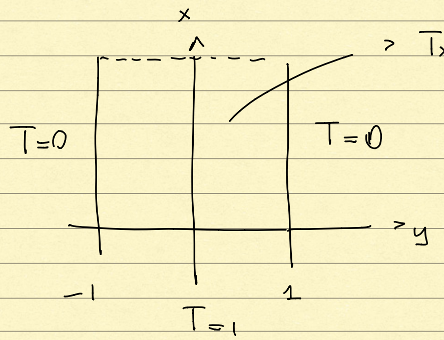


$T_x = T_{yy}$

$T_0(x,y) = X(x)Y(y) \sim$

$$\frac{X'}{X} = \frac{Y''}{Y} = -\lambda^2 < 0$$

$Y'' + \lambda^2 Y = 0$ $\qquad$ $Y = A\cos(\lambda y) + B\sin(\lambda y)$

$T_0\big|_{y=\pm 1} = 0$

$Y(\pm 1) = 0$

$$\begin{cases} A\cos(\lambda) + B\sin(\lambda) = 0 \\ A\cos(\lambda) - B\sin(\lambda) = 0 \end{cases}$$

Sol. $A = 0$ $\qquad$ $\lambda_n = n\pi$ $\qquad$ $n \in \mathbb{N}$ $\qquad$ $\lambda_n^2 = n^2\pi^2$

$$Y_n(y) = B_n \sin(n\pi y) \quad \Rightarrow \quad \frac{X'}{X} = -\lambda_n^2 = -n^2\pi^2$$

$$X_n(x) = \exp(-n^2\pi^2 x)$$

$$T_0(x,y) = \sum_{n=0}^{\infty} X_n(x)Y_n(y) = \overbrace{\sum_{n=0}^{\infty} B_n \sin(n\pi y)e^{-n^2\pi^2 x}}^{\text{serie di Fourier}}$$

I B_n vengono determinati imponendo $T_0(0,y) = 1$

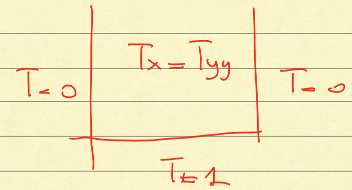
$$1 = \sum_{n=0}^{\infty} B_n \sin(n\pi y) \quad \Rightarrow \quad \sin(m\pi y) = \sum_{n=0}^{\infty} B_n \sin(n\pi y)\sin(m\pi y)$$

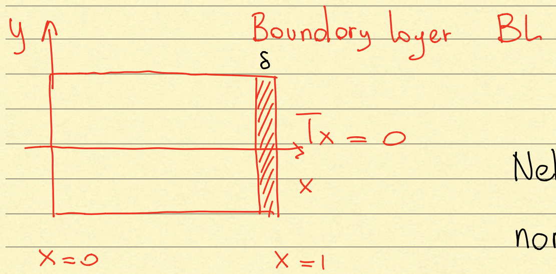
$$\int_0^1 \sin(m\pi y)\,dy = \sum_{n=0}^{\infty} B_n \underbrace{\int_0^1 \sin(n\pi y)\sin(m\pi y)\,dy}_{\delta_{nm}/2} \qquad \delta_{nm} \begin{cases} 1 & \text{se } n = m \\ 0 & \text{se } n \neq m \end{cases}$$

$$\int_0^1 \sin(n\pi y)\,dy = B_n \cdot \frac{1}{2} \qquad \sim \frac{\cos(n\pi y)}{n\pi}\Bigg|_0^1 = \frac{B_n}{2}$$

$$\frac{2}{n\pi}\left[1 - (-1)^n\right] = B_n$$

$$T_0 = \sum_1 B_n \sin(n\pi y) e^{-n^2\pi^2 x}$$





Nel BL la riscalatura effettuata non vale

SOLUZIONE NEL BL

Torniamo all'eq.ne generale $\quad T_x = \varepsilon^2 T_{xx} + T_{yy}$

$x \in [0,1]$ $\qquad$ Introduco una nuova variabile ξ

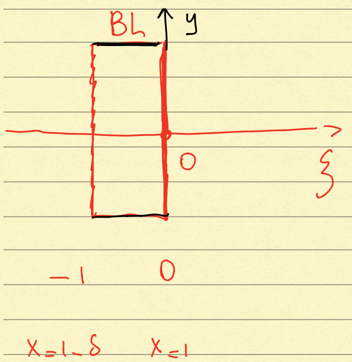
$$x - 1 = \delta\xi \qquad \begin{cases} x = 1 & \xi = 0 \\ x = 1-\delta & \xi = -1 \end{cases} \quad \Rightarrow \quad \begin{matrix} x \in (1-\delta, 1) \\ \xi \in (-1, 0) \end{matrix}$$

$$\frac{\partial}{\partial x} = \frac{1}{\delta}\frac{\partial}{\partial \xi} \qquad T(x,y) = T_b(\xi, y)$$

$$\frac{1}{\delta}\frac{\partial T_b}{\partial \xi} = \frac{\varepsilon^2}{\delta^2}\frac{\partial^2 T_b}{\partial \xi^2} + \frac{\partial^2 T_b}{\partial y^2} \qquad \text{Hyp.} \quad \delta = \varepsilon^2$$

$$\frac{1}{\varepsilon^2}\frac{\partial T_b}{\partial \xi} = \frac{\cancel{\varepsilon^2}}{\varepsilon^{4\,2}}\frac{\partial^2 T_b}{\partial \xi^2} + \frac{\partial^2 T_b}{\partial y^2} \Rightarrow \boxed{\frac{\partial T_b}{\partial \xi} = \frac{\partial^2 T_b}{\partial \xi^2} + \varepsilon^2 \frac{\partial^2 T_b}{\partial y^2}} \quad (*)(*)$$

Prendiamo l'ordine 0 di (**)



Imposto il pb.ma

$$\boxed{\begin{cases} Tb_{\xi} = Tb_{\xi\xi} \\ Tb(\xi, \pm 1) = 0 \\ Tb_{\xi}(0, y) = 0 \end{cases}}$$

$$\frac{\partial}{\partial x} = \frac{1}{\varepsilon^2}\frac{\partial}{\partial \xi}$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{in } x = 1$$

$$\frac{\partial Tb}{\partial \xi} = 0 \quad \text{in } \xi = 0$$

Soluzione sono tutte le funzioni $\boxed{A(y) \text{ t.c. } A(\pm 1) = 0}$

$$Tb(\xi, y) = A(y)$$

In conclusione dobbiamo "RACCORDARE" $Tb(\xi,y)$ e $To(x,y)$

(Matching)

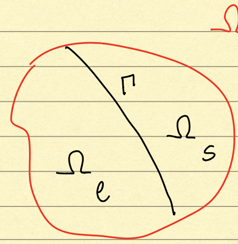
$\Downarrow$

Basta imporre che $To(1-\delta, y) = Tb(-1, y) = A(y)$

$$A(y) = \sum_{n=0}^{\infty} B_n \sin(n\pi y)\, e^{-n^2\pi^2(1-\delta)}$$

PROBLEMA DI STEFAN

$\Omega \subset \mathbb{R}^3$



Consiste nello studiare l'evoluz.ne di un dominio $\Omega = \Omega_\ell \cup \Omega_s$

Ω_ℓ fase liquida (e.g. acqua)

Ω_s fase solida (e.g. ghiaccio)

Γ interfaccia della transizione di fase

• I p.ti che compongono Ω sono fermi (caso statico)

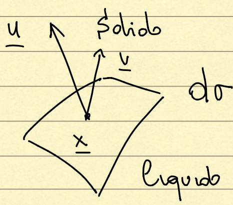
• Γ non è sup. materiale

In Ω_e $\qquad \rho_e C_e \dfrac{\partial T_e}{\partial t} - k_e \Delta T_e = 0 \qquad T_e(\underline{x},t)$

Ω_s $\qquad \rho_s C_s \dfrac{\partial T_s}{\partial t} - k_s \Delta T_s = 0 \qquad T_s(\underline{x},t)$

Su Γ $\qquad T_e = T_s = 0$

Devo imporre un'altra condizione su Γ, ossia il bilancio di flusso termico su Γ (Condizione di Stefan)



$d\sigma \subset \Gamma$

$$\underbrace{\rho_j \lambda \underline{v}\cdot\underline{u}}_{\text{Calore necessario per la transiz. di fase}} = \underbrace{-k_e \nabla T_e \cdot \underline{u} + k_s \nabla T_s \cdot \underline{u}}_{\text{flusso netto di calore in } \underline{x}}$$

λ • Calore latente $\qquad [\lambda] = \dfrac{J}{kg}$

Supponiamo che Γ sia espressa in forma cartesiana $\qquad S(\underline{x},t) = 0$

$\underline{u} = \pm \dfrac{\nabla S}{\|\nabla S\|} \qquad S(\underline{x}(t),t) = 0 \qquad$ derivo rispetto a t

$$S_t + \nabla S \cdot \underline{v} = 0$$

$\nabla S = \pm \|\nabla S\| \underline{u} \qquad S_t \pm \|\nabla S\| \underline{v}\cdot\underline{u} = 0 \qquad \underline{v}\cdot\underline{u} = \dfrac{-S_t}{\pm\|\nabla S\|} = \mp \dfrac{S_t}{\|\nabla S\|}$

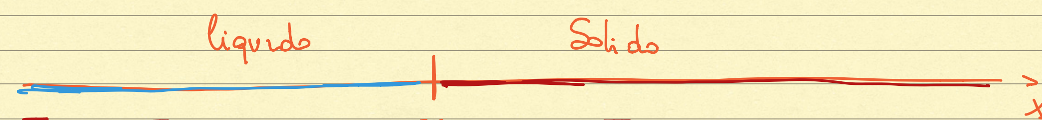
$$\mp\frac{S_t}{\cancel{\|\nabla S\|}}\,\rho_j\,\lambda = \left(-k_e \nabla T_e + k_s \nabla T_s\right)\cdot \pm\frac{\nabla S}{\cancel{\|\nabla S\|}}$$

$$\boxed{-S_t\,\rho_j\,\lambda = \left(-k_e \nabla T_e + k_s \nabla T_s\right)\cdot \nabla S}$$

Condiz. di Stefan

$$\boxed{T_e = T_s = 0 \quad \text{su} \quad S(\underline{x}, t)}$$

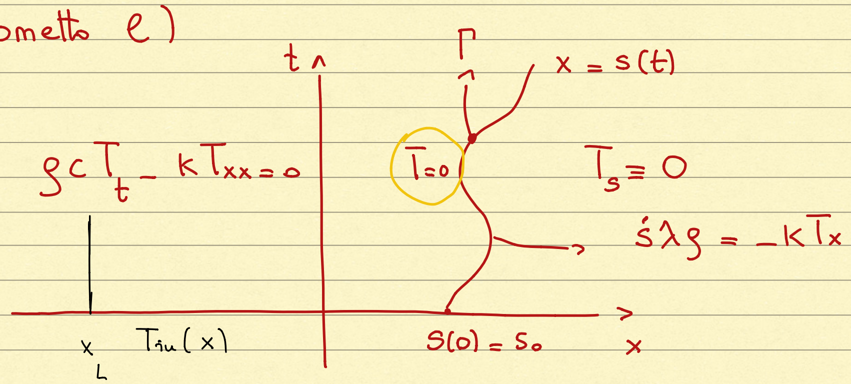
CASO 1D (ad una sola fase (liquida))



$\mathbb{R} = \Omega$

$\rho c T_t - k T_{xx} = 0$ $\quad x = S(t) \quad$ $T_s \equiv 0$

(ometto e)



Cond. $\quad -S_t\,\lambda\,\rho = -k\nabla T\cdot\nabla S \qquad S(\underline{x}, t) = x - s(t)$

$$\nabla S = 1 \qquad S_t = -\dot{s}(t)$$

Pb.mo Stefan 1D

$$\begin{cases} \rho c T_t - k T_{xx} = 0 \\ T(x,0) = T_{in}(x) \\ T(s,t) = 0 \\ k T_x(s,t) = -\lambda \rho \dot{s} \end{cases}$$

devo aggiungere o una condizione su T per $x \to -\infty$

" " " su T per $x_L < S_0$

SCALING Pb.mo

$x = L\tilde{x}$ $\quad t = \left(\frac{\rho C L^2}{K}\right)\tilde{t}$ tempo caratt. diffusivo

$T = \left(\frac{\lambda}{c}\right)\tilde{T}$ $\quad s = L\tilde{s}$

x esercizio

$$\boxed{\begin{cases} \tilde{T}_{\tilde{t}} - \tilde{T}_{\tilde{x}\tilde{x}} = 0 \\ \tilde{T}(\tilde{x},0) = \tilde{T}_{in} = \frac{T_{in}}{(\lambda/c)} \\ \tilde{T}(\tilde{s},\tilde{t}) = 0 \\ \tilde{T}_x(\tilde{s},\tilde{t}) = -\dot{\tilde{s}} \end{cases}}$$

Problema di stefan

1D ad una fase

$$K T_x(s,t) = -\rho\lambda\dot{s} \overset{\text{adim.}}{\Longrightarrow} \left(\frac{\cancel{\lambda}\cancel{K}}{\cancel{c}\cancel{L}}\right)\tilde{T}_{\tilde{x}} = -\left(\frac{\cancel{\rho}\cancel{\lambda}\cancel{L}\cancel{K}}{\cancel{\rho}\cancel{c}\cancel{L^2}}\right)\dot{\tilde{s}}$$

Cercheremo per questo pb.mo soluzioni autosimilari: $T(x,t) = f(\gamma(t)\,\vartheta(x))$

In particolare cercheremo fun. del $\quad f(x\,\gamma(t))$