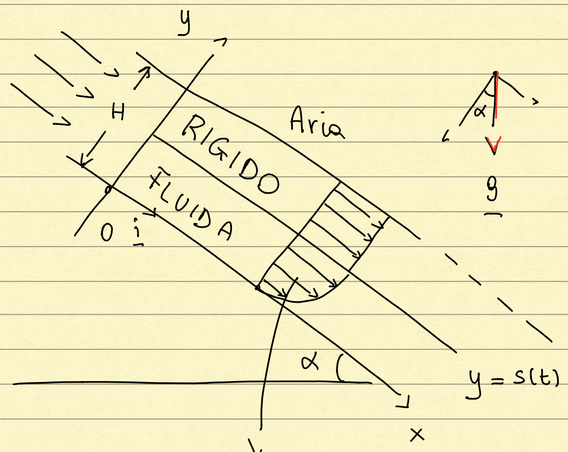


Forza di massa

$$\rho \underline{f} = (\rho g \sin\alpha, -\rho g \cos\alpha)$$

H è incognita e dipende dal flusso di fluido Q

Al solito supponiamo $\underline{V} = V(y,t)\,\underline{i}$ $\quad \operatorname{div} \underline{V} = \frac{\partial V}{\partial x} = 0$

$$\begin{cases} \rho \cancel{\frac{\partial V}{\partial t}} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\underbrace{\left[\mu V_y + \tau\right]}_{S_{12}} + \rho g \sin\alpha \\ 0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha \end{cases}$$

C.C. $\quad V_y(s,t) = 0 \qquad V(0,t) = 0$

STAZIONARIO

$S_{12} = \mu V_y + \tau \qquad S_{12}\big|_s = \tau$

INTEGRO

$$\begin{cases} \mu V_{yy} + \rho g \sin\alpha - \cancel{p_x} = 0 \\ p_y + \rho g \cos\alpha = 0 \end{cases} \qquad p = -\rho g \cos\alpha\, y + K$$

p_a = pressione atmosferica $\qquad p_a = -\rho g \cos\alpha\, H + K$

$$p = p_a + \rho g \cos\alpha\,(H - y) \implies p_x = 0$$

$\mu V_{yy} = -\rho g \sin\alpha \quad \to \quad \mu V_y = -\rho g \sin\alpha\, y + K \qquad V_y(s) = 0$

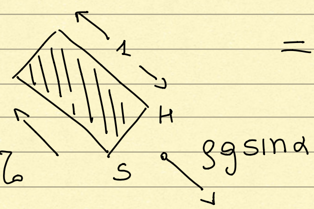
$\mu V_y = \rho g \sin\alpha\,(s - y) \geq 0$ se $y \in [0, s]$ fase fluida

$$\mu V = -\frac{\rho g \sin\alpha\,(s-y)^2}{2} + K \quad \to \quad V(0) = 0$$

$$\boxed{V(y) = \frac{\rho g \sin\alpha}{2\mu}\left[s^2 - (s-y)^2\right]}$$

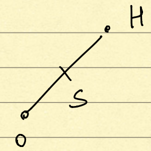
S è incognita

Impongo il bilancio della quantità di moto ad una porzione unitaria della parte rigida



$$\Rightarrow \underbrace{\tau_0 = (H - s)\,\rho g \sin\alpha}_{\text{Questo bilancio mi fornisce } s}$$

$$s = \left(H - \frac{\tau_0}{\rho g \sin\alpha}\right)$$

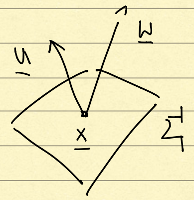


$s < H$ automatico

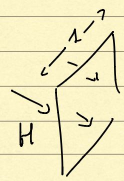
$s > 0 \iff \rho g \sin\alpha > \dfrac{\tau_0}{H} \qquad H > \dfrac{\tau_0}{\rho g \sin\alpha}$

Come determino H? Con la conoscenza del flusso in $x = 0$

$$Q = \text{flusso} = \int_0^H V(y)\,dy$$



flusso att. $\Sigma = \int_\Sigma \underline{w}\cdot\underline{u}\, d\sigma$



$$Q = \int_0^H V(y)\,dy = \text{int. x parti}$$

$[0,s]$ $\quad V(y) = \dfrac{\rho g \sin\alpha}{2\mu}\left[s^2 - (s-y)^2\right]$

$[0,s]$ $\quad V_y = \dfrac{\rho g \sin\alpha}{\mu}(s-y)$

$$Q = yV(y)\Big|_0^H - \int_0^H y V_y\, dy = H V(s) - \int_0^s y \frac{\rho g \sin\alpha}{\mu}(s-y)\,dy$$

$$Q = H \frac{\rho g \sin\alpha}{2\mu} s^2 - \frac{\rho g \sin\alpha}{\mu}\left[\frac{y^2 s}{2} - \frac{y^3}{3}\right]_0^s$$

$$Q = \frac{\rho g \sin\alpha}{2\mu} H s^2 - \frac{\rho g \sin\alpha}{\mu}\frac{s^3}{6} = \frac{\rho g \sin\alpha}{6\mu}\left[3Hs^2 - s^3\right]$$

$$s = \left(H - \frac{\tau}{\rho g \sin\alpha}\right) \qquad Q = \frac{\rho g \sin\alpha\, H^3}{6\mu}\left[3\left(\frac{s}{H}\right)^2 - \left(\frac{s}{H}\right)^3\right]$$

$$\frac{s}{H} = 1 - \frac{\tau}{\rho g \sin\alpha\, H} \qquad H = \frac{\tau}{(\rho g \sin\alpha)\left(1 - \frac{s}{H}\right)} \qquad \frac{s}{H} = z \in (0,1)$$

$$Q = Q(z) = \frac{\cancel{\rho g \sin\alpha}\,\tau_0^3}{6\mu(\rho g \sin\alpha)^{\cancel{3}\,2}}\,\frac{(3z^2 - z^3)}{(1-z)^3} = \underbrace{\frac{\tau_0^3}{6\mu(\rho g \sin\alpha)^2}}_{\psi}\left[\frac{3z^2 - z^3}{(1-z)^3}\right]$$

$$Q = \psi\left(\frac{3z^2 - z^3}{(1-z)^3}\right) : [0,1) \to \mathbb{R}$$

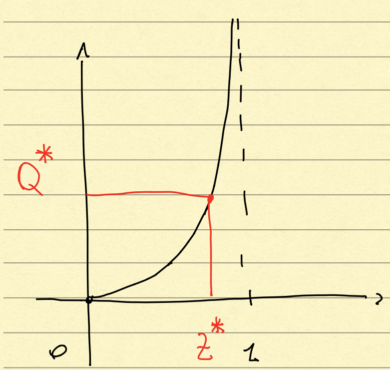
$$Q(0) = 0$$

$$Q(1^-) = +\infty$$

$$Q'(z) = \psi\,\frac{(6z - 3z^2)(1-z)^{\cancel{3}} + 3\cancel{(1-z)^2}(3z^2 - z^3)}{(1-z)^{\cancel{6}\,4}}$$

$$Q'(z) = \psi\,\frac{6z \cancel{-6z^2} \cancel{-3z^2} \cancel{+3z^3} + \cancel{9z^2} \cancel{-3z^3}}{(1-z)^4} = \psi\,\frac{6z}{(1-z)^4} \geq 0$$

$$Q''(z) = \psi\,\frac{6(1-z)^4 + 4(1-z)^3 \cdot 6z}{(1-z)^8} > 0$$

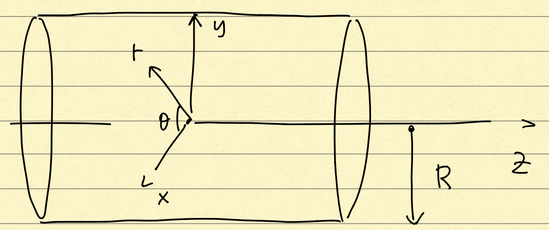


Q è strett. monotono in $(0,1)$

$\forall\, Q^* > 0 \quad \exists!\ z^* = \frac{S^*}{H^*} \in (0,1)$

$$\Rightarrow H^* = \frac{\tau_0}{\rho g \sin\alpha\,(1 - z^*)}$$

Flusso alla Poiseuille in geometria cilindrica



COORD. CILINDRICHE

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$\underline{v} = v(r)\,\underline{e}_z$ $\qquad$ $p(r,\theta,z)$

Il bilancio di massa è automaticamente soddisfatto

$$\rho \frac{dv}{dt} = -\nabla p + \mathrm{div}(\mathbb{S}) + \rho \underline{f} \quad \Longrightarrow$$ Scriverlo in coord. cilindriche

$$0 = \frac{\partial p}{\partial r} \qquad 0 = \frac{\partial p}{\partial \theta} \qquad \underline{f} = 0$$

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\mu \frac{\partial v}{\partial r} - b\right)\right] \quad \Longrightarrow \quad p = p(z)$$

p_z = costante $\Longrightarrow$ $\begin{cases} p_z = -\dfrac{\Delta p}{L} = -G < 0 \\ G > 0 \end{cases}$ $\quad$ L lunghezza condotta

$$-Gr = \left[r(\mu v_r - b)\right]_r \quad \Bigg| \quad c - \frac{Gr^2}{2} = r(\mu v_r - b)$$

$$\frac{\partial v}{\partial r}(s) = 0$$

$$\frac{C}{r} - \frac{Gr}{2} = \mu v_r - \tau_0 \qquad\qquad \frac{C}{S} - \frac{GS}{2} = -\tau_0$$

$$C = \frac{GS^2}{2} - \tau_0 S$$

$$\mu v_r = \frac{G}{2}\frac{S^2}{r} - \frac{\tau_0 S}{r} + \tau_0 - \frac{Gr}{2}$$

$$\mu v_r = \frac{G}{2}\left(\frac{S^2}{r} - r\right) - \tau_0\left(\frac{S}{r} - 1\right) \qquad [S, R]$$

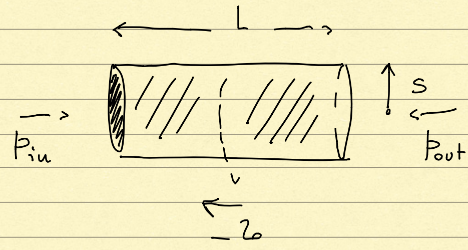
Integro tra r e R $\qquad v(R) = 0$ (No slip)

$$-\mu v(r) = \frac{G}{2}\left(S^2 \ln \tilde{r} - \frac{\tilde{r}^2}{2}\right)\Bigg|_r^R - \tau_0\left(S \ln \tilde{r} - \tilde{r}\right)\Bigg|_r^R \qquad r > S$$

$$\mu v(r) = \frac{G}{2}\left(\frac{R^2 - r^2}{2} - S^2 \ln\left(\frac{R}{r}\right)\right) + \tau_0\left(S \ln\left(\frac{R}{r}\right) - (R - r)\right)$$

Devo det. S

Impongo il bilancio delle forze



$$p_{in}\cancel{\pi} S^2 - p_{out}\cancel{\pi} S^2 - 2\cancel{\pi} S \cdot L \cdot \tau_0 = 0$$

$$\underbrace{\frac{\Delta p}{L}}_{G} S = 2\tau_0 \qquad \boxed{S = \frac{2\tau_0}{G}}$$

$S < R \implies$ cond. di flusso $\quad G > \left(\frac{2\tau}{R}\right) \quad \tau = \frac{GS}{2}$

$$\mu V(r) = \frac{G}{2}\left(\frac{R^2 - r^2}{2} - \cancel{S^2 \ln\left(\frac{R}{r}\right)}\right) + \frac{G}{2}\left(\cancel{S^2 \ln\left(\frac{R}{r}\right)} - S(R-r)\right)$$

$$\boxed{V(r) = \frac{G}{2\mu}\left[\frac{R^2 - r^2}{2} - S(R-r)\right] \qquad \frac{\partial V}{\partial r} = \frac{G}{2\mu}\left[S - r\right]}$$

$V(S)$ = velocità del nucleo rigido $= \frac{G}{4\mu}\left[R^2 - S^2 - 2S(R-S)\right]$

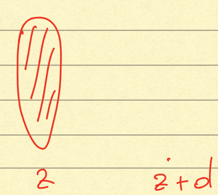
$$V(S) = \frac{GR^2}{4\mu}\left[1 - \left(\frac{S}{R}\right)^2 - 2\left(\frac{S}{R}\right)\left(1 - \frac{S}{R}\right)\right] = \frac{GR^2}{4\mu}\left[1 - \left(\frac{S}{R}\right)\right]^2$$

RELAZIONE CON IL FLUSSO Q

(Equazione di Buckingham)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$Q = \int_{\Sigma} v \, d\sigma \qquad \Sigma = \left\{ r : \begin{array}{l} r \in [0,R] \\ \theta \in [0,2\pi] \end{array} \right\}$$



$$Q = \int_0^{2\pi} d\theta \int_0^R r\, v(r)\, dr = \text{Int. per parti}$$

$$\frac{Q}{\cancel{2}\pi} = \cancel{\left[\frac{r^2}{2} v\right]_0^R} - \int_0^R \frac{r^2}{\cancel{2}} v_r \, dr \qquad V_r = \frac{G}{2\mu}(s - r)$$

$$\frac{Q}{\pi} = -\frac{G}{2\mu} \int_s^R r^2 (s - r) \, dr = -\frac{G}{2\mu}\left[\frac{r^3}{3} s - \frac{r^4}{4}\right]_s^R$$

$$\frac{Q}{\pi} = \frac{G}{2\mu}\left[\frac{R^4 - s^4}{4} - \frac{R^3 s}{3} + \frac{s^4}{3}\right]$$

$$\frac{Q}{\pi} = \frac{GR^4}{8\mu}\left[1 - \left(\frac{s}{R}\right)^4 - \frac{4}{3}\left(\frac{s}{R}\right) + \frac{4}{3}\left(\frac{s}{R}\right)^4\right]$$

$$\frac{Q}{\pi} = \frac{GR^4}{8\mu}\left[1 + \frac{1}{3}\left(\frac{s}{R}\right)^4 - \frac{4}{3}\left(\frac{s}{R}\right)\right] \qquad \frac{s}{R} = \xi \in (0,1)$$

$Q = U\pi R^2$ $\qquad$ U velocità cost. in ingresso

$$1 = \frac{GR^2}{8\mu U}\left(1 + \frac{\xi^4}{3} - \frac{4}{3}\xi\right) \qquad G = \frac{2\tau_0}{s}$$

$$1 = \frac{\cancel{2}\tau_0 R^2}{s \, \cancel{8}^{4} \mu U}(\ldots\ldots) \qquad 1 = \frac{\tau_0 R}{4\mu U\left(\frac{s}{R}\right)}(\ldots\ldots)$$

$$1 = \frac{\tau_0 R}{4\mu U}\cdot\frac{1}{\xi}\left(1+\frac{\xi^4}{3}-\frac{4}{3}\xi\right) \qquad \frac{\tau_0 R}{\mu U} = Bu = \text{numero di Bingham}$$

$$Bu = \frac{\tau_0}{\underbrace{\left(\mu\frac{U}{R}\right)}_{\simeq S_{12}}} \qquad \text{Eq. di Buckingham}$$

$$4\xi = Bu\left(1+\frac{\xi^4}{3}-\frac{4}{3}\xi\right)$$

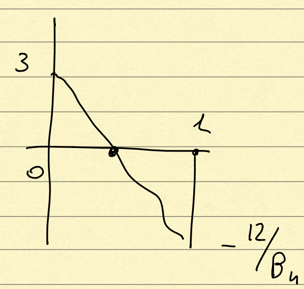
$$1+\frac{\xi^4}{3}-\frac{4}{3}\xi-4\xi\frac{1}{Bu}=0$$

$$\xi^4-4\xi\left(1+\frac{3}{Bu}\right)+3=0 \qquad \text{(Eq. di Buckingham)}$$

$$f(\xi)=\xi^4-4\xi\left(1+\frac{3}{Bu}\right)+3 \;:\; [0,1] \to \mathbb{R}$$

$$f(0)=3>0$$

$$f(1)=-\frac{12}{Bu}<0 \qquad \Rightarrow \exists \text{ almeno } 1 \text{ zero}$$



$$f'(\xi)=4\xi^3-4-\frac{12}{Bu}=4(\xi^3-1)-\frac{12}{Bu}<0$$

$\forall\, Bu^* > 0 \quad \exists!$ soluzione $\xi^* \in (0,1) \qquad \xi=\frac{S}{R}$