$z = |z| e^{i\theta} = \rho e^{i\theta}$

$(e^{i\theta})^n = e^{in\theta}$ $\rightsquigarrow$ Da dimostrare (*)

$e^{i\theta} = \cos\theta + i\sin\theta$

$$(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = \underbrace{\cos^2\theta - \sin^2\theta}_{\cos(2\theta)} + \underbrace{(2\sin\theta\cos\theta)}_{\sin(2\theta)} i = \cos(2\theta) + i\sin(2\theta) = e^{i2\theta}$$

Per il caso $n = 2$ la formula (*) è dimostrata

Suppongo che (*) sia vera per $n$ e la dimostro per $(n+1)$

$$(e^{i\theta})^{n+1} = \underbrace{(e^{i\theta})^n}_{\text{Vale * per } n} \cdot e^{i\theta} = e^{in\theta} \cdot e^{i\theta} = [\cos(n\theta) + i\sin(n\theta)] \cdot [\cos\theta + i\sin\theta] =$$

$$= \underbrace{(\cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta))}_{\cos(n\theta+\theta)} + i\underbrace{(\sin(n\theta)\cos(\theta) + \sin(\theta)\cdot\cos(n\theta))}_{\sin(n\theta+\theta)} =$$

$$= \cos((n+1)\theta) + i\sin((n+1)\theta) = e^{i(n+1)\theta}$$

In definitiva abbiamo dimostrato che $\boxed{(e^{i\theta})^n = e^{i\theta n}}$ $\forall n \in \mathbb{R}$

## RADICE DI UN NUMERO COMPLESSO

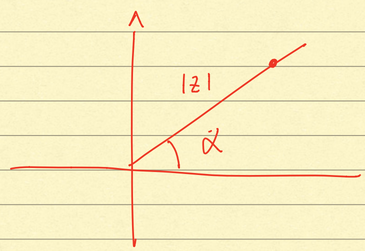
$z^n = w$ con $n \in \mathbb{R}$ $n \neq 0$ $w = \mathbb{C}$

$w$ è dato e quindi avrò la sua rappresent. esponenziale $w = re^{i\varphi}$

Io voglio trovare i numeri complessi $z$ t.c. $z^n = w$

$z$ è la mia incognita $\quad z = \rho e^{i\theta}$ (devo det. $\rho$ e $\theta$)

$$(z^n) = (\rho e^{i\theta})^n \underset{\text{uso } (*)}{=} \boxed{\rho^n e^{in\theta} = r e^{i\varphi}} \quad \begin{cases} \rho^n = r \\ n\theta = \varphi + 2k\pi \end{cases}$$




r e $\varphi$ sono dati

$$\begin{cases} \rho = \sqrt[n]{r} \\ \theta = \dfrac{\varphi}{n} + \dfrac{2k\pi}{n} \end{cases}$$

Ci saranno allora esattamente n radici (complesse)

$$z_0 = \left(\sqrt[n]{r}\right) e^{i\left(\frac{\varphi}{n}\right)} \qquad z_1 = \left(\sqrt[n]{r}\right) e^{i\left(\frac{\varphi}{n} + \frac{2\pi}{n}\right)}$$

$$z_2 = \left(\sqrt[n]{r}\right) e^{i\left(\frac{\varphi}{n} + \frac{4\pi}{n}\right)} \quad \ldots\ldots \quad z_{n-1} = \left(\sqrt[n]{r}\right) e^{i\left(\frac{\varphi}{n} + \frac{2(n-1)\pi}{n}\right)}$$

Le radici di $z^n = w$ sono esattamente $n$

Es

$z^4 = 1$ Quali soluz. ho nel campo dei reali

$t = z^2 \implies t^2 = 1 \quad t = \pm 1 \quad z^2 = \pm 1 \implies z_0 = 1, \; z_2 = -1$

Troviamo le soluz. nel campo complesso

$$\underset{w}{z^4} = 1 \iff (\rho e^{i\theta})^4 = (re^{i\varphi}) = 1 e^{i\cdot 0}$$

$$\rho^4 e^{i4\theta} = 1 \cdot e^{i\cdot 0} \quad \begin{cases} \rho^4 = 1 \to \rho = 1 \\ \boxed{4\theta = 0 + 2k\pi} \end{cases}$$

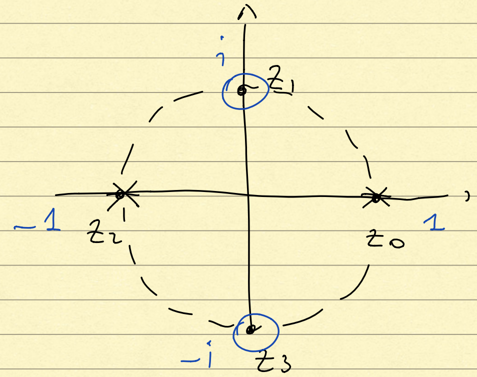
$$z_0 = 1 \cdot e^{i \cdot 0} = 1$$

$$\theta = \frac{k\pi}{2}$$

$$z_1 = 1 \cdot e^{i\pi/2} = i$$

$$z_2 = 1 \cdot e^{i\pi} = -1$$

$$z_3 = 1 \cdot e^{i\frac{3}{2}\pi} = -i$$




Es. «Calcolare le radici di $z^3 = 8i$ »   $z = \rho e^{i\theta}$   $8i = w = re^{i\varphi}$

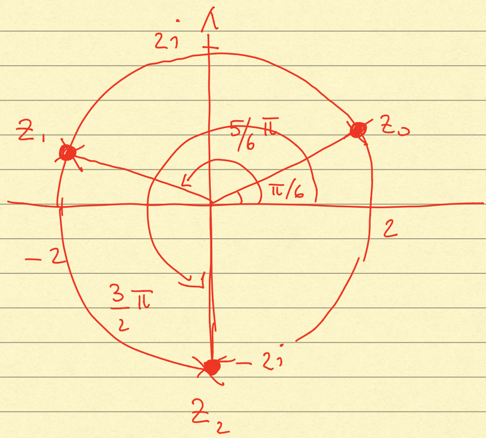
$$\underbrace{\rho^3} e^{i3\theta} = 8i = 8 \cdot e^{i\varphi} = \underbrace{8}e^{i\frac{\pi}{2}}$$

$$i = 0 + 1 \cdot i$$

$$|8i| = |8| \cdot |i| = |8| = 8 \qquad |i| = \sqrt{0^2 + 1^2} = 1$$

$$\begin{cases} \rho^3 = 8 \\ 3\theta = \frac{\pi}{2} + 2k\pi \end{cases} \qquad \begin{cases} \rho = 2 \\ \theta = \frac{\pi}{6} + \frac{2k\pi}{3} \end{cases}$$

$$\boxed{\begin{array}{ll} z_0 = 2e^{i\pi/6} & k=0 \\ z_1 = 2e^{i\overbrace{\left(\frac{\pi}{6} + \frac{2}{3}\pi\right)}^{\frac{5}{6}\pi}} & k=1 \\ z_2 = 2e^{i\underbrace{\left(\frac{\pi}{6} + \frac{4}{3}\pi\right)}_{\frac{3}{2}\pi}} & k=2 \end{array}}$$




Esprimo $z_0, z_1, z_2$ in forma cartesiana

$$z_0 = 2\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = \sqrt{3} + i$$

$$z_1 = 2\left[-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = -\sqrt{3} + i$$

$$z_2 = -2i$$

Es. Calcolare tutte le soluzioni di

$$\boxed{z^4 - 2i\sqrt{3}\, z^2 - 4 = 0} \qquad z^2 = t$$

$$t^2 - 2i\sqrt{3}\, t - 4 = 0 \qquad t_{1,2} = \frac{\cancel{2} i\sqrt{3} \pm \sqrt{(2i\sqrt{3})^2 + \cancel{4}\cdot 4}}{\cancel{2}}$$

$$t_1 = i\sqrt{3} + 1 \longrightarrow \boxed{z^2 = i\sqrt{3} + 1} = r_1 e^{i\varphi_1}$$

$$t_2 = i\sqrt{3} - 1 \longrightarrow \boxed{z^2 = i\sqrt{3} - 1} = r_2 e^{i\varphi_2}$$

$$r_1 = \sqrt{(\sqrt{3})^2 + 1} = 2 \qquad \varphi_1 = \frac{\pi}{3} \qquad z = |z| e^{i\theta}$$

$$r_2 = 2 \qquad \varphi_2 = \frac{2\pi}{3} \qquad z^2 = |z|^2 e^{i2\theta} \qquad |z|^2 = 2$$

$$z^2 = 2e^{\pi/3 \cdot i} \rightsquigarrow |z| = \sqrt{2} \qquad \begin{cases} 2\theta = \pi/3 + 2k\pi & \theta = \dfrac{\pi}{6} + k\pi \\ \boxed{\theta_0 = \dfrac{\pi}{6} \qquad \theta_1 = \dfrac{7}{6}\pi} \end{cases}$$

$$z^2 = 2e^{\frac{2}{3}\pi i} \rightsquigarrow |z| = \sqrt{2} \qquad \begin{cases} 2\theta = \dfrac{2}{3}\pi + 2k\pi & \theta = \dfrac{\pi}{3} + k\pi \\ \boxed{\theta_2 = \dfrac{\pi}{3} \qquad \theta_3 = \dfrac{4}{3}\pi} \end{cases}$$

Quindi le mie soluzioni sono

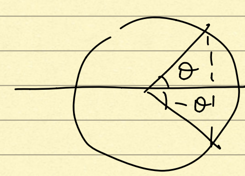
$$z_0 = \sqrt{2}\, e^{i\pi/6} = \sqrt{2}\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = \sqrt{\frac{3}{2}} + i\frac{1}{\sqrt{2}}$$

$$z_1 = \sqrt{2}\, e^{7/6\,\pi i} = \ldots$$

$$z_2 = \sqrt{2}\, e^{\pi/3\, i} = \ldots$$

Finire per esercizio!!

$$z_3 = \sqrt{2}\, e^{4/3\,\pi i} = \ldots$$



Es. Calcolare tutte le soluzioni di $z^3 = 9\bar{z}$

$z = \rho e^{i\theta}$ $\rightsquigarrow$ $(\rho e^{i\theta})^3 = 9 \cdot \rho e^{-i\theta}$

$\bar{z} = \rho e^{-i\theta}$

$\rho^3 e^{i3\theta} = 9\rho e^{-i\theta}$

$\begin{cases} \rho^3 = 9\rho \\ 3\theta = -\theta + 2k\pi \end{cases}$ $\rightsquigarrow$ $\begin{cases} \rho^3 - 9\rho = 0 & \rho(\rho^2 - 9) = 0 \\ \rho = 0 \quad e \quad \rho = \pm 3 & \boxed{\rho = 3} \end{cases}$

$\rho = -3$ ↯

Sicuramente $z_0 = 0$ è soluzione

Le altre soluzioni hanno $\rho = 3$

$4\theta = 2k\pi$ $\qquad \theta = \frac{k\pi}{2}$ $\qquad k = 0, 1, 2, 3$

$z_1 = 3$ $\qquad z_2 = 3 \cdot i$ $\qquad z_3 = -3$ $\qquad z_4 = -3i$

ESERC. : « Calcolare tutte le soluzioni di

$$\boxed{z^3 \bar{z} + 3z^2 - 4 = 0}$$

$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$

$z \cdot \bar{z} = |z|^2$

$z^2 \cdot (z \cdot \bar{z}) + 3z^2 - 4 = 0$ $\qquad z^2 \cdot |z|^2 + 3z^2 - 4 = 0$

$z^2 \cdot (|z|^2 + 3) = 4$ (*)

$\left| z^2 \cdot (|z|^2 + 3) \right| = |4| = 4$ (++)

Oss. $|w \cdot t| = |w| \cdot |t|$

$w = |w| e^{i\theta}$ $\quad |w \cdot t| = \left| |w| e^{i\theta} \cdot |t| e^{i\varphi} \right| =$

$t = |t| e^{i\varphi}$

$= |w||t| \underbrace{\left| e^{i(\theta + \varphi)} \right|}_{=1} = |w| \cdot |t|$

Ritornando alla mia eq. (++)

$\boxed{|z^2| \cdot (|z|^2 + 3) = 4}$ (+++) $\quad z = \rho e^{i\theta}$

$z^2 = \rho^2 e^{i2\theta}$

Oss. $|z^2| = |z|^2$

$|z^2| = |\rho^2 e^{i2\theta}| = \rho^2$ $\implies |z^2| = |z|^2$

$|z|^2 = \rho^2$

Allora (+++) la riscrivo come $|z|^2 \cdot (|z|^2 + 3) = 4$

Pongo $\quad q = |z|^2 \in \mathbb{R}$

$$q(q+3)=4 \iff q^2+3q-4=0$$

$$(q+4)(q-1)=0 \quad \begin{cases} q=1=|z|^2 \Rightarrow |z|^2=1 \\ q=\cancel{-4=|z|^2} \quad q \geq 0 \end{cases}$$

Necessariamente in (*) $|z|^2=1$

$$\Rightarrow \quad z^2(1+3)=4 \quad \Rightarrow \quad z^2=1 \quad \begin{cases} z=1 \\ z=-1 \end{cases}$$

<u>ESERCIZIO</u> Disegnare nel piano l'insieme dei numeri complessi definiti da

$$\begin{cases} \mathrm{Re}\,(i z^2 - i \bar{z}^2) \geq -4 \\ \left| z - \left(\frac{\sqrt{2}}{2} + i\sqrt{2}\right) \right| \leq \sqrt{2} \end{cases}$$

$z = x+iy$

$\bar{z} = x-iy$ $\quad x,y \in \mathbb{R}$

$z^2 = (x+iy)^2 = x^2-y^2+2xyi$

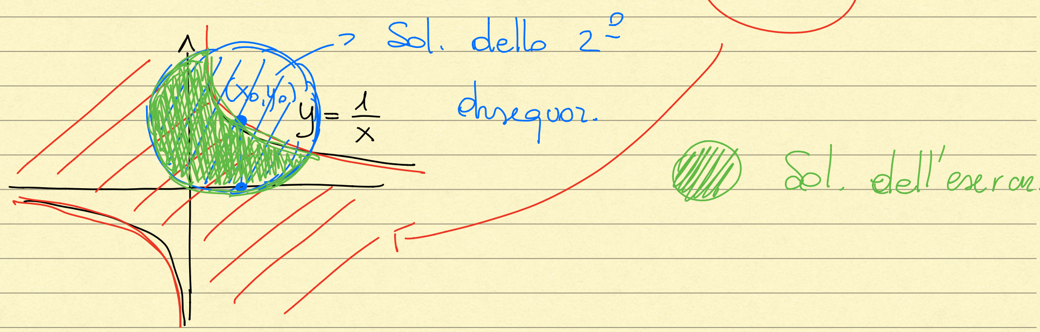
$iz^2 - i\bar{z}^2$

$\bar{z}^2 = (x-iy)^2 = x^2-y^2-2xyi$

$$\underbrace{iz^2 - i\bar{z}^2}_{} = i(z^2-\bar{z}^2) = i(4xyi) = 4xyi^2 = \underbrace{-4xy}_{}$$

$$\mathrm{Re}\,(iz^2 - i\bar{z}^2) = \mathrm{Re}\,(-4xy) = -4xy \geq -4 \iff xy \leq 1$$

$xy = 1$



Oss. $|z - z_0| = R$ nel piano complesso rappresenta un cerchio di centro $z_0 = x_0 + iy_0$ e raggio $R$

$|z - z_0| \leq R$ rappresenta l'interno (bordo compreso) del cerchio centrato in $z_0$ e raggio $R$

$$\left| z - \left( \frac{\sqrt{2}}{2} + i\sqrt{2} \right) \right| \leq \sqrt{2}$$

$$z_0 = \underbrace{\frac{\sqrt{2}}{2}}_{\to\, x_0} + i\,\underbrace{\sqrt{2}}_{\to\, y_0}$$

Se faccio $x_0 \cdot y_0 = 1$