characterizes the relative majority in terms of anonymity, neutrality and positive responsiveness; Asan and Sanver (2006) characterize absolute qualified majority rules in terms of anonymity, neutrality and Maskin monotonicity; Sanver (2009) presents a unified exposition of the separate characterizations of relative and absolute majority rules, all of them involving anonymity and neutrality. Anonymity and neutrality are also properties used by Asan and Sanver (2002), Woeginger (2003) and Miroiu (2004) to characterize the relative majority when the structure of society is variable, that is, the number of individuals is not fixed. In the general case for the number of alternatives and when individual preferences are linear orders, Maskin (1995) characterizes the majority rule using anonymity, neutrality, and some maximal transitivity condition; Can and Storcken (2012) characterize the ν-majority correspondences (that is, set-valued rules) in terms, among other things, of anonymity and neutrality.

In the framework of social choice functions, that is, functions which associate a unique alternative with a preference profile, Moulin proves that anonymous and neutral social choice functions exist if and only if the number of alternatives n cannot be written as sum of non-trivial divisors of the number h of individuals (Moulin, 1983, Problem 1, p.25), and also that anonymous and neutral h-majority social choice functions exist if and only if

$$\gcd(h, n!) = 1, \tag{1}$$

where $\gcd(h, n!)$ denotes the greatest common divisor between h and $n!$ (Moulin, 1983, Theorem 1, p.23). Note that the coprimality condition (1), being equivalent to the requirement that each prime divisor of h must be greater than n, is quite rarely satisfied: while for two alternatives just h odd is asked, if alternatives are three or four, then h has to be odd and not divisible by three[1]. In a remarkable paper dealing only with majority principle, Greenberg (1979, Corollary 3) proves that ν-majority social choice functions exist if and only if

$$\nu > \frac{n-1}{n}h. \tag{2}$$

While it is known that condition (2) is necessary and sufficient also for the existence of ν-majority rules[2], at the best of our knowledge, in the literature there is no result about which conditions on the parameters h, n and ν guarantee the existence of anonymous and neutral rules and anonymous and neutral ν-majority rules. The major contribution of the paper is just the determination of such conditions as described by Theorems A and B below[3].

Theorem A. *There exists an anonymous and neutral rule if and only if* $\gcd(h, n!) = 1$.

[1] When alternatives are two, Campbell and Kelly (2011, 2013) prove that social choice functions satisfying monotonicity and suitable weak versions of anonymity and neutrality are consistent with the majority principle, whether the number of individuals is odd or even.

[2] For the sake of completeness, following Can and Storken (2013, Example 4), we prove again that fact in Section 5.

[3] Theorem A is just a rephrasing of Theorem 5, while Theorem B of Theorem 12.

Theorem B. *There exists an anonymous and neutral ν-majority rule if and only if* $\gcd(h, n!) = 1$ *and* $\nu > \frac{n-1}{n}h$.

Theorem A shows that anonymity and neutrality, though natural and appealing, are not requirements frequently reachable by rules since having them together is equivalent to the strong coprimality condition (1). However, this quite negative result is far to be discouraging. On the contrary, it suggests new research directions which we are going to discuss in the last section of the paper. Theorem B shows instead that taking together the necessary and sufficient conditions for the existence of anonymous and neutral rules (condition (1)) and for the existence of majority rules (condition (2)), we get a set of necessary and sufficient conditions for the existence of anonymous and neutral majority rules. While one of the two implication is straightforward, the other one is not obvious and a bit unexpected. We also emphasize that Theorem B implies a generalization of the already quoted theorem by Moulin for social choice functions (Theorem 15).

In order to get the stated results, a careful and detailed analysis of the set of preference profiles is needed. Following an approach already explored by Eğecioğlu (2009), we carry on that analysis by means of the theory of finite symmetric groups[4]. The way we use the fundamental concepts and theorems of group theory seems to be promising and able to generate many interesting results in social choice theory. In fact, beyond Theorems A and B, we further prove that assumption (1) is necessary and sufficient for the existence of anonymous and neutral rules having the property that, for every preference profile, the corresponding social preference is consistent with all majority thresholds not generating cycles for that profile, that is, consistent with transitivity (Theorem 14). Moreover, we provide a general method to construct and count all the anonymous and neutral rules[5] and all the anonymous and neutral majority rules (Proposition 16). That method is largely mechanical (with algorithmic implementations, in principle, possible) but it involves a great amount of computations whose complexity strongly increases when the number of individuals and alternatives increase. However, at least for small values of n and h, it can be really applied to generate concrete examples. Thus, after having discussed all the needed preliminary notation and results, in Section 10 we explicitly describe how that method works in some simple cases. For instance, we show that if five members of a committee decide to strict rank three alternatives via an anonymous and neutral 4-majority rule, then they have $2^{21}3^{20}$ rules to choose from. We also prove that, in order to make that choice, committee members simply have to find an agreement on which social outcome, consistent with the 4-majority principle, should be associated with each preference profile in a particular set of preference profiles having only 33 elements. Of course, such an agreement must be found on the basis of further shared principles different from anonymity, neutrality and 4-majority.

2 Definitions and notation

2.1 Symmetric groups

Let X be a nonempty finite set. We denote by $|X|$ the order of X and by $\mathfrak{F}(X)$ the set of functions from X to X. Given $f_1, f_2 \in \mathfrak{F}(X)$, we denote by $f_1 f_2$ the element of $\mathfrak{F}(X)$ defined as follows: for every $x \in X$, $f_1 f_2(x) = f_1(f_2(x))$. In other words, we denote the (right-to-left) composition of two functions by juxtaposition. Given $f_1, f_2 \in \mathfrak{F}(X)$, we call $f_1 f_2$ the product between f_1 and f_2. The subset of $\mathfrak{F}(X)$ made up by the bijective functions is denoted by $\mathrm{Sym}(X)$. Under the product of functions $\mathrm{Sym}(X)$ is a finite group called the *symmetric group* on X, whose neutral element is the identity function, denoted by id_X or simply by id.

Fixed $k \in \mathbb{N}$, we denote $\mathrm{Sym}(\{1, \ldots, k\})$ simply by S_k and call its elements *permutations* on k objects. Any notation and basic results for permutations used in the paper are standard (see, for instance, Wielandt (1964) and Rose (1978)).

[4]The use of group theory in social choice theory is not a novelty. Kelly (1991), for instance, discusses the role of symmetry in the arrovian framework through suitable subgroups of the symmetric group. Many results proved within the topological approach to social choices developed by Chichilnisky (1980) require the use of algebraic concepts and in particular that of symmetric group. The geometric approach and the symmetry arguments introduced by Donald Saari to understand paradoxes in voting, have been recently cast in a fully algebraic framework by Daugherty et al. (2009), inaugurating what is now called algebraic voting theory.

[5]When alternatives are two and individual and social preferences can express indifference between them, the problem to count the anonymous and neutral rules was solved by Perry and Powers (2008).

2.2 Linear orders and permutations

Fix $n \in \mathbb{N}$ and let $N = \{1, \dots, n\}$. Denote by $\mathcal{L}(N)$ the set of *linear orders* on N, that is, the set of transitive, complete and antisymmetric binary relations on N. Consider the set of vectors with n distinct components in N given by

$$\mathcal{V}(N) = \left\{ (x_j)_{j=1}^n \in N^n : x_{j_1} = x_{j_2} \Rightarrow j_1 = j_2 \right\},$$

and think the vector $(x_j)_{j=1}^n \in \mathcal{V}(N)$ as a column vector, that is,

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1, \dots, x_n]^T.$$

There is a natural bijection between $\mathcal{V}(N)$ and $\mathcal{L}(N)$ given by function $f : \mathcal{V}(N) \to \mathcal{L}(N)$ which associates with the vector $(x_j)_{j=1}^n \in \mathcal{V}(N)$ the relation

$$\{(x_i, x_j) \in N \times N : i, j \in \{1, \dots, n\}, i \leq j\} \in \mathcal{L}(N).$$

We identify $\mathcal{R} \in \mathcal{L}(N)$ with the vector $f^{-1}(\mathcal{R}) \in \mathcal{V}(N)$. Note also that $|\mathcal{V}(N)| = |\mathcal{L}(N)| = n!$.

Given $\mathcal{R} \in \mathcal{L}(N)$ and $\psi \in S_n$, we define the relation $\psi\mathcal{R} \in \mathcal{L}(N)$ as follows: for every $x, y \in N$, $(x, y) \in \psi\mathcal{R}$ if and only if $(\psi^{-1}(x), \psi^{-1}(y)) \in \mathcal{R}$. It is easily checked that, identifying $\mathcal{R}$ with $(x_j)_{j=1}^n \in \mathcal{V}(N)$, our definition is readable as

$$\psi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \psi(x_1) \\ \vdots \\ \psi(x_n) \end{bmatrix}.$$

Note that, for every $\mathcal{R} \in \mathcal{L}(N)$ and $\psi_1, \psi_2 \in S_n$, $\psi_1\mathcal{R} = \mathcal{R}$ if and only if $\psi_1 = id$, and $(\psi_1\psi_2)\mathcal{R} = \psi_1(\psi_2\mathcal{R})$. As a consequence, for each fixed $\mathcal{R}_0 \in \mathcal{L}(N)$, the map $g : S_n \to \mathcal{L}(N)$ defined by $g(\psi) = \psi\mathcal{R}_0$ is bijective. In particular, for every $\mathcal{R} \in \mathcal{L}(N)$, there exists $\psi \in S_n$ such that $\mathcal{R} = \psi\mathcal{R}_0$.

Given $C \subseteq \mathcal{L}(N)$ and $\psi \in S_n$, let $\psi C = \{\psi\mathcal{R} \in \mathcal{L}(N) : \mathcal{R} \in C\}$. Note that, for every $C \subseteq \mathcal{L}(N)$ and $\psi_1, \psi_2 \in S_n$, $|\psi_1 C| = |C|$, and $(\psi_1\psi_2)C = \psi_1(\psi_2 C)$.

2.3 Individual preferences and rules

From now on, let $h, n \in \mathbb{N}$ with $h, n \geq 2$ be fixed. Let $H = \{1, \dots, h\}$ be the set of individuals and $N = \{1, \dots, n\}$ be the set of alternatives. A *preference* on N is an element of $\mathcal{L}(N)$. Given $p_0 \in \mathcal{L}(N)$ and $x, y \in N$, we say that x is at least as good as y according to p_0, if $(x, y) \in p_0$ and x is preferred to y according to p_0 if $(x, y) \in p_0$ and $(y, x) \notin p_0$. A *preference profile* is an element of $\mathcal{L}(N)^h$. The set $\mathcal{L}(N)^h$ is denoted by $\mathcal{P}$. If $p \in \mathcal{P}$ and $i \in H$, the i-th component of p is denoted by p_i and represents the preference of individual i. Any $p \in \mathcal{P}$ can be identified with the matrix whose i-th column is the column vector representing the i-th component of p. Note that $|\mathcal{P}| = n!^h$.

A *rule* or *social welfare function* is a function from $\mathcal{P}$ to $\mathcal{L}(N)$. The set of all rules is denoted by $\mathcal{F}$.

Consider now the group $G = S_h \times S_n$. For every $(\varphi, \psi) \in G$ and $p \in \mathcal{P}$, define $p^{(\varphi,\psi)} \in \mathcal{P}$ as the preference profile such that, for every $i \in H$,

$$\left(p^{(\varphi,\psi)}\right)_i = \psi p_{\varphi^{-1}(i)}. \tag{3}$$

The profile $p^{(\varphi,\psi)}$ is thus the profile obtained by p as if alternatives and individuals were renamed according to the following rules: for every $i \in H$, individual i is renamed $\varphi(i)$; for every $x \in N$, alternative x is renamed $\psi(x)$. For instance, if $n = 3, h = 5$ and

$$p = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 3 & 3 & 1 & 1 \end{bmatrix}, \quad \varphi = (134)(25), \quad \psi = (12),$$

we have

$$p^{(\varphi,id)} = \begin{bmatrix} 3 & 2 & 3 & 2 & 1 \\ 2 & 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 3 \end{bmatrix}, \quad p^{(id,\psi)} = \begin{bmatrix} 3 & 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 2 & 2 \end{bmatrix}, \quad p^{(\varphi,\psi)} = \begin{bmatrix} 3 & 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 2 & 1 \\ 2 & 2 & 2 & 3 & 3 \end{bmatrix}.$$

Since we have given no meaning to $(p_i)^{(\varphi,\psi)}$ for a single preference $p_i \in \mathcal{L}(N)$, we will write the i-th component of the profile $p^{(\varphi,\psi)}$ simply as $p_i^{(\varphi,\psi)}$, instead of $\big(p^{(\varphi,\psi)}\big)_i$.

A rule F is said *anonymous and neutral* if, for every $p \in \mathcal{P}$ and $(\varphi,\psi) \in G$,

$$F(p^{(\varphi,\psi)}) = \psi F(p),$$

that is, collective decisions are independent on alternative and individual names. The set of anonymous and neutral rules is denoted by $\mathcal{F}^{\mathrm{an}}$.

Given $\nu \in \mathbb{N} \cap (h/2, h]$, let us define, for every $p \in \mathcal{P}$, the set

$$C_\nu(p) = \big\{q_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : (x,y) \in p_i, (y,x) \notin p_i\}| \geq \nu \Rightarrow (x,y) \in q_0, (y,x) \notin q_0\big\},$$

that is, the set of preferences having x preferred to y whenever, according to the preference profile p, at least ν individuals prefer x to y. Note that, for every $\nu, \nu' \in \mathbb{N} \cap (h/2, h]$ with $\nu \leq \nu'$ and $p \in \mathcal{P}$, $C_\nu(p) \subseteq C_{\nu'}(p)$.

A rule F is said a *ν-majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_\nu(p)$. The set of ν-majority rules is denoted by $\mathcal{F}_\nu$. Of course,

$$\mathcal{F}_\nu = \times_{p \in \mathcal{P}} C_\nu(p), \tag{4}$$

and if $\nu, \nu' \in \mathbb{N} \cap (h/2, h]$ with $\nu \leq \nu'$, then $\mathcal{F}_\nu \subseteq \mathcal{F}_{\nu'}$. As already explained in the introduction, our purpose is to investigate which conditions on n, h and ν are necessary and sufficient to get $\mathcal{F}^{\mathrm{an}} \neq \varnothing$, $\mathcal{F}_\nu \neq \varnothing$ and $\mathcal{F}^{\mathrm{an}} \cap \mathcal{F}_\nu \neq \varnothing$.

3 Properties of the set of preference profiles

In the present section we begin an analysis of the structure of the preference profile set $\mathcal{P}$ to be continued in Section 9. As explained in the introduction, our approach follows the one by Eğecioğlu (2009). We start with a basic result which allows to exploit many facts from group theory.

Proposition 1. *The function $f : G \to \mathfrak{F}(\mathcal{P})$ defined, for every $(\varphi,\psi) \in G$, as*

$$f(\varphi,\psi) : \mathcal{P} \to \mathcal{P}, \quad p \mapsto p^{(\varphi,\psi)}, \tag{5}$$

maps G into $\mathrm{Sym}(\mathcal{P})$ *and induces a group homomorphism from G to* $\mathrm{Sym}(\mathcal{P})$.

Proof. First of all, we note that, by definition (3), we have $f(id, id) = id$. Then we show that, for every $(\varphi_1,\psi_1), (\varphi_2,\psi_2) \in G$,

$$f((\varphi_1,\psi_1)(\varphi_2,\psi_2)) = f(\varphi_1,\psi_1) f(\varphi_2,\psi_2), \tag{6}$$

that is, for every $p \in \mathcal{P}$ and (φ_1,ψ_1), $(\varphi_2,\psi_2) \in G$,

$$p^{(\varphi_1\varphi_2,\psi_1\psi_2)} = \left(p^{(\varphi_2,\psi_2)}\right)^{(\varphi_1,\psi_1)}. \tag{7}$$

Indeed, for every $i \in H$ and $(x,y) \in N$, by definition (3), we have

$$(x,y) \in p_i^{(\varphi_1\varphi_2,\psi_1\psi_2)} \quad \Leftrightarrow \quad \big((\psi_1\psi_2)^{-1}(x), (\psi_1\psi_2)^{-1}(x)\big) \in p_{(\varphi_1\varphi_2)^{-1}(i)}$$

$$\Leftrightarrow \quad \left(\psi_2^{-1}\left(\psi_1^{-1}(x)\right), \psi_2^{-1}\left(\psi_1^{-1}(y)\right)\right) \in p_{\varphi_2^{-1}\left(\varphi_1^{-1}(i)\right)} \quad \Leftrightarrow \quad \left(\psi_1^{-1}(x), \psi_1^{-1}(y)\right) \in p^{(\varphi_2,\psi_2)}_{\varphi_1^{-1}(i)}$$

$$\Leftrightarrow \quad (x,y) \in \left(p^{(\varphi_2,\psi_2)}\right)^{(\varphi_1,\psi_1)}_i.$$

As a consequence, for every $(\varphi, \psi) \in G$, we get

$$f(\varphi, \psi)f(\varphi^{-1}, \psi^{-1}) = f(\varphi^{-1}, \psi^{-1})f(\varphi, \psi) = f(id, id) = id.$$

Thus, $f(\varphi, \psi)$ is a function of $\mathcal{P}$ into itself with inverse $f(\varphi^{-1}, \psi^{-1})$, and therefore $f(\varphi, \psi) \in \mathrm{Sym}(\mathcal{P})$. Finally, note that the fact that f is a homomorphism from the group G to the group $\mathrm{Sym}(\mathcal{P})$ is now exactly the content of equality (6). □

According to the language of group theory, the function $f : G \to \mathrm{Sym}(\mathcal{P})$ defined as in (5) is an *action* of the group G on the set $\mathcal{P}$. Given $p \in \mathcal{P}$ and $g \in G$, we write p^g instead of $f(g)(p)$. For every $p \in \mathcal{P}$, the set $\{p^g \in \mathcal{P} : g \in G\}$ is called the *orbit* of p and is denoted by p^G. It is well known that the set of the orbits $\mathcal{P}^G = \{p^G : p \in \mathcal{P}\}$ is a partition[6] of $\mathcal{P}$. We denote the order of $\mathcal{P}^G$ by R. Any vector $(p^j)_{j=1}^R \in \mathcal{P}^R$ such that $\mathcal{P}^G = \{p^{j\,G} : j \in \{1, \dots, R\}\}$, is called *a system of representatives* of the orbits. The set of all the systems of representatives is nonempty and denoted by $\mathfrak{S}$. For every $p \in \mathcal{P}$, the *stabilizer* of p in G is the subgroup of G defined by

$$\mathrm{Stab}_G(p) = \{g \in G : p^g = p\}.$$

It is well known that the order of the orbit p^G can be expressed in terms of the stabilizer of p by

$$|p^G| = \frac{|G|}{|\mathrm{Stab}_G(p)|}, \tag{8}$$

and, in particular, the order of each orbits divides $|G| = n!h!$.

Proposition 2 below is the key ingredient in each existence theorem of the paper and it is proved under the strong coprimality condition $\gcd(h, n!) = 1$. As we will show later, that condition is very natural in the context of anonymous and neutral rules.

Proposition 2. *Let* $\gcd(h, n!) = 1$ *and* $p \in \mathcal{P}$. *Then* $\mathrm{Stab}_G(p) \leq S_h \times \{id\}$.

Proof. Suppose that $g = (\varphi, \psi) \in \mathrm{Stab}_G(p)$ and prove that $\psi = id$. Let $\varphi = \gamma_1 \cdots \gamma_r$ be a decomposition of φ into $r \geq 1$ disjoint cycles $\gamma_j \in S_h$ of order $b_j \geq 1$ with $\sum_{j=1}^r b_j = h$. Since $\mathrm{Stab}_G(p)$ is a subgroup of G, for every $j \in \{1, \dots, r\}$, we also have $g^{b_j} = (\varphi^{b_j}, \psi^{b_j}) \in \mathrm{Stab}_G(p)$. Then, for every $j \in \{1, \dots, r\}$, $p^{(\varphi^{b_j}, \psi^{b_j})} = p$, that is,

$$\psi^{b_j} p_i = p_{\varphi^{b_j}(i)} \quad \text{for all } i \in H.$$

Since φ^{b_j} has at least b_j fixed points, picking one of them, say i_0, we get $\psi^{b_j} p_{i_0} = p_{i_0}$ and thus $\psi^{b_j} = id$. It follows that, for every $j \in \{1, \dots, r\}$, $|\psi| \mid b_j$ and then $|\psi| \mid h$. On the other hand, $\psi \in S_n$ implies $|\psi| \mid n!$, as well. As a consequence, $|\psi| \mid \gcd(h, n!) = 1$ which gives $\psi = id$. □

4 Anonymous and neutral rules

In this section we use the action of $G = S_h \times S_n$ on $\mathcal{P}$ and the properties of its stabilizer, established in Proposition 2, to reach a fundamental result: when $\gcd(h, n!) = 1$, we can construct an anonymous and neutral rule freely assigning its values on a system of representatives of the orbits.

Proposition 3. *Let* $\gcd(h, n!) = 1$. *For every* $(p^j)_{j=1}^R \in \mathfrak{S}$ *and* $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$, *there exists a unique* $F \in \mathcal{F}^{\mathrm{an}}$ *such that, for every* $j \in \{1, \dots, R\}$, $F(p^j) = q_j$.

Proof. Let $(p^j)_{j=1}^R \in \mathfrak{S}$ and $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$. Since $\{p^{j\,G} : j \in \{1, \dots, R\}\}$ is a partition of $\mathcal{P}$, given $p \in \mathcal{P}$, there exist $j \in \{1, \dots, R\}$ and $(\varphi, \psi) \in G$ such that $p = p^{j\,(\varphi,\psi)}$ even though that representation is not necessarily unique. We show that if there exist $j_1, j_2 \in \{1, \dots, R\}$ and $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in G$ such that $p^{j_1\,(\varphi_1,\psi_1)} = p^{j_2\,(\varphi_2,\psi_2)}$, then $j_1 = j_2$ and $\psi_1 = \psi_2$. In fact, by definition of system of representatives, we immediately have that $j_1 = j_2$. Thus, we get $p^{j_1\,(\varphi_1,\psi_1)} = p^{j_1\,(\varphi_2,\psi_2)}$ and equality (7) gives

$$\left(p^{j_1\,(\varphi_1,\psi_1)}\right)^{(\varphi_2^{-1},\psi_2^{-1})} = p^{j_1\,(\varphi_2^{-1}\varphi_1,\psi_2^{-1}\psi_1)} = p^{j_1}.$$

[6] A partition of a nonempty set X is a set of nonempty pairwise disjoint subsets of X whose union is X.

Thus $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1) \in \mathrm{Stab}_G(p^{j_1})$ and, by Proposition 2, we obtain $\psi_2^{-1}\psi_1 = id$, that is, $\psi_1 = \psi_2$.

Consider now the rule F defined, for every $p \in \mathcal{P}$ as $F(p) = \psi q_j$, where $j \in \{1, \ldots, R\}$ and $(\varphi, \psi) \in G$ are such that $p = p^{j\,(\varphi,\psi)}$. Note that, because of the previous remark, this definition is unambiguous. Moreover, for every $j \in \{1, \ldots, R\}$, $F(p^j) = q_j$. Let us prove that $F \in \mathcal{F}^{\mathrm{an}}$. Consider $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$ and let $p = p^{j\,(\varphi_*,\psi_*)}$ for some $j \in \{1, \ldots, R\}$ and $(\varphi_*, \psi_*) \in G$. By definition of F and by the equality (7), we conclude that

$$F(p^{(\varphi,\psi)}) = F\left(\left(p^{j\,(\varphi_*,\psi_*)}\right)^{(\varphi,\psi)}\right) = F(p^{j\,(\varphi\varphi_*,\psi\psi_*)}) = (\psi\psi_*)q_j = \psi(\psi_* q_j) = \psi F(p^{j\,(\varphi_*,\psi_*)}) = \psi F(p).$$

In order to prove the uniqueness of F, it suffices to note that if $F' \in \mathcal{F}^{\mathrm{an}}$ is such that, for every $j \in \{1, \ldots, R\}$, $F'(p^j) = q_j$, then $F'(p^{j\,(\varphi,\psi)}) = \psi q_j = F(p^{j\,(\varphi,\psi)})$ for all $j \in \{1, \ldots, R\}$ and $(\varphi, \psi) \in G$. Thus, for every $p \in \mathcal{P}$, $F'(p) = F(p)$. □

Let $\gcd(h, n!) = 1$, $(p^j)_{j=1}^R \in \mathfrak{S}$ and $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$: we denote by $F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$ the unique $F \in \mathcal{F}^{\mathrm{an}}$ such that, for every $j \in \{1, \ldots, R\}$, $F(p^j) = q_j$.

Proposition 4. *Let* $\gcd(h, n!) = 1$ *and* $(p^j)_{j=1}^R \in \mathfrak{S}$*. Then the function* $f : \mathcal{L}(N)^R \to \mathcal{F}^{\mathrm{an}}$ *defined, for every* $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$*, as* $f\left((q_j)_{j=1}^R\right) = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$*, is bijective. In particular,* $|\mathcal{F}^{\mathrm{an}}| = n!^R$*.*

Proof. Straightforward. □

Theorem 5. $\mathcal{F}^{\mathrm{an}} \neq \varnothing$ *if and only if* $\gcd(h, n!) = 1$*.*

Proof. From Proposition 3, it immediately follows the "if" part. In order to prove the "only if" part, assume $\gcd(h, n!) \neq 1$ and suppose, by contradiction, that there exists $F \in \mathcal{F}^{\mathrm{an}}$. Let c be an integer such that $2 \leq c \leq n$ and $c \mid h$, and let $m = \frac{h}{c}$. Let

$$\psi = (1 \ldots c) \in S_n \quad \text{and} \quad \varphi = (1 \ldots c)(c+1 \ldots 2c) \cdots ((m-1)c + 1 \ldots h) \in S_h.$$

Note that $|\psi| = c$ and, being $c \geq 2$, we have $\psi \neq id$. Note also that φ is a product of m cycles of length c and, for every $i \in H$, c divides the integer $\varphi(i) - (i + 1)$.

Let $p_0 = [1, \ldots, n]^T$ and define $p \in \mathcal{P}$ by $p_i = \psi^{i-1}p_0$ for all $i \in H$. We claim that $p^{(\varphi,\psi)} = p$, that is, for every $i \in H$, $p^{(\varphi,\psi)}_{\varphi(i)} = p_{\varphi(i)}$. In fact, $p^{(\varphi,\psi)}_{\varphi(i)} = \psi p_i = \psi^i p_0$ and $p_{\varphi(i)} = \psi^{\varphi(i)-1}p_0$. Since $|\psi| \mid \varphi(i) - (i + 1)$, we have $\psi^{\varphi(i)-(i+1)} = id$. Then $\psi^i = \psi^{\varphi(i)-1}$ and thus $p^{(\varphi,\psi)}_{\varphi(i)} = p_{\varphi(i)}$. As a consequence, $F(p) = F(p^{(\varphi,\psi)}) = \psi F(p)$, which implies the contradiction $\psi = id$. □

5 Majority rules

We devote this section to explain under which arithmetical conditions on h, n and ν a ν-majority rule exists. As already discussed, the results are not original but it is useful to have them expressed in terms of our notation and inside our framework (for the proofs, we follow Can and Storken, 2013, Example 4).

Define, for every $p \in \mathcal{P}$ and $x, y \in N$, the set

$$H(p, x, y) = \{i \in H : (x, y) \in p_i, (y, x) \notin p_i\},$$

and note that, if $x = y$, then $H(p, x, y) = \varnothing$ while if $x \neq y$, then $H(p, x, y) = \{i \in H : (x, y) \in p_i\}$. Define also, for every $\nu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$, the relation on N

$$\Sigma_\nu(p) = \{(x, y) \in N \times N : |H(p, x, y)| \geq \nu\}.$$

Of course, $C_\nu(p) = \{q_0 \in \mathcal{L}(N) : q_0 \text{ extends } \Sigma_\nu(p)\}$ and, in particular, $C_\nu(p) \neq \varnothing$ if and only if $\Sigma_\nu(p)$ is acyclic.

Proposition 6. *Let* $\nu \in \mathbb{N} \cap (h/2, h]$ *such that* $\nu > \frac{n-1}{n}h$*. Then, for every* $p \in \mathcal{P}$*,* $C_\nu(p) \neq \varnothing$*.*

Proof. We get the proof showing that $\Sigma_\nu(p)$ is acyclic. Assume by contradiction that there exist $l \in \mathbb{N}$ with $l \geq 2$ and distinct $x_1, \dots, x_l \in N$ such that, for every $j \in \{1, \dots, l-1\}$, $(x_j, x_{j+1}) \in \Sigma_\nu(p)$ and $(x_l, x_1) \in \Sigma_\nu(p)$. Note that, in particular, $l \leq n$. Define also $x_{l+1} = x_1$. Then, for every $j \in \{1, \dots, l\}$,

$$|H(p, x_j, x_{j+1})| \geq \nu > \frac{n-1}{n} h \geq \frac{l-1}{l} h. \tag{9}$$

Observe now that if it were $\bigcap_{j=1}^{l} H(p, x_j, x_{j+1}) = \varnothing$ then, for every $i \in H$, we would obtain

$$|\{j \in \{1, \dots, l\} : i \in H(p, x_j, x_{j+1})\}| \leq l - 1.$$

Thus, by (9) we would get

$$(l-1)h < \sum_{j=1}^{l} |H(p, x_j, x_{j+1})| = |\{(j,i) \in \{1, \dots, l\} \times H : i \in H(p, x_j, x_{j+1})\}|$$

$$= \sum_{i \in H} |\{j \in \{1, \dots, l\} : i \in H(p, x_j, x_{j+1})\}| \leq (l-1)h,$$

which is a contradiction. As a consequence, we have $\bigcap_{j=1}^{l} H(p, x_j, x_{j+1}) \neq \varnothing$ and there exists $i^* \in H$ such that, for every $j \in \{1, \dots, l\}$, $(x_j, x_{j+1}) \in p_{i^*}$. Then the linear order p_{i^*} contains a cycle, which is a contradiction. □

Proposition 7. *Let $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu \leq \frac{n-1}{n} h$. Then, there exists $p \in \mathcal{P}$ such that $C_\nu(p) = \varnothing$.*

Proof. Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ with $\nu \leq \frac{n-1}{n} h$. Write $h = qn + r$, where $q, r \in \mathbb{N} \cup \{0\}$ and $r \leq n-1$, and consider $p_0 = [1, \dots, n]^T$. Let $\sigma = (1 \dots n) \in S_n$ and define $p \in \mathcal{P}$ by $p_i = \sigma^i p_0$ for all $i \in H$. First of all, let us prove that, for every $k \in \mathbb{N}$ with $k \leq n$ and $x \in N$,

$$|\{i \in \{1, \dots, k\} : (x, \sigma(x)) \in \sigma^i p_0\}| \geq k - 1. \tag{10}$$

To that purpose, observe first that, for every $y \in N$, $(y, \sigma(y)) \in p_0$ if and only if $y \neq n$, and that, for every $m \in \mathbb{N}$, $\sigma^m(1) = n$ if and only if $m \equiv -1 (\mathrm{mod}\, n)$. Fix now $x \in N$ and $k \leq n$, and let j be the unique element in $\{0, \dots, n-1\}$ such that $x = \sigma^j(1)$. Then

$$\{i \in \{1, \dots, k\} : (x, \sigma(x)) \in \sigma^i p_0\} = \{i \in \{1, \dots, k\} : \big(\sigma^{j-i}(1), \sigma(\sigma^{j-i}(1))\big) \in p_0\}$$

$$= \{i \in \{1, \dots, k\} : \sigma^{j-i}(1) \neq n\} = \{1, \dots, k\} \setminus \{j+1\}$$

that implies (10). Since, for every $i \in H$ with $i \leq h - n$, we have that $p_i = p_{i+n}$, using (10) we get, for every $x \in N$,

$$|H(p, x, \sigma(x))| \geq (n-1)q + \max\{0, r-1\}.$$

However, it is easily checked that

$$(n-1)q + \max\{0, r-1\} = (n-1)q + \left\lfloor \frac{n-1}{n} r \right\rfloor = \left\lfloor \frac{n-1}{n}(qn + r) \right\rfloor \geq \nu.$$

As a consequence, for every $x \in N$, $|H(p, x, \sigma(x))| \geq \nu$ and so the relation $\Sigma_\nu(p)$ contains the cycle $1, 2, \dots, n$, that is, $C_\nu(p) = \varnothing$. □

By the definition of $\mathcal{F}_\nu$ and Propositions 6 and 7, the following result immediately follows.

Theorem 8. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. Then $\mathcal{F}_\nu \neq \varnothing$ if and only if $\nu > \frac{n-1}{n} h$.*

For every $p \in \mathcal{P}$, let us now define

$$\nu(p) = \min\{\nu \in \mathbb{N} \cap (h/2, h] : C_\nu(p) \neq \varnothing\}.$$

Of course, $\nu(p)$ is well defined as, by Proposition 6, for every $p \in \mathcal{P}$, $C_h(p) \neq \varnothing$ and $\nu(p) \in \mathbb{N} \cap (h/2, h]$. A rule F is said a *minimal majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_{\nu(p)}(p)$. In other words, minimal majority rules have the property that, for every preference profile, the corresponding social preference is consistent with all majority thresholds not generating cycles for that profile.

The set of minimal majority rules is denoted by $\mathcal{F}_{\min}$. Thus

$$\mathcal{F}_{\min} = \times_{p \in \mathcal{P}} C_{\nu(p)}(p)$$

and, by definition, $\mathcal{F}_{\min} \neq \varnothing$ independently on any arithmetic condition on h and n. Note that, for every $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu > \frac{n-1}{n} h$, we have $\mathcal{F}_{\min} \subseteq \mathcal{F}_\nu \subseteq \mathcal{F}_h$.

6 Anonymous and neutral majority rules

In this section we characterize in terms of h, n and ν when the sets $\mathcal{F}_\nu^{\mathrm{an}} = \mathcal{F}^{\mathrm{an}} \cap \mathcal{F}_\nu$ and $\mathcal{F}_{\min}^{\mathrm{an}} = \mathcal{F}^{\mathrm{an}} \cap \mathcal{F}_{\min}$ are nonempty, and give a method to construct and, in particular, count all of their elements.

Lemma 9. *For every $p \in \mathcal{P}$, $x, y \in N$ and $(\varphi, \psi) \in G$, $H(p^{(\varphi,\psi)}, x, y) = \varphi\left(H(p, \psi^{-1}(x), \psi^{-1}(y))\right)$ and $|H(p^{(\varphi,\psi)}, x, y)| = |H(p, \psi^{-1}(x), \psi^{-1}(y))|$.*

Proof. The first relation is trivially verified when $x = y$. If instead $x \neq y$, then we have that

$$i \in H(p^{(\varphi,\psi)}, x, y) \Leftrightarrow (x, y) \in p_i^{(\varphi,\psi)} = \psi p_{\varphi^{-1}(i)} \Leftrightarrow (\psi^{-1}(x), \psi^{-1}(y)) \in p_{\varphi^{-1}(i)}$$

$$\Leftrightarrow \varphi^{-1}(i) \in H(p, \psi^{-1}(x), \psi^{-1}(y)) \Leftrightarrow i \in \varphi\left(H(p, \psi^{-1}(x), \psi^{-1}(y))\right).$$

The second relation immediately follows by the fact that $\varphi \in S_h$. □

Lemma 10. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. Then, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, $C_\nu(p^{(\varphi,\psi)}) = \psi C_\nu(p)$ and, $|C_\nu(p^{(\varphi,\psi)})| = |C_\nu(p)|$. In particular, $\nu(p) = \nu(p^{(\varphi,\psi)})$.*

Proof. We prove first that, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$,

$$\psi C_\nu(p) \subseteq C_\nu(p^{(\varphi,\psi)}). \tag{11}$$

Let us consider then $p \in \mathcal{P}$, $(\varphi, \psi) \in G$, $q_0 \in C_\nu(p)$ and show that $\psi q_0 \in C_\nu(p^{(\varphi,\psi)})$. Assume that $x, y \in N$ are such that $|H(p^{(\varphi,\psi)}, x, y)| \geq \nu$. By Lemma 9, we have $|H(p, \psi^{-1}(x), \psi^{-1}(y))| \geq \nu$ and since $q_0 \in C_\nu(p)$, we have $(\psi^{-1}(x), \psi^{-1}(y)) \in q_0$, that is, $(x, y) \in \psi q_0$. Then $\psi q_0 \in C_\nu(p^{(\varphi,\psi)})$.

We are left to show that, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, we also have $\psi C_\nu(p) \supseteq C_\nu(p^{(\varphi,\psi)})$, that is, $\psi^{-1} C_\nu(p^{(\varphi,\psi)}) \subseteq C_\nu(p)$. This is immediately seen because, by (11) and by (7), we have

$$\psi^{-1} C_\nu(p^{(\varphi,\psi)}) \subseteq C_\nu\left(\left(p^{(\varphi,\psi)}\right)^{(\varphi^{-1},\psi^{-1})}\right) = C_\nu(p).$$

Finally, since $|C_\nu(p)| = |\psi C_\nu(p)|$ and $C_\nu(p^{(\varphi,\psi)}) = \psi C_\nu(p)$, we also get $|C_\nu(p^{(\varphi,\psi)})| = |C_\nu(p)|$. In particular $\nu(p) = \nu(p^{(\varphi,\psi)})$. □

Proposition 11. *Let $\gcd(h, n!) = 1$, $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu > \frac{n-1}{n} h$, and $(p^j)_{j=1}^R \in \mathfrak{S}$. Then the function $f : \times_{j=1}^R C_\nu(p^j) \to \mathcal{F}^{\mathrm{an}}$ defined, for every $(q_j)_{j=1}^R \in \times_{j=1}^R C_\nu(p^j)$, as $f\left((q_j)_{j=1}^R\right) = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$ has nonempty domain, is injective and its image is $\mathcal{F}_\nu^{\mathrm{an}}$. In particular, $|\mathcal{F}_\nu^{\mathrm{an}}| = \prod_{j=1}^R |C_\nu(p^j)| \geq 1$.*

Proof. By Proposition 6, the condition $\nu > \frac{n-1}{n} h$ implies that f has nonempty domain; by Proposition 3, the definition of f is well posed; the injectivity of f is trivial.

Let us prove now that $\mathrm{Im}(f) \subseteq \mathcal{F}_\nu^{\mathrm{an}}$. Let $F \in \mathrm{Im}(f)$ and show that $F \in \mathcal{F}_\nu^{\mathrm{an}}$. Surely $F \in \mathcal{F}^{\mathrm{an}}$ and there exists $(q_j)_{j=1}^R \in \times_{j=1}^R C_\nu(p^j)$ such that $F = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$. Consider now any $p \in \mathcal{P}$. Then there are $j \in \{1, \ldots, R\}$ and $(\varphi, \psi) \in G$ such that $p = p^{j\,(\varphi,\psi)}$. As, for every $j \in \{1, \ldots, R\}$, we know that $F(p^j) \in C_\nu(p^j)$, using Lemma 10 we have that

$$F(p) = F(p^{j\,(\varphi,\psi)}) = \psi F(p^j) \in \psi C_\nu(p^j) = C_\nu(p^{j\,(\varphi,\psi)}) = C_\nu(p).$$

Then, for every $p \in \mathcal{P}$, $F(p) \in C_\nu(p)$, that is, $F \in \mathcal{F}_\nu$.

In order to prove that $\mathcal{F}_\nu^{\mathrm{an}} \subseteq \mathrm{Im}(f)$, let $F \in \mathcal{F}_\nu^{\mathrm{an}}$ and define, for every $j \in \{1 \ldots, R\}$, $q_j = F(p^j)$. Then we immediately have $(q_j)_{j=1}^R \in \times_{j=1}^R C_\nu(p^j)$ and $F = f\left((q_j)_{j=1}^R\right)$, so that $F \in \mathrm{Im}(f)$. □

Theorem 12. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. Then $\mathcal{F}_\nu^{\mathrm{an}} \neq \varnothing$ if and only if $\gcd(h, n!) = 1$ and $\nu > \frac{n-1}{n} h$.*

Proof. The "if" part follows from Proposition 11. The "only if" part follows instead from Theorems 5 and 8. □

Proposition 13. *Let $\gcd(h, n!) = 1$ and $(p^j)_{j=1}^R \in \mathfrak{S}$. Then the function $f : \times_{j=1}^R C_{\nu(p^j)}(p^j) \to \mathcal{F}^{\mathrm{an}}$ defined, for every $(q_j)_{j=1}^R \in \times_{j=1}^R C_{\nu(p^j)}(p^j)$, as $f\left((q_j)_{j=1}^R\right) = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$ has nonempty domain, is injective and its image is $\mathcal{F}_{\min}^{\mathrm{an}}$. In particular, $|\mathcal{F}_{\min}^{\mathrm{an}}| = \prod_{j=1}^R |C_{\nu(p^j)}(p^j)| \geq 1$.*

Proof. Since $C_{\nu(p)}(p) \neq \varnothing$ for all $p \in \mathcal{P}$, we immediately get that f has nonempty domain; the injectivity of f is trivial. In order to prove that $\mathrm{Im}(f) = \mathcal{F}^{\mathrm{an}}_{\min}$ we have only to observe that Lemma 10 implies that, for all $j \in \{1, \dots, R\}$ and $p \in p^{j\,G}$, $\nu(p) = \nu(p^j)$. Then, we conclude by the same argument used in the proof of Proposition 11. □

Theorem 14. $\mathcal{F}^{\mathrm{an}}_{\min} \neq \varnothing$ *if and only if* $\gcd(h, n!) = 1$.

Proof. The "if" part follows from Proposition 13. The "only if" part follows instead from Theorem 5. □

7 Anonymous and neutral majority social choice functions

The last existence result of the paper is about social choice functions: Theorem 15 below is an immediate consequence of Theorem 12, and generalizes Theorem 1 in Moulin (1983, p.23). We recall that a *social choice function* is a function $f : \mathcal{P} \to N$; it is called *anonymous and neutral* if, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, $f(p^{(\varphi,\psi)}) = \psi(f(p))$; given $\nu \in \mathbb{N} \cap (h/2, h]$, it is called a *$\nu$-majority* if, for every $p \in \mathcal{P}$ and $x \in N$, we have $|H(p, x, f(p))| < \nu$.

Theorem 15. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. There exists an anonymous and neutral ν-majority social choice function if and only if* $\gcd(h, n!) = 1$ *and* $\nu > \frac{n-1}{n} h$.

Proof. The "only if" part follows Theorem 1 in Moulin (1983, p.23) and Corollary 3 in Greenberg (1979). In order to prove the "if" part assume that $\gcd(h, n!) = 1$ and $\nu > \frac{n-1}{n} h$ and, using Theorem 12, consider $F \in \mathcal{F}^{\mathrm{an}}_{\nu}$. Consider then the social choice function $f : \mathcal{P} \to N$ defined as follows: for every $p \in \mathcal{P}$, $f(p)$ is the unique maximum of the linear order $F(p)$. It is immediate to verify that f is an anonymous and neutral ν-majority social choice function. □

8 General formulas for counting the rules

Summarizing the content of theorems and propositions proved in Sections 4, 5 and 6, we get the following proposition which provides formulas for the order of all the sets of rules defined along the paper.

Proposition 16. *Let $\nu \in \mathbb{N} \cap (h/2, h]$ and $(p^j)_{j=1}^R \in \mathfrak{S}$. Then*

$$|\mathcal{F}| = n!^{(n!^h)}, \tag{16.1}$$

$$|\mathcal{F}_\nu| = \textstyle\prod_{j=1}^R |C_\nu(p^j)|^{|p^{j\,G}|}, \tag{16.2}$$

$$|\mathcal{F}_{\min}| = \textstyle\prod_{j=1}^R |C_{\nu(p^j)}(p^j)|^{|p^{j\,G}|}, \tag{16.3}$$

$$|\mathcal{F}^{\mathrm{an}}| = \begin{cases} n!^R & \text{if } \gcd(h, n!) = 1 \\ 0 & \text{if } \gcd(h, n!) \neq 1 \end{cases}, \tag{16.4}$$

$$|\mathcal{F}^{\mathrm{an}}_\nu| = \begin{cases} \prod_{j=1}^R |C_\nu(p^j)| & \text{if } \gcd(h, n!) = 1 \text{ and } \nu > \frac{n-1}{n} h \\ 0 & \text{if } \gcd(h, n!) \neq 1 \text{ or } \nu \leq \frac{n-1}{n} h \end{cases}, \tag{16.5}$$

$$|\mathcal{F}^{\mathrm{an}}_{\min}| = \begin{cases} \prod_{j=1}^R |C_{\nu(p^j)}(p^j)| & \text{if } \gcd(h, n!) = 1 \\ 0 & \text{if } \gcd(h, n!) \neq 1 \end{cases}. \tag{16.6}$$

Proof. In order to get (16.1), simply note that $|\mathcal{P}| = n!^h$. By definition (4) of $\mathcal{F}_\nu$ and Lemma 10, we have

$$|\mathcal{F}_\nu| = \prod_{p \in \mathcal{P}} |C_\nu(p)| = \prod_{j=1}^R \prod_{p \in p^j G} |C_\nu(p)| = \prod_{j=1}^R |C_\nu(p^j)|^{|p^j G|}$$

and (16.2) is proved. An analogous argument proves (16.3). Formula (16.4) follows from Proposition 4 and Theorem 5. Formula (16.5) follows from Proposition 11 and Theorem 12. Finally, (16.6) follows from Proposition 13 and Theorem 14. □

9 Further properties of the set of preference profiles

When $\gcd(h, n!) = 1$, we can construct and count rules via Propositions 4, 11, 13, and 16. In order to apply those results, a system of representatives is needed but its computation is hard in general. In this section we are going to prove some further properties of the set of preference profiles that are useful to simplify that task.

Given $k \in \mathbb{N}$, we define the set

$$\Pi(k) = \bigcup_{r=1}^{k} \left\{ (b_j)_{j=1}^{r} \in \mathbb{N}^r : \sum_{j=1}^{r} b_j = k,\, b_1 \geq \ldots \geq b_r \right\}$$

whose elements are called *partitions* of k. In other words, a partition of k is a decreasing list of positive integers whose sum is k. We call each component of $b \in \Pi(k)$ a *part* of b. Given $b \in \Pi(k)$, $r(b)$ denotes the number of parts of b and, for every $j \in \{1, \ldots, k\}$, $a_j(b)$ counts how many parts of b are equal to j. Observe that $\sum_{j=1}^{k} a_j(b) = r(b)$ and $\sum_{j=1}^{k} j a_j(b) = k$. If $m \in \mathbb{N}$, we also define the set $\Pi_m(k) = \{b \in \Pi(k) : r(b) \leq m\}$. Note that $m \geq k$ implies $\Pi_m(k) = \Pi(k)$.

Given now $b \in \Pi_{n!}(h)$, we say that $p \in \mathcal{P}$ has *block type* b if there exist $B_1, \ldots, B_{r(b)} \subseteq H$ and distinct $q_1, \ldots, q_{r(b)} \in \mathcal{L}(N)$ such that:

- $\{B_k\}_{k=1}^{r(b)}$ is a partition of H,
- for every $k \in \{1 \ldots, r(b)\}$, $|B_k| = b_k$,
- for every $i \in H$ and $k \in \{1 \ldots, r(b)\}$, $p_i = q_k$ if and only if $i \in B_k$.

The set of preference profiles having block type b is denoted by $\mathcal{P}(b)$. It is immediate to prove that $\{\mathcal{P}(b) : b \in \Pi_{n!}(h)\}$ has order $|\Pi_{n!}(h)|$ and is a partition of $\mathcal{P}$. Moreover, for every $b \in \Pi_{n!}(h)$ and $p \in \mathcal{P}(b)$, we have that $p^G \subseteq \mathcal{P}(b)$. In particular, $\mathcal{P}(b)$ is union of orbits.

As shown by the following propositions, the concept of block type allows to get a deeper insight into the properties of the orbits. In fact, Proposition 17 shows that the order of the orbit of a preference profile depends only on its block type. Proposition 18 provides instead a formula for counting the orbits contained in the set of preference profiles having the same block type. The proofs of those propositions are in the Appendix.

Proposition 17. *Let* $\gcd(h, n!) = 1$*,* $b \in \Pi_{n!}(h)$ *and* $p \in \mathcal{P}(b)$*. Then*

$$|p^G| = \frac{n!\, h!}{\prod_{j=1}^{h} j!^{\,a_j(b)}}. \tag{12}$$

Proposition 18. *Let* $\gcd(h, n!) = 1$ *and* $b \in \Pi_{n!}(h)$*. Then the number of orbits in* $\mathcal{P}(b)$ *is*

$$\frac{\binom{n!}{r(b)} r(b)!}{n! \prod_{j=1}^{h} a_j(b)!}.$$

Let us propose now a simple formula for the number R of orbits under the assumption $\gcd(h, n!) = 1$. That formula, already proved by Eğecioğlu (2009, Section 4.4)[7], is obtained here in a different manner, using Proposition 18 and the following equality, holding for every $m, k \in \mathbb{N}$:

$$\binom{m+k-1}{k-1} = \sum_{b \in \Pi_k(m)} \frac{\binom{k}{r(b)} r(b)!}{\prod_{j=1}^{m} a_j(b)\,!}. \tag{13}$$

Equality (13) is an original arithmetic relation linking a classical balls-in-boxes counting with partitions. We leave its proof in the Appendix.

Proposition 19. *Let* $\gcd(h, n!) = 1$*. Then the number* R *of orbits in* $\mathcal{P}$ *is*

$$\frac{1}{n!} \binom{h+n!-1}{n!-1}. \tag{14}$$

[7] Actually, Eğecioğlu (2009) claims that the formula for R was previously proved by Giritligil and Doğan (An Impossibility Result on Anonymous and Neutral Social Choice Functions, preprint). However, we could not find that reference.

Proof. Since $\{\mathcal{P}(b) : b \in \Pi_{n!}(h)\}$ is a partition of $\mathcal{P}$, by Proposition 18, we have that

$$R = \frac{1}{n!} \sum_{b \in \Pi_{n!}(h)} \frac{\binom{n!}{r(b)} r(b)!}{\prod_{j=1}^{h} a_j(b)\,!}.$$

Applying equality (13) with $k = n!$ and $m = h$, we immediately get (14). □

10 Some applications

In the present section we show how the results stated in the previous sections can be concretely applied. We focus on some particular values of n and h satisfying the condition $\gcd(h, n!) = 1$.

Assume first $n = 2$ and h odd. Let $k \in \mathbb{N}$ be such that $h = 2k + 1$ and note that $\mathbb{N} \cap (h/2, h] = \{k + 1, \ldots, h\}$ and

$$\Pi_{2!}(h) = \{(k + j, k + 1 - j) : j \in \{1, \ldots, k\}\} \cup \{(2k + 1)\}.$$

In particular, $|\Pi_{2!}(h)| = k + 1$. On the other hand, by Proposition 19, the number of orbits is $R = k + 1$. As a consequence, for every $b \in \Pi_{2!}(h)$, $\mathcal{P}(b)$ is made up by exactly one orbit so that we can build a system of representatives of the orbits simply choosing, for every $b \in \Pi_{2!}(h)$, a preference profile having block type b. Given $j \in \{1, \ldots, k + 1\}$, let $p^j \in \mathcal{P}$ be such that, for every $i \in \{1, \ldots, h\}$,

$$p_i^j = \begin{cases} [1, 2]^T & \text{if } i \le k + j \\ [2, 1]^T & \text{if } i > k + j \end{cases}$$

Note that the block type of p^j is $(k+j, k+1-j)$ if $j \le k$, while it is $(2k+1)$ if $j = k+1$. As a consequence, $(p^j)_{j=1}^{k+1}$ is a system of representatives of the orbits. Let us fix now $j \in \{1, \ldots, k + 1\}$. Using Proposition 17, we get $|p^{j\,G}| = \frac{2(2k+1)!}{(k+j)!(k+1-j)!}$. Moreover, for every $\nu \in \{k + 1, \ldots, h\}$, $C_\nu(p^j) = \mathcal{L}(\{1, 2\})$ if $j < \nu - k$, while $C_\nu(p^j) = \left\{[1, 2]^T\right\}$ if $j \ge \nu - k$. In particular, $\nu(p^j) = k + 1$. Then, applying Proposition 16, we finally obtain

$$|\mathcal{F}| = 2^{(2^{2k+1})}, \quad |\mathcal{F}_\nu| = \begin{cases} 1 & \text{if } \nu = k + 1 \\ \prod_{j=1}^{\nu-k-1} 2^{\frac{2(2k+1)!}{(k+j)!(k+1-j)!}} & \text{if } \nu \ge k + 2 \end{cases}, \quad |\mathcal{F}_{\min}| = 1,$$

$$|\mathcal{F}^{\text{an}}| = 2^{k+1}, \quad |\mathcal{F}^{\text{an}}_\nu| = 2^{\nu-k-1}, \quad |\mathcal{F}^{\text{an}}_{\min}| = 1.$$

Note also that, $\mathcal{F}_{k+1} = \mathcal{F}_{\min} = \mathcal{F}^{\text{an}}_{k+1} = \mathcal{F}^{\text{an}}_{\min} = \{F_{\text{maj}}\}$, where F_{maj} is the simple majority rule.

Assume now $n = 3$ and $h = 5$. We have that $N = \{1, 2, 3\}$, $\mathbb{N} \cap (\frac{h}{2}, h] = \{3, 4, 5\}$ and $\nu > \frac{n-1}{n}h$ if and only if $\nu \in \{4, 5\}$. Moreover,

$$\Pi_{3!}(5) = \Pi(5) = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\}.$$

By Proposition 19, we know that the number of orbits in $\mathcal{P}$ is 42, and by Propositions 17 and 18 we have that:

- $\mathcal{P}(5)$ is one orbit of order 6,
- $\mathcal{P}(4, 1)$ is the union of 5 orbits of order 30,
- $\mathcal{P}(3, 2)$ is the union of 5 orbits of order 60,
- $\mathcal{P}(3, 1, 1)$ is the union of 10 orbits of order 120,
- $\mathcal{P}(2, 2, 1)$ is the union of 10 orbits of order 180,
- $\mathcal{P}(2, 1, 1, 1)$ is the union of 10 orbits of order 360,
- $\mathcal{P}(1, 1, 1, 1, 1)$ is one orbit of order 720.

Consider now the following preference profiles

$$p^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, p^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, p^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, p^4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix},$$

$$p^5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, p^6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, p^7 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, p^8 = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix},$$

$$p^9 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 2 & 2 \end{bmatrix}, p^{10} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix}, p^{11} = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 2 & 2 \end{bmatrix}, p^{12} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix},$$

$$p^{13} = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, p^{14} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, p^{15} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, p^{16} = \begin{bmatrix} 1 & 1 & 1 & 3 & 1 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix},$$

$$p^{17} = \begin{bmatrix} 1 & 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix}, p^{18} = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, p^{19} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 2 & 1 \end{bmatrix}, p^{20} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 3 & 2 & 2 \end{bmatrix},$$

$$p^{21} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, p^{22} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 1 & 2 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, p^{23} = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, p^{24} = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 1 & 1 & 3 \end{bmatrix},$$

$$p^{25} = \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 1 & 1 & 2 \end{bmatrix}, p^{26} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 & 1 \\ 3 & 3 & 2 & 2 & 3 \end{bmatrix}, p^{27} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 3 & 2 \\ 3 & 3 & 2 & 2 & 1 \end{bmatrix}, p^{28} = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 1 \\ 3 & 3 & 1 & 1 & 3 \end{bmatrix},$$

$$p^{29} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & 2 \\ 3 & 3 & 1 & 1 & 1 \end{bmatrix}, p^{30} = \begin{bmatrix} 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 1 & 1 & 2 \\ 3 & 3 & 2 & 2 & 1 \end{bmatrix}, p^{31} = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 2 & 2 & 1 \end{bmatrix}, p^{32} = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 & 3 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix},$$

$$p^{33} = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix}, p^{34} = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, p^{35} = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 3 & 3 \\ 3 & 3 & 3 & 2 & 1 \end{bmatrix}, p^{36} = \begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 3 & 3 & 2 & 2 \end{bmatrix},$$

$$p^{37} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, p^{38} = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 1 & 2 & 1 \end{bmatrix}, p^{39} = \begin{bmatrix} 1 & 1 & 3 & 1 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 1 & 2 & 2 \end{bmatrix}, p^{40} = \begin{bmatrix} 1 & 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 1 & 1 & 2 \end{bmatrix},$$

$$p^{41} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & 1 \\ 3 & 3 & 2 & 1 & 2 \end{bmatrix}, p^{42} = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 3 \\ 3 & 3 & 1 & 2 & 1 \end{bmatrix}.$$

Using the fact that two preference profiles having different block type cannot be in the same orbit and simple algebraic arguments, it can be proved that:

- p^1 is a representative of the unique orbit in $\mathcal{P}(5)$,
- p^j for $2 \leq j \leq 6$ are representatives of the 5 orbits in $\mathcal{P}(4,1)$,
- p^j for $7 \leq j \leq 11$ are representatives of the 5 orbits in $\mathcal{P}(3,2)$,
- p^j for $12 \leq j \leq 21$ are representatives of the 10 orbits in $\mathcal{P}(3,1,1)$,
- p^j for $22 \leq j \leq 31$ are representatives of the 10 orbits in $\mathcal{P}(2,2,1)$,
- p^j for $32 \leq j \leq 41$ are representatives of the 10 orbits in $\mathcal{P}(2,1,1,1)$,
- p^{42} is a representative of the unique orbit in $\mathcal{P}(1,1,1,1,1)$.

In order to give to the reader the possibility to check what affirmed above, we illustrate the strategy to show that the p^j, for $22 \leq j \leq 31$, are representatives of the 10 orbits in $\mathcal{P}(2,2,1)$. The argument is analogous for the other cases. First of all, let $p_0 = [1,2,3]^T$ and write all the columns of p^j in the form σp_0 for suitable $\sigma \in S_3$. We find the following equalities:

$$p^{22} = [p_0, p_0, (12)p_0, (12)p_0, (13)p_0]\,, \qquad p^{23} = [p_0, p_0, (12)p_0, (12)p_0, (23)p_0]$$

$$p^{24} = [p_0, p_0, (13)p_0, (13)p_0, (12)p_0]\,, \qquad p^{25} = [p_0, p_0, (13)p_0, (13)p_0, (23)p_0]$$

$$p^{26} = [p_0, p_0, (23)p_0, (23)p_0, (12)p_0]\,, \qquad p^{27} = [p_0, p_0, (23)p_0, (23)p_0, (13)p_0]$$

$$p^{28} = [p_0, p_0, (123)p_0, (123)p_0, (12)p_0]\,, \quad p^{29} = [p_0, p_0, (123)p_0, (123)p_0, (13)p_0]$$

$$p^{30} = [p_0, p_0, (132)p_0, (132)p_0, (13)p_0]\,, \quad p^{31} = [p_0, p_0, (132)p_0, (132)p_0, (123)p_0]$$

Of course, we get the desired result proving that, for every $22 \leq j < k \leq 31$, $p^k \notin p^{j\,G}$. Let us describe only the case $k = 31$, being the other cases similar. Assume, by contradiction, that there exist $(\varphi, \psi) \in G = S_5 \times S_3$ and $22 \leq j < 31$ such that $p^{31} = p^{j\,(\varphi,\psi)}$. Then, it has to be $\psi p^j_5 = (123)p_0$. As $\{p^j_5 : 22 \leq j < 31\} = \{(12)p_0, (13)p_0, (23)p_0\}$, it is immediate to prove that $\psi \in \{(12), (13), (23)\}$. As a consequence, since p_0 appears twice in each p^j, at least one among $(12)p_0$, $(13)p_0$, and $(23)p_0$ should appear twice in p^{31}, but that is not the case and the contradiction is found. Let us finally note that the strategy here described can also be used to build a system of representatives of the orbits. Indeed, it is exactly what we used to determine $p^1, \ldots, p^{42}$.

For every $j \in \{1, \ldots, 42\}$ and $\nu \in \{3, 4, 5\}$, the computation of the set $C_\nu(p^j)$ is now easy and mechanical. Again, for the sake of clarity, we explain how to compute $C_\nu(p^j)$ in one case, namely, $j = 20$ and $\nu = 4$. We observe that in p^{20} we have that 1 is preferred to 2 four times, 1 is preferred to 3 five times, and 2 is preferred to 3 three times. Thus, the 4-majority condition applies only to 1 with respect to 2 and to 1 with respect to 3, imposing us to find the set of linear orders $q_0 \in \mathcal{L}(N)$ such that $(1,2), (1,3) \in q_0$. Thus, we get $C_4(p^{20}) = \{[1,2,3]^T, [1,3,2]^T\}$.

The following table describes the results of those computations. Of course, in agreement with Propositions 6 and 7, $C_4(p^j) \neq \varnothing$ and $C_5(p^j) \neq \varnothing$ for all j, while $C_3(p^j) = \varnothing$ for some j.

	C_3	C_4	C_5
p^1	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$
p^2	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$
p^3	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^4	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$
p^5	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$
p^6	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$
p^7	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$
p^8	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^9	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$
p^{10}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$
p^{11}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$
p^{12}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{13}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$
p^{14}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$
p^{15}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{16}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{17}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{18}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{19}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{20}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$
p^{21}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{22}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{23}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$
p^{24}	$\{[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{25}	$\{[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{26}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$
p^{27}	$\{[1,3,2]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{28}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$
p^{29}	$\{[2,3,1]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{30}	$\{[3,1,2]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{31}	$\varnothing$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{32}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{33}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{34}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{35}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{36}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{37}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{38}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{39}	$\{[1,3,2]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{40}	$\varnothing$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{41}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{42}	$\{[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$

By Proposition 13, we know that selecting a rule in $\mathcal{F}^{\text{an}}_{\min}$ is equivalent to choosing, for every j, an element in $C_{\nu(p^j)}(p^j)$ representing the social outcome associated with p^j. Of course, for those j such that $C_{\nu(p^j)}(p^j)$ is a singleton, that choice is completely determined. Then, as the table shows, if five individuals desire to strict rank three alternatives using an anonymous and neutral minimal majority rule, then they only need to find an agreement on which element in $C_{\nu(p^j)}(p^j)$ has to be associated with p^j for all j such that $|C_{\nu(p^j)}(p^j)|\geq 2$, that is, for $j\in\{31,40\}$. Similar considerations also hold for the sets $\mathcal{F}^{\text{an}}_4$ and $\mathcal{F}^{\text{an}}_5$, using now Proposition 11. Note that there are 33 values of j such that $|C_4(p^j)|\geq 2$, and 41 values of j such that $|C_5(p^j)|\geq 2$.

By Proposition 16 and checking on the table, we also get

$$|\mathcal{F}| = 6^{7776}, \quad |\mathcal{F}_5| = 2^{6690}3^{7590}, \quad |\mathcal{F}_4| = 2^{4740}3^{5460}, \quad |\mathcal{F}_3| = 0, \quad |\mathcal{F}_{\min}| = 2^{360}3^{540},$$

$$|\mathcal{F}^{\rm an}| = 6^{42}, \quad |\mathcal{F}^{\rm an}_5| = 2^{31}3^{37}, \quad |\mathcal{F}^{\rm an}_4| = 2^{21}3^{20}, \quad |\mathcal{F}^{\rm an}_3| = 0, \quad |\mathcal{F}^{\rm an}_{\min}| = 18.$$

11 Concluding comments

We dealt with the problem to understand whether the members of a committee could strict rank a given family of alternatives obeying the principles of anonymity, neutrality and majority. The existence results presented in the paper are proved thanks to a preliminary theoretical analysis of the set of preference profiles developed with some tools from group theory. Indeed, principles of anonymity and neutrality are naturally associated with an action of the group $G = S_h \times S_n$ on the set of preference profiles.

We firmly believe that the way we apply group theory for dealing with anonymous and neutral rules could be developed into further interesting directions. Consider, for instance, a collective decision problem where a committee is divided into two or more sub-committees whose members are known and impartially treated within or where one or more alternatives are favourite or distinguished. Situations characterized by partial anonymity and partial neutrality can be analysed via the action of a suitable subgroup U of G on the set of preference profiles. As Proposition 1 can be proved replacing G by any of its subgroups, most of the algebraic machinery we used still works in the new framework. Given the number of individuals and alternatives, not necessarily satisfying the coprimality condition (1), an interesting problem is then to understand which extent of anonymity and neutrality is allowed. Using our approach that corresponds to find the subgroups U of G for which there exists a rule F such that, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in U$, $F(p^{(\varphi,\psi)}) = \psi F(p)$.

Note also that in our paper individual and social preferences are modelled as linear orders. However, Proposition 1 can be easily adapted to the interesting case where indifference is allowed and preferences are modelled as orders. Then, even though new tools and methods seem to be required to fully treat the new setting, some parts of the theory we developed can be used to carry on that analysis.

We finally observe that our paper has an essentially constructive flavour as particularly evidenced by the analysis of the case with three alternatives and five individuals. That suggests the possibility to implement algorithmically our results in order to explicitly determine and use the various kinds of rules.

Appendix

Proof of Proposition 17. Since $p \in \mathcal{P}(b)$ we know that there are a partition $B = \{B_k\}_{k=1}^{r(b)}$ of H with $|B_k| = b_k$ and distinct $q_1, \dots q_{r(b)} \in \mathcal{L}(N)$ such that, for every $i \in H$ and $k \in \{1 \dots, r(b)\}$,

$$p_i = q_k \quad \text{if and only if} \quad i \in B_k. \tag{15}$$

Consider the subgroup U of S_h which leaves fixed each B_k, that is,

$$U = \{\varphi \in S_h : \forall k \in \{1, \dots, r(b)\}, \varphi(B_k) = B_k\}\,.$$

Clearly U is isomorph to $\times_{k=1}^{r(b)}\mathrm{Sym}(B_k)$ and then also to $\times_{j=1}^{h} S_j^{a_j(b)}$.

We show that $\mathrm{Stab}_G(p) = U \times \{id\}$. Let $(\varphi, id) \in U \times \{id\}$ and see that $p^{(\varphi,id)} = p$, that is, for every $i \in H$, $p_{\varphi(i)} = p_i$. Since B is a partition of H, we can verify that, for every $k \in \{1, \dots, r(b)\}$ and $i \in B_k$, $p_{\varphi(i)} = p_i$. But if $i \in B_k$ also $\varphi(i) \in B_k$, by definition of U, and thus $p_i = q_k = p_{\varphi(i)}$, by (15). Next let $g \in \mathrm{Stab}_G(p)$. Then, by Proposition 2, we have that $g = (\varphi, id)$ for some $\varphi \in S_h$ and thus $p_{\varphi(i)} = p_i$ for all $i \in H$. We need to see that $\varphi \in U$, that is, $\varphi(B_k) \subseteq B_k$ for all $k \in \{1, \dots, r(b)\}$. But if $i \in B_k$ we have $p_i = q_k$ and thus also $p_{\varphi(i)} = q_k$, which by (15) gives $\varphi(i) \in B_k$.

It follows that

$$|\mathrm{Stab}_G(p)| = |U| = \left|\times_{j=1}^{h} S_j^{a_j(b)}\right| = \prod_{j=1}^{h} j!^{\,a_j(b)}$$

which, by (8), gives immediately (12). □

Proof of Proposition 18. For every $p \in \mathcal{P}(b)$ and $j \in \{1, \ldots, h\}$, define

$$Q_j(p) = \left\{q_0 \in \mathcal{L}(N) : |\{i \in H : p_i = q_0\}| = j\right\}.$$

It is immediately observed that, for every $p \in \mathcal{P}(b)$, $j \in \{1, \ldots, h\}$ and $(\varphi, \psi) \in G$,

$$Q_j(p^{(\varphi,\psi)}) = \psi Q_j(p). \tag{16}$$

Moreover, for every $p \in \mathcal{P}(b)$, define $Q(p) = (Q_j(p))_{j=1}^h$ and note that it belongs to the set

$$\mathcal{Q}(b) = \left\{(Q_j)_{j=1}^h : \forall j, k \in \{1, \ldots, h\} \text{ with } j \neq k, Q_j \subseteq \mathcal{L}(N), |Q_j| = a_j(b), Q_j \cap Q_k = \varnothing\right\}.$$

Since $\sum_{j=1}^h a_j(b) = r(b)$, we have that

$$|\mathcal{Q}(b)| = \binom{n!}{a_1(b)}\binom{n! - a_1(b)}{a_2(b)} \cdots \binom{n! - r(b) + a_h(b)}{a_h(b)} = \frac{\binom{n!}{r(b)} r(b)!}{\prod_{j=1}^h a_j(b)!}.$$

For every $Q = (Q_j)_{j=1}^h \in \mathcal{Q}(b)$, let $\mathcal{P}(Q) = \{p \in \mathcal{P} : Q(p) = Q\}$. Note that the set $\mathcal{P}(Q)$ is well defined and nonempty because $\sum_{j=1}^h j a_j(b) = h$. Since the profiles in $\mathcal{P}(Q)$ are obtained one from the other simply acting with a permutation of the individuals, $\mathcal{P}(Q)$ is contained in a unique orbit. Consider then the function $f : \mathcal{Q}(b) \to \{p^G : p \in \mathcal{P}(b)\}$ defined, for every $Q \in \mathcal{Q}(b)$, as the unique orbit containing $\mathcal{P}(Q)$ and prove that, for every $p \in \mathcal{P}(b)$,

$$f^{-1}(p^G) = \left\{\left(Q_j(p^{(id,\psi)})\right)_{j=1}^h \in \mathcal{Q}(b) : \psi \in S_n\right\}.$$

Let $Q = \left(Q_j(p^{(id,\psi)})\right)_{j=1}^h$ where $\psi \in S_n$. Then $p^{(id,\psi)} \in \mathcal{P}(Q)$ and therefore $p^G = f(Q)$. This shows that $\left\{\left(Q_j(p^{(id,\psi)})\right)_{j=1}^h \in \mathcal{Q}(b) : \psi \in S_n\right\} \subseteq f^{-1}(p^G)$. To get the other inclusion let $Q \in f^{-1}(p^G)$: then $f(Q) = p^G$, which implies $\mathcal{P}(Q) \subseteq p^G$. Picking $q \in \mathcal{P}(Q)$, we have that there exists $(\varphi, \psi) \in G$ such that $q = p^{(\varphi,\psi)}$. Then, by (16), it follows that, for every $j \in \{1, \ldots, h\}$,

$$Q_j = Q_j(q) = Q_j(p^{(\varphi,\psi)}) = \psi Q_j(p) = Q_j(p^{(id,\psi)}),$$

which says $Q = \left(Q_j(p^{(id,\psi)})\right)_{j=1}^h$.

Let us show now that, for every $p \in \mathcal{P}(b)$,

$$\left|\left\{\left(Q_j(p^{(id,\psi)})\right)_{j=1}^h \in \mathcal{Q}(b) : \psi \in S_n\right\}\right| = n!$$

by proving that, for every $\psi_1, \psi_2 \in S_n$,

$$\left(Q_j(p^{(id,\psi_1)})\right)_{j=1}^h = \left(Q_j(p^{(id,\psi_2)})\right)_{j=1}^h \quad \Rightarrow \quad \psi_1 = \psi_2.$$

Assume that, for every $j \in \{1, \ldots, h\}$, $Q_j(p^{(id,\psi_1)}) = Q_j(p^{(id,\psi_2)})$. Then there exists $\varphi \in S_h$ such that $p^{(id,\psi_1)} = [p^{(id,\psi_2)}]^{(\varphi,id)}$, which by (7), gives $p^{(\varphi,\psi_1^{-1}\psi_2)} = p$. Thus, $(\varphi, \psi_1^{-1}\psi_2) \in \mathrm{Stab}_G(p)$ and by Proposition 2, it follows that $\psi_1^{-1}\psi_2 = id$, that is, $\psi_1 = \psi_2$.

As a consequence

$$|\{p^G : p \in \mathcal{P}(b)\}| = \frac{1}{n!}|\mathcal{Q}(b)| = \frac{\binom{n!}{r(b)} r(b)!}{n! \prod_{j=1}^h a_j(b)!},$$

and the proof is complete. □

Proof of equality (13). It is well known (see, for instance, Feller (1957)) that the number $W(m, k)$ of ways of distributing $m \in \mathbb{N}$ indistinguishable balls into $k \in \mathbb{N}$ distinguishable boxes is given by

$$W(m, k) = \binom{m + k - 1}{k - 1}.$$

We also show that

$$W(m,k) = \sum_{b\in\Pi_k(m)} \frac{\binom{k}{r(b)} r(b)!}{\prod_{j=1}^m a_j(b)\,!} \tag{17}$$

Let $r \in \mathbb{N}$ with $r \leq k$ be fixed. We associate with each partition $b \in \Pi(m)$ such that $r(b) = r$ some distributions of the m balls into the k boxes: form bunches of b_1 balls, b_2 balls, up to b_r balls exhausting the m available balls. Note that there is only a way to do that because the balls are indistinguishable.

Now choose r boxes among the k available to distribute the r bunches of balls each in a different box: since the boxes are distinguishable you have k choices for the first box in which you put the b_1 balls, $k-1$ choices for the second box in which you put the b_2 balls, up to $k-r+1$ choices for the box in which you put the last bunch of b_r balls. Note that $a_j(b) \in \mathbb{N} \cup \{0\}$ counts the number of bunches of order j for $j \in \{1, \ldots, m\}$: since the balls are indistinguishable the $a_j(b)$ bunches of order j may be interchanged among themselves producing the same final result. Thus we reach $\frac{\binom{k}{r} r!}{\prod_{j=1}^m a_j(b)\,!}$ different distributions associated with $b \in \Pi(m)$ with $r(b) = r$. It is clear that, with r fixed, different choices of the partition b produce different distributions. It is also clear that the varying of r leads to different distributions, because r is the number of boxes involved in the distribution of the balls. In other words, we have in total reached

$$\sum_{r=1}^{k} \sum_{b\in\Pi(m),r(b)=r} \frac{\binom{k}{r} r!}{\prod_{j=1}^m a_j(b)\,!} = \sum_{b\in\Pi_k(m)} \frac{\binom{k}{r(b)} r(b)!}{\prod_{j=1}^m a_j(b)\,!}$$

ways of distributing the m indistinguishable balls into the k distinguishable boxes. On the other hand, if any distribution is given, we may think that it arises from our procedure. Namely, look into each box, count the balls inside and extract them. Then, consider the number r of boxes containing at least one ball and arrange the number of balls found in the boxes in a non-decreasing order reaching a partition $b \in \Pi(m)$, with $r(b) = r$. Now among the possibilities of our procedure starting from $b \in \Pi(m)$, there is the one consisting in putting the bunches in the boxes where they originally were. Thus, we have shown (17). □

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