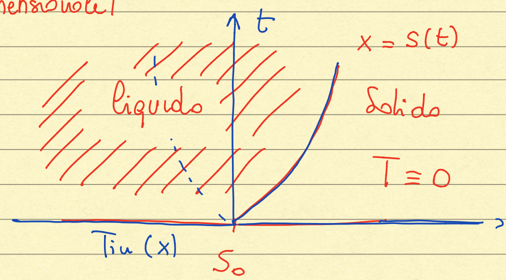
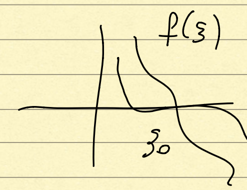
Pb.ma Stefan (adimensionale)

$$\begin{cases} T_t - T_{xx} = 0 \\ T(x,0) = T_{in} \\ T(s,t) = 0 \\ T_x(s,t) = -\dot{s} \end{cases}$$



SOLUZIONI AUTOSIMILARI $\quad T(x,t) = f(\gamma(t)x) = f(\xi) \qquad \xi = \gamma(t)\, x$

$T_t = f'(\xi)\,\dot{\gamma}\, x \qquad T_x = f'(\xi)\cdot\gamma(t) \qquad T_{xx} = f''\gamma^2$



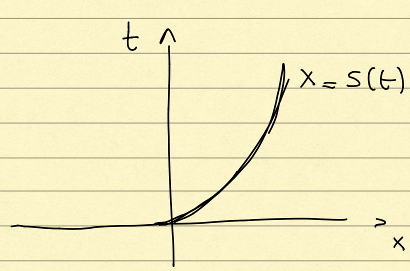
$f'\dot{\gamma}\,\dfrac{\xi}{\gamma} = f''\gamma^2 \qquad \dfrac{f''}{f'\xi} = \dfrac{\dot{\gamma}}{\gamma^3} = c$ (costante)

Integro $\dfrac{\dot{\gamma}}{\gamma^3} = c \qquad -\dfrac{1}{2\gamma^2} = ct$ (prendo la costante additiva $= 0$)

Se $t \geq 0 \implies c = -\lambda^2 < 0 \qquad \gamma^2 = \dfrac{1}{2t\lambda^2} \qquad \gamma(t) = \dfrac{1}{\lambda\sqrt{2t}}$

$T(s,t) = 0 \qquad f(\gamma(t)s(t)) = 0 \qquad \gamma(t)s(t) = \xi_0$ t.c. $f(\xi_0) = 0$

$\gamma(t)s(t) = \xi_0 \qquad s(t) = \dfrac{\xi_0}{\gamma(t)} = \underbrace{\xi_0\lambda\sqrt{2t}}\; =: \underbrace{2\alpha}\sqrt{t}$



$2\alpha = \xi_0\lambda\sqrt{2} \qquad \dfrac{\xi_0\lambda}{\sqrt{2}} = \alpha$

Incognita a questo stadio

$\dfrac{f''}{f'} = -\lambda^2\xi \qquad \dfrac{d}{d\xi}\left(\ln|f'|\right) = -\lambda^2\xi \qquad \ln|f'| = -\dfrac{\lambda^2\xi^2}{2} + c$

$|f'| = e^c e^{-\frac{\lambda^2\xi^2}{2}}$ tolgo il $|\ldots|$ e considero il segno +

(se prendessi il segno − ottengo gli stessi risultati)

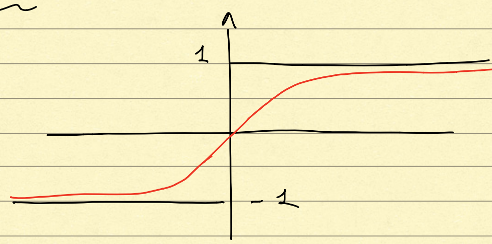
$f' = e^c e^{-\frac{\lambda^2\xi^2}{2}}$ Integro tra ξ_0 e ξ

$$\int_{\xi_0}^{\xi} f' = \boxed{f(\xi) - \cancel{f(\xi_0)} = e^c \int_{\xi_0}^{\xi} e^{-\frac{\lambda^2\eta^2}{2}}\, d\eta}$$

Sost. $\frac{\lambda\eta}{\sqrt{2}} = \theta \qquad d\theta = \frac{\lambda}{\sqrt{2}}\, d\eta$

$$f(\xi) = e^c \int_{\frac{\xi_0\lambda}{\sqrt{2}}}^{\frac{\xi\lambda}{\sqrt{2}}} e^{-\theta^2} d\theta = e^c \int_{\alpha}^{\frac{\xi\lambda}{\sqrt{2}}} e^{-\theta^2} d\theta \cdot \frac{\sqrt{2}}{\lambda} = \frac{e^c\sqrt{2}}{\lambda} \int_{\alpha}^{\frac{\xi\lambda}{\sqrt{2}}} e^{-\theta^2} d\theta$$

Introduciamo $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\theta^2} d\theta$



$\lim_{z \to \pm\infty} \mathrm{erf}(z) = \pm 1$ $\quad$ erf(z) è dispari

$\mathrm{erf}(z) - \mathrm{erf}(\bar{z}) = \frac{2}{\sqrt{\pi}} \int_{\bar{z}}^{z} e^{-\theta^2} d\theta$ $\quad$ erf (funzione degli errori)

Utilizziamo erf per scrivere $f(\xi)$ $\quad f(\xi) = \overbrace{\frac{e^c\sqrt{2}}{\lambda} \cdot \frac{\sqrt{\pi}}{2}}^{= A} \left[\mathrm{erf}\left(\frac{\xi\lambda}{\sqrt{2}}\right) - \mathrm{erf}(\alpha)\right]$

$$\xi = \gamma(t) \cdot x = \frac{x}{\lambda\sqrt{2t}} \Rightarrow \frac{\xi\lambda}{\sqrt{2}} = \frac{x\cancel{\lambda}}{\cancel{\lambda}\sqrt{2t}\cdot\sqrt{2}} = \frac{x}{2\sqrt{t}}$$

$$f(\xi) = \boxed{\begin{cases} T(x,t) = A\left[\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) - \mathrm{erf}(\alpha)\right] \\ S(t) = 2\alpha\sqrt{t} \end{cases}}$$

Soluzioni autosimilari del pb.mo di Stefan 1D ad 1 fase (Devo det A e α)

Per determinare A e α impongo la cond. iniziale e la condizione

$T_x(s,t) = -\dot{s}$ (Condizione di Stefan)

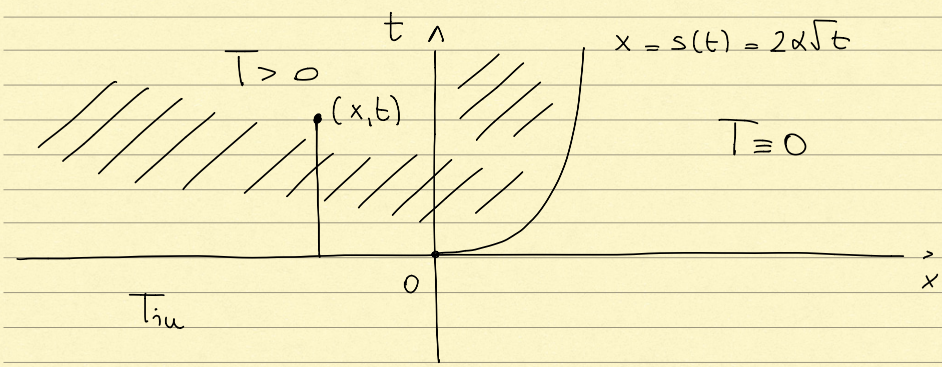
$$T_x = A\left[\frac{\cancel{2}}{\sqrt{\pi}}\, e^{-\frac{x^2}{4t}} \cdot \frac{1}{\cancel{2}\sqrt{t}}\right] = \frac{A}{\sqrt{\pi}\sqrt{t}}\, e^{-x^2/4t} \qquad \dot{s} = \frac{\alpha}{\sqrt{t}}$$

$$T_x(s,t) = \frac{A}{\sqrt{\pi}\cancel{\sqrt{t}}}\, e^{-\alpha^2} = -\frac{\alpha}{\cancel{\sqrt{t}}} \quad \Rightarrow \quad A = -\alpha\sqrt{\pi}\, e^{\alpha^2}$$

$$\boxed{\begin{cases} T(x,t) = \alpha\sqrt{\pi}\, e^{\alpha^2}\left[\operatorname{erf}(\alpha) - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] \\ s(t) = 2\alpha\sqrt{t} \end{cases}}$$

L'unica incognita rimasta è α

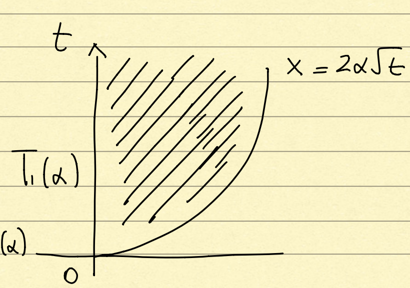
CASO 1 $\left(\alpha > 0 \quad e \quad x < s(t) \quad x < 2\alpha\sqrt{t}\right) \rightarrow \frac{x}{2\sqrt{t}} < \alpha$



Per determinare la condizione T_{in} considero $x < 0$ e calcolo

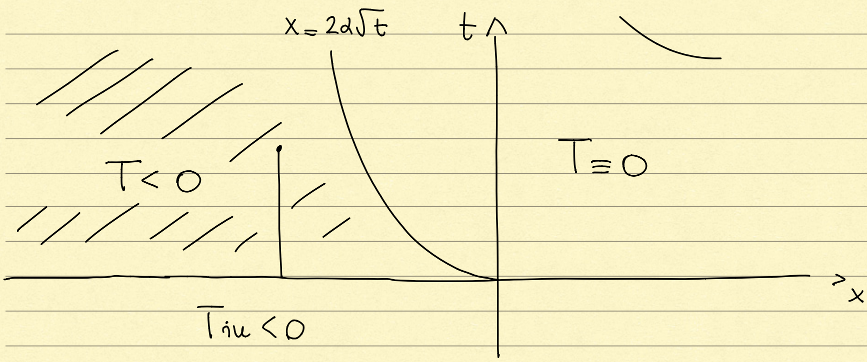
$$\lim_{t \to 0^+} \alpha\sqrt{\pi}\, e^{\alpha^2}\left[\operatorname{erf}(\alpha) - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] = \alpha\sqrt{\pi}\, e^{\alpha^2}\left[\operatorname{erf}(\alpha) + 1\right] = T_0(\alpha) \overset{!}{=} T_{in}$$

CASO 2 $\left(\alpha > 0 \qquad 0 < x < 2\alpha\sqrt{t}\right)$



$$T_1(\alpha) = \lim_{x \to 0^+} \alpha\sqrt{\pi}\, e^{\alpha^2}\left[\operatorname{erf}(\alpha) - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] = \alpha\sqrt{\pi}\, e^{\alpha^2} \operatorname{erf}(\alpha)$$

CASO 3 $\quad (\alpha < 0 \quad x < 2\alpha\sqrt{t})$ Sottoraffreddamento



$$\frac{x}{2\sqrt{t}} < \alpha < 0$$

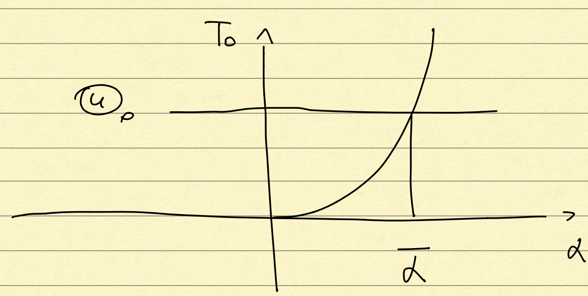
$$T = \alpha\sqrt{\pi}e^{\alpha^2}\underbrace{\left[erf(\alpha) - erf\left(\frac{x}{2\sqrt{t}}\right)\right]}_{>0} < 0$$

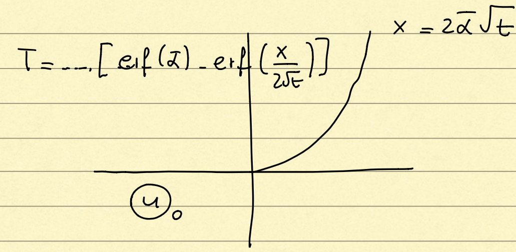
$$T_{in} = \lim_{t \to 0^+} \alpha\sqrt{\pi}e^{\alpha^2}\left[erf(\alpha) - erf\left(\frac{x}{2\sqrt{t}}\right)\right] = \alpha\sqrt{\pi}e^{\alpha^2}\left[erf(\alpha) + 1\right] = T_0(\alpha)$$

Pbmo nei 3 casi

(Caso 1)

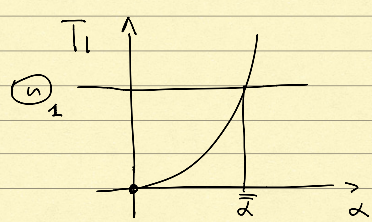
1) Dato una temp. $\Theta_0 > 0$ trovare α per cui $T_0(\alpha) = \Theta_0$





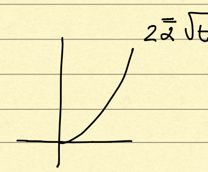
CASO 2 $\quad$ Dato una temperatura Θ_1 determinare α t.c. $T_1(\alpha) = \Theta_1$

$\alpha > 0$



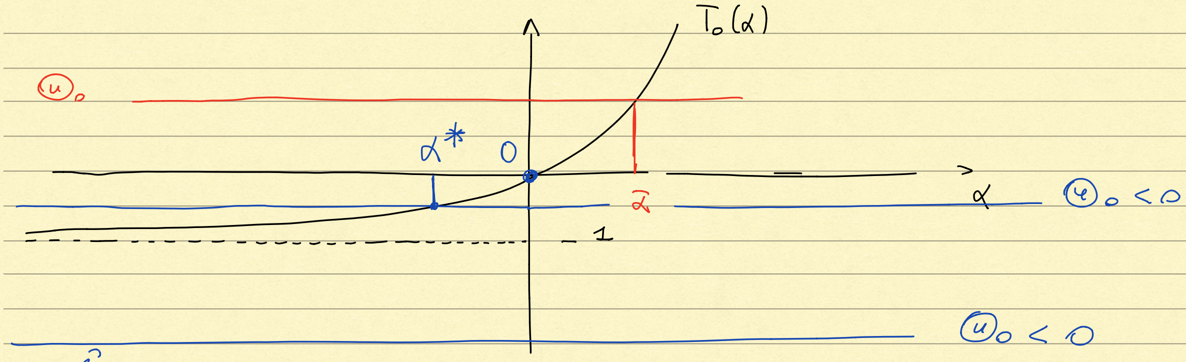
$$T_1 = \alpha\sqrt{\pi}e^{\alpha^2}\left[erf(\alpha)\right]$$

t.c. $\Theta_1 = T_1(\bar{\bar{\alpha}})$



CASO 3 $(\alpha < 0)$ $\quad T = \alpha e^{\alpha^2}\sqrt{\pi}\left[\operatorname{erf}(\alpha) - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] < 0$

$T_0(\alpha) = \alpha e^{\alpha^2}\sqrt{\pi}\left[\operatorname{erf}(\alpha) + 1\right] : (-\infty, 0]$



In questo caso α^* $\nexists$