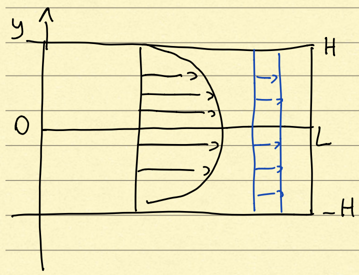


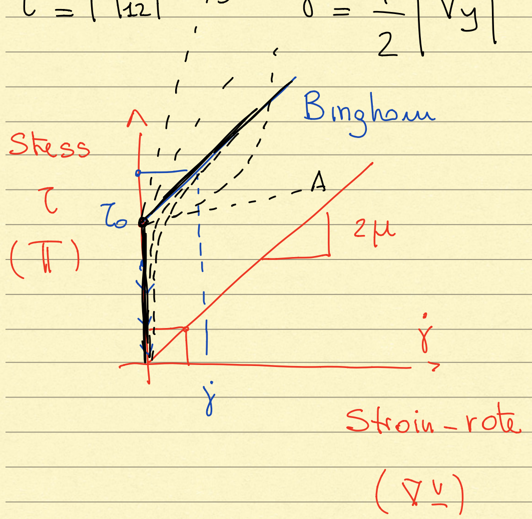
$$V(y) = \frac{\Delta P}{2\mu L}\left(H^2 - y^2\right) \qquad V_y = -\frac{\Delta P}{L\mu}\, y$$

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D} \qquad \mathbb{D} = \frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^T\right)$$

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} 0 & V_y \\ V_y & 0 \end{pmatrix} \qquad \mathbb{T} = \begin{pmatrix} -p & + \mu V_y \\ \mu V_y & -p \end{pmatrix}$$

T_{12} = sforzo di taglio = $\mu V_y = -\dfrac{\Delta P}{L}\, y$ — massimo alla parete / nullo al centro

$$\tau = |T_{12}| \qquad \dot{\gamma} = \frac{1}{2}|V_y| \qquad \Rightarrow \qquad \boxed{\tau = 2\mu\dot{\gamma}}$$



Pendenza è ∝ alla viscosità

τ_0 < SFORZO DI SOGLIA / YIELD STRESS

Bingham (fluido non-Newtoniano). Al di sopra di τ_0 il comportamento è comunque Newtoniano

$$(\tau - \tau_0)_+ = 2\mu \dot{\gamma}$$

Nel caso 1D $\quad T_{12} = \mu V_y + \mathrm{sgn}(V_y)\tau_0$

$$\mathrm{sgn}(\xi) = \begin{cases} 1 & \xi \geq 0 \\ -1 & \xi < 0 \end{cases}$$

$$T_{12} = \mathrm{sgn}(V_y)\left[\mu \frac{V_y}{\mathrm{sgn}(V_y)} + \tau_0\right] = \mathrm{sgn}(V_y)\left[\mu |V_y| + \tau_0\right]$$

Equazione costitutiva del Bingham

$$\tau = |T_{12}| = \mu |V_y| + \tau_0 \quad \Rightarrow \quad \boxed{(\tau - \tau_0)_+ = 2\mu \dot{\gamma}}$$

Equazione costitutiva Bingham 3D

$$\mathbb{T} = \underbrace{-p\mathbb{I}}_{} + \underbrace{2\mu \mathbb{D}}_{} \qquad \mathbb{D} = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$$

NEWTONIANO

parte sferica del tensore $\mathbb{T}$ — parte deviatoria $\mathbb{S} = 2\mu \mathbb{D}$

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S}$$

Oss. Nel Newtoniano $\mathrm{tr}\,\mathbb{D} = 0$ (se il fluido è omogeneo ρ = costante)

Se $\rho = \mathrm{Cost.} \Rightarrow \mathrm{div}\, \underline{v} = 0$

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} 2u_x & \dots & \dots \\ \dots & 2v_y & \dots \\ \dots & \dots & 2w_z \end{pmatrix} \qquad \mathrm{tr}\,\mathbb{D} = \mathrm{div}\,\underline{v} = 0$$

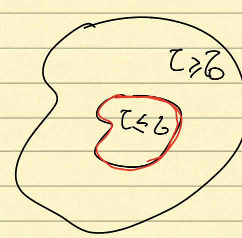
Come scrivo $\mathbb{S}$ nel Bingham 3D ?? Introduco le norme di $\mathbb{S}$ e $\mathbb{D}$

$$\tau = \sqrt{\frac{1}{2}\mathbb{S}:\mathbb{S}} \qquad \mathbb{S}:\mathbb{S} = \sum_{i,j=1}^{3} S_{ij}^2$$

$$\dot{\gamma} = \sqrt{\frac{1}{2}\mathbb{D}:\mathbb{D}} \qquad \mathbb{D}:\mathbb{D} = \sum_{i,j=1}^{3} D_{ij}^2$$

Eq.ne costitutiva 3D $\qquad \mathbb{T} = -p\mathbb{I} + \mathbb{S}$

$$\begin{cases} \mathbb{S} = \left(2\mu + \dfrac{\tau_0}{\dot{\gamma}}\right)\mathbb{D} & \text{se } \tau \geq \tau_0 \\ \mathbb{D} = 0 \quad \text{(corpo rigido)} & \text{se } \tau \leq \tau_0 \end{cases}$$

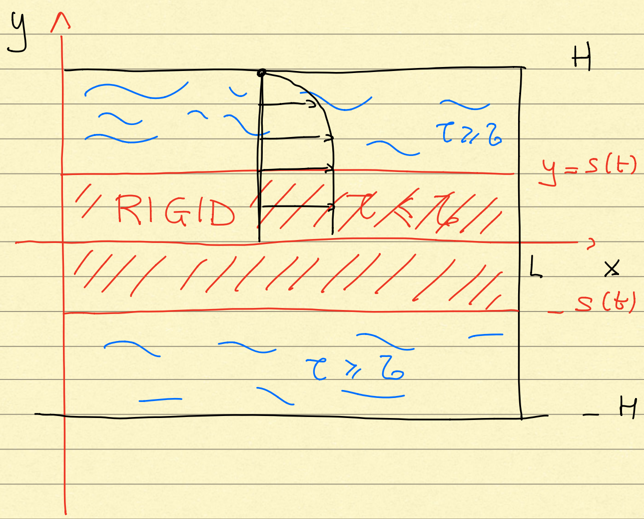


Se ritorno al caso 1D $\qquad \underline{v} = v(y,t)\,\underline{i}$

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} 0 & v_y \\ v_y & 0 \end{pmatrix} \qquad \dot{\gamma} = \sqrt{\frac{1}{\cancel{2}}\,\frac{1}{4}\left(\cancel{2}v_y^2\right)} = \frac{1}{2}|v_y|$$

$$\boxed{S_{12} = T_{12}} = \left(2\mu + \frac{\tau_0}{\frac{1}{2}|v_y|}\right)\frac{1}{2}v_y = \boxed{\mu v_y + \tau_0\,\mathrm{sgn}(v_y)}$$

MOTO ALLA "POISEUILLE" di un fluido di Bingham (1D)



Assumo che la parte sotto soglia sia una regione

$[-s(t), s(t)] \subset [-H, H]$

Fluido

$[-H, -s] \cup [s, H]$

Per simmetria mi limito a $[0, H]$ $\qquad \dfrac{\partial v}{\partial y}$

$y = s(t)$ è la yield-surface

Su $y = s(t)$ mi aspetto che $\tau = \tau_0$

$$\tau = \mu |v_y| + \tau_0 \qquad \tau\Big|_s = \underbrace{\cancel{\tau_0} = \mu |v_y|\Big|_s + \cancel{\tau_0}}$$

$$\Downarrow$$

$v(y,t) \implies \boxed{v_y(s,t) = 0}$ Condizione su $s(t)$ $\quad 1^a$ condizione

$\underline{v} = v(y,t)\,\underline{i}$

Modello nella parte fluida $\qquad \rho\left(\underline{v}_t + (\nabla \underline{v})\underline{v}\right) = -\nabla p + \mathrm{div}(\mathbb{S}) + \cancel{\rho \underline{f}}$

$$\begin{cases} \rho \dfrac{\partial v}{\partial t} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial S_{12}}{\partial y} \\ 0 = -\dfrac{\partial p}{\partial y} + \cancel{\dfrac{\partial S_{12}}{\partial x}} \end{cases} \implies \begin{cases} \rho v_t = -p_x + \dfrac{\partial}{\partial y}\left(\mu v_y + \mathrm{sgn}(v_y)\tau_0\right) \\ 0 = p_y \end{cases}$$

Questa eq.ne vale su $[s, H]$

Oss. In $[s,H)$ $\quad V_y < 0 \quad \mathrm{sgn}(V_y) = -1$

L'eq.ne da risolvere è

$$\begin{cases} \rho V_t = -p_x + \dfrac{\partial}{\partial y}\left(\mu V_y - \tau_0\right) \\ p_y = 0 \quad \Rightarrow \quad p = p(x,t) \qquad p_x = \text{Cost.} \end{cases}$$

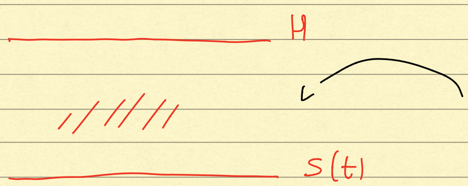
Costante (al max dipendente dal tempo). Questo significa che p è lineare in x

$$\Delta p = p_{in} - p_{out}$$

$$p = Ax + B$$

$$p_{|0} = p_{in} \quad \Rightarrow \quad \boxed{p = -\frac{\Delta p}{L}\, x + p_{in}} \qquad \frac{\Delta p}{L} = G > 0$$

$$p_{|L} = p_{out}$$



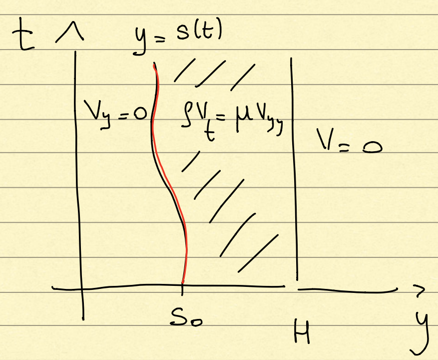
NO-SLIP $\quad V(H,t) = 0$

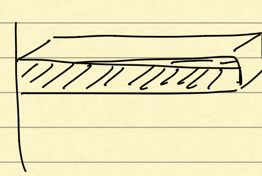
$$\rho V_t = \mu V_{yy} + G$$

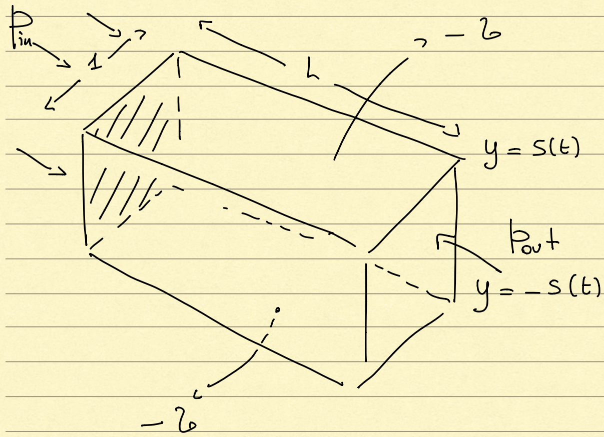
YIELD $\quad V_y(s,t) = 0$

?! Ulteriore cond. per V

???







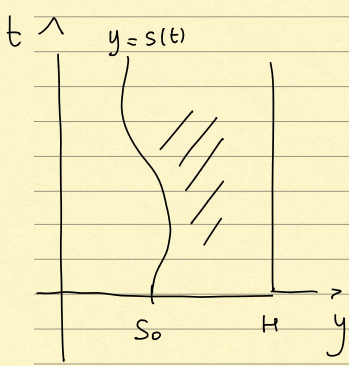
Applico $\vec{F} = m\vec{a}$ al blocco rigido

$$\cancel{2}S\cdot\cancel{1}\cdot p_{in} - \cancel{2}S\cdot\cancel{1}\, p_{out} - \cancel{2}\tau L\cdot\cancel{1} = \rho\,\text{Volume}\cdot V_t(s,t) = \rho\,\cancel{2}S\cdot\cancel{1}\cdot L\, V_t(s,t)$$

$$S\Delta p - \tau L = \rho V_t(s,t)\, S L$$ (divido per SL)

$$\rho V_t(s,t) = \left(\frac{\Delta p}{L} - \tau/S\right) \Rightarrow \boxed{V_t(s,t) = \frac{1}{\rho}\left(G - \frac{\tau}{S}\right)}$$

Formulazione pb. 1D



$$\boxed{\begin{cases} \rho V_t = \mu V_{yy} + G & y \in [s(t), H] \quad t \geq 0 \\ V(y,0) = V_0(y) & y \in [S_0, H] \\ s(0) = S_0 & S_0 \in (0, H) \\ V(H,t) = 0 & t \geq 0 \\ V_y(s,t) = 0 & t \geq 0 \\ V_t(s,t) = \frac{1}{\rho}\left(G - \frac{\tau}{S}\right) & t \geq 0 \end{cases}}$$

(*)

SOLUZ. STAZIONARIA di (*)

$$\begin{cases} 0 = \mu V_{yy} + G \\ V\big|_H = 0 \qquad V_y\big|_s = 0 \\ G - \dfrac{\tau}{s} = 0 \quad \rightsquigarrow \quad \left(s = \dfrac{\tau}{G} \right) \end{cases}$$

posiz. dell' interfaccia $y = s$ nel caso stazionario

Condiz. di flusso

$$s \in (0, H) \qquad \frac{\tau}{G} < H \qquad G > \frac{\tau}{H}$$

$$\left.\begin{aligned} \mu V_y &= -Gy + K \\ 0 &= -Gs + K \end{aligned}\right\} \Rightarrow \quad \mu V_y = G(s-y)$$

Integro di nuovo e trovo
$$\begin{cases} \mu V = -\dfrac{G(s-y)^2}{2} + K \\ 0 = -\dfrac{G(s-H)^2}{2} + K \end{cases}$$

$$V(y) = \frac{G}{2\mu}\left[(s-H)^2 - (s-y)^2\right] \qquad y \in [s, H]$$

La velocità nella parte rigida è $V(s) = \dfrac{G}{2\mu}\left[(s-H)^2\right]$

$$V(s) = \frac{G}{2\mu}\left[\left(\frac{\tau}{G} - H\right)^2\right]$$

Per risolvere (*) considero la trasf. $V_t = z$

$$\left(\rho V_t = \mu V_{yy} + G\right)_t \longrightarrow \left(\rho V_{tt} = \mu V_{yyt}\right)$$

$$\boxed{\rho z_t = \mu z_{yy}}$$

$$\boxed{z(y,0) = z_0(y) = V_t(y,0) = \frac{1}{\rho}\left[\mu V_0''(y) + G\right]}$$

$$\rho V_t(y,0) = \mu V_{yy}(y,0) + G = \mu V_0''(y) + G$$

$$V(H,t) = 0 \implies V_t(H,t) = 0 \implies \boxed{z(H,t) = 0}$$

$$V_t(s,t) = \boxed{\frac{1}{\rho}\left[G - \frac{b}{s}\right] = z(s,t)} \qquad V_y(s(t),t) = 0$$

$$V_y(s,t) = 0 \implies \left(V_{yy}\dot{s} + V_{yt}\right)\Big|_s = 0$$

$$z_y(s,t) = -V_{yy}(s,t)\dot{s}$$

$$\rho V_t = \mu V_{yy} + G \longrightarrow \mu V_{yy}(s,t) = \rho V_t(s,t) - G$$

$$V_t(s,t) = \frac{1}{\rho}\left[G - \frac{\tau}{s}\right] \Rightarrow \mu V_{yy}(s,t) = \cancel{G} - \frac{\tau}{s} - \cancel{G}$$

$$V_{yy}(s,t) = -\frac{\tau}{\mu s} \qquad \boxed{z_y(s,t) = \frac{\tau \dot{s}}{\mu s}}$$

Riassumendo il pb. per z

$$\begin{cases} \rho z_t = \mu z_{yy} \\ z(y,0) = z_0(y) \\ z(H,t) = 0 \\ z(s,t) = \frac{1}{\rho}\left[G - \frac{\tau}{s}\right] \\ z_y(s,t) = \frac{\tau \dot{s}}{\mu s} \\ S(0) = S_0 \end{cases}$$

$\rightsquigarrow$ l'$\exists$ è l'$!$ di questo problema si risolve con tecniche di p.to Fisso

COMPARINI 1992

A one-dimensional Bingh. flow

J. Math. Anal. Appl.