$$\boxed{\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e} \quad (L1)$$

x	$y = \left(1+\frac{1}{x}\right)^x$
1	...
2	
3	
.	
100	
1000	2.7172 ....

$$\left(\frac{a}{b}\right)^{-1} = \left(\frac{b}{a}\right)$$

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = ?? \quad (e) \quad = (*)$$

$$(*) \begin{bmatrix} \text{sost. di variabile} \\ x = -s \\ x \to -\infty \\ s \to +\infty \end{bmatrix} = \lim_{s \to +\infty} \left(1 - \frac{1}{s}\right)^{-s} = \lim_{s \to +\infty} \left(\frac{s-1}{s}\right)^{-s} =$$

$$= \lim_{s \to +\infty} \left(\frac{s}{s-1}\right)^{s} = \lim_{s \to +\infty} \left(1 + \frac{1}{s-1}\right)^{s} = \begin{bmatrix} \text{SOST.} \\ \frac{1}{s-1} = \frac{1}{z} \\ s \to +\infty \\ z \to +\infty \end{bmatrix} \nearrow s = z+1 =$$

$$= \lim_{z \to +\infty} \left(1 + \frac{1}{z}\right)^{z+1} = \lim_{z \to +\infty} \underbrace{\left(1 + \frac{1}{z}\right)^{z}}_{\to e} \cdot \underbrace{\left(1 + \frac{1}{z}\right)^{1}}_{\to 1} = \boxed{e}$$

$$\boxed{\lim_{x \to \pm\infty} \left(1 + \frac{t}{x}\right)^x = e^t} \quad (L2) \qquad t \in \mathbb{R}$$

$$* = \begin{bmatrix} \frac{t}{x} = \frac{1}{s} \quad ts = x \\ s \to \pm\infty \\ t \to +\infty \end{bmatrix} = \lim_{s \to \pm\infty} \left(1 + \frac{1}{s}\right)^{ts} =$$

$$\lim_{s \to \pm\infty} \left[\underbrace{\left(1 + \frac{1}{s}\right)^s}_{\to e} \right]^t = e^t$$

ESEMPIO LIMITE NOTEVOLE

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \frac{0}{0} = \boxed{\begin{cases} \text{Non } \nexists \\ 0 \\ \ell \in \mathbb{R} \quad \ell \neq 0 \\ \pm\infty \end{cases}} = 1$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \left[\begin{array}{ll} \text{Sost.} & \text{In generale} \\ s = \frac{1}{x} & s \to \pm\infty \\ x \to 0^+ & s \to +\infty \\ x \to 0^- & s \to -\infty \end{array} \right] = \lim_{s \to \pm\infty} s \ln\left(1 + \frac{1}{s}\right) =$$

$$= \lim_{s \to \pm\infty} \ln\left[\left(1 + \frac{1}{s}\right)^s\right] = \ln\left[\underbrace{\lim_{s \to \pm\infty} \left(1 + \frac{1}{s}\right)^s}_{e}\right] = \ln(e) = 1$$

$$\boxed{\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1 \quad \text{(L3)}}$$

Esempio

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \left[\begin{array}{lll} \text{Sost.} & x \to 0 & \\ e^x - 1 = s & s \to 0 & e^x = s+1 \\ x \to 0^+ & s \to 0^+ & x = \ln(s+1) \\ x \to 0^- & s \to 0^- & \end{array} \right]$$

$$\lim_{s \to 0} \frac{s}{\ln(s+1)} = \lim_{s \to 0} \frac{1}{\underbrace{\left[\frac{\ln(s+1)}{s}\right]}_{\to 1}} = \frac{1}{1} = 1$$

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad (L4)}$$

Es.

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \alpha \in \mathbb{R}$$

(Se $\alpha = 0$ il risultato del limite è 0)

$\alpha \neq 0$

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \left[\begin{array}{ll} \text{Sost.} & \\ (1+x)^\alpha = e^t & \left[(1+x)^\alpha\right]^{1/\alpha} = \left(e^t\right)^{1/\alpha} \\ x \to 0 & \\ t \to 0 & (1+x) = e^{t/\alpha} \\ & x = e^{t/\alpha} - 1 \end{array} \right]$$

$$\lim_{t\to 0} \frac{e^t-1}{e^{t/\alpha}-1} = \lim_{t\to 0} \underbrace{\frac{e^t-1}{t}}_{\to 1} \cdot \frac{t}{e^{t/\alpha}-1} =$$

$$= \lim_{t\to 0} 1 \cdot \frac{1}{\left(\frac{e^{t/\alpha}-1}{t/\alpha}\right)} \cdot \alpha = \lim_{z\to 0} \frac{\alpha}{\underbrace{\left(\frac{e^z-1}{z}\right)}_{\to 1}} = \frac{\alpha}{1} = \alpha$$

Con la sost.

$z = t/\alpha$

$$\boxed{\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \quad (L5)} \qquad \boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad (L6)}$$

L1 - . . . - L6 (Noti, conosciuti)

ESEMPI $\quad \lim_{x\to 0} \frac{a^x-1}{x} = \ldots \qquad \lim_{x\to 0} \frac{\log_a(x+1)}{x} = \ldots$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \cdot \ln a}$$

$$\lim_{x \to 0} \left(\underbrace{\frac{e^{x \ln a} - 1}{\ln a \, x}}_{\to 1} \right) \ln a = \ln(a) \qquad \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

$$x \to 0$$

$$x \cdot \ln a \to 0$$

$$\log_a(1+x) = \frac{\log_e(1+x)}{\log_e a} = \frac{\ln(1+x)}{\ln a}$$

$$\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \lim_{x \to 0} \left(\overbrace{\frac{\ln(1+x)}{\ln a \cdot x}}^{\to 1} \right) = \frac{1}{\ln a}$$

Ho dimost. che $\displaystyle \lim_{x \to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln a}$

ESERC. Calcolo limiti

$$\lim_{x \to -1} \frac{x^{1/3}+1}{x+1} = \lim_{x \to -1} \frac{\cancel{(x^{1/3}+1)}}{\cancel{(x^{1/3}+1)}(x^{2/3}-x^{1/3}+1)} = \frac{1}{3}$$

$$(a^3 \pm b^3) = (a \pm b)(a^2 \mp ab + b^2)$$

$$(x+1) = \underbrace{(x^{1/3}+1)(x^{2/3}-x^{1/3}+1)}$$

Es

$$\lim_{x \to 1} \frac{\overbrace{(x+1)}^{a} - \overbrace{2\sqrt{x}}^{b}}{(x-1)^2} \cdot \left[\frac{\overbrace{(x+1)}^{a} + \overbrace{2\sqrt{x}}^{b}}{(x+1)+2\sqrt{x}}\right]$$

$$\lim_{x \to 1} \frac{(x+1)^2 - (2\sqrt{x})^2}{(x-1)^2 \cdot \left[(x+1)+2\sqrt{x}\right]} =$$

$$\lim_{x \to 1} \frac{\cancel{x^2+1+2x-4x}}{\cancel{(x-1)^2}\left[(x+1)+2\sqrt{x}\right]} = \frac{1}{4}$$

Es

$$\lim_{x \to +\infty} \left(\frac{1}{x}\right)^{1/3} \cdot \left(1+x^2\right)^{1/2} =$$

$$\lim_{x \to +\infty} \frac{\left(1+x^2\right)^{1/2}}{x^{1/3}} = \lim_{x \to +\infty} \frac{x\left(1+\frac{1}{x^2}\right)^{1/2}}{x^{1/3}} =$$

$$= \lim_{x \to +\infty} x^{\frac{2}{3}} \left(1+\frac{1}{x^2}\right)^{1/2} = +\infty$$

Es. $\lim_{x \to +\infty} \sqrt{x}\left(\sqrt{x} - \sqrt{1+x}\right) = \infty \cdot (\infty - \infty)$??

$$= \lim_{x \to +\infty} \frac{\sqrt{x}\,\overbrace{\left(\sqrt{x} - \sqrt{1+x}\right)}^{a\,-\,b} \cdot \overbrace{\left(\sqrt{x} + \sqrt{1+x}\right)}^{a\,+\,b}}{\left(\sqrt{x} + \sqrt{1+x}\right)} =$$

$$= \lim_{x \to +\infty} \left\{ \frac{\sqrt{x}\left[\cancel{x} - (1+\cancel{x})\right]}{\sqrt{x} + \sqrt{1+x}} \right\} = \lim_{x \to +\infty} -\frac{\sqrt{x}}{\sqrt{x} + \sqrt{1+x}}$$

$$= \lim_{x \to +\infty} - \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\left[1 + \sqrt{1 + \frac{1}{x}}\right]} = -\frac{1}{2}$$

Es

$$\lim_{x \to 81} \frac{3 - x^{1/4}}{9 - x^{1/2}} = \left(\begin{array}{ll} t = x^{1/4} & t^2 = x^{1/2} \\ x \to 81 & \\ t \to 3 & \end{array} \right)$$

$$= \lim_{t \to 3} \left(\frac{3 - t}{9 - t^2} \right) = \lim_{t \to 3} \frac{\cancel{(3 - t)}}{\cancel{(3 - t)}(3 + t)} = \frac{1}{6}$$

Es

$$\lim_{x \to 0} \frac{\sin^4 x}{(1 - \cos x)^2} = (**)$$

Rk

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2} =$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2 (1+\cos x)} = \lim_{x \to 0} \underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1} \cdot \frac{1}{1+\cos x} =$$

$$= 1/2$$

$$(*)(*) = \lim_{x \to 0} \frac{x^4}{(1 - \cos x)^2} \cdot \frac{\sin^4 x}{x^4} =$$

$$= \lim_{x \to 0} \left(\underbrace{\frac{x^2}{1 - \cos x}}_{\to 2}\right)^2 \cdot \left(\underbrace{\frac{\sin x}{x}}_{\to 1}\right)^4 = 2^2 = 4$$

Es. $$\lim_{x \to 0} \frac{3x + \tan x}{\sin x + \tan^2 x} = 3$$

Rn

$$\lim_{x \to 0} \left(\frac{\tan x}{x}\right) = \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \frac{1}{\cos x} = 1$$

$$\lim_{x \to 0} \frac{\cancel{x}\left(3 + \frac{\tan x}{x}\right)}{\cancel{x}\left(\underbrace{\frac{\sin x}{x}}_{\downarrow\, 1} + \underbrace{\frac{\tan x}{x}}_{\downarrow\, 1} \cdot \underbrace{\tan x}_{\downarrow\, 0}\right)} = \frac{3}{1} = 3$$

<u>Es.</u> $$\lim_{x \to \pm\infty} |x| \cdot \log \frac{(x^2 + x + 1)}{(x^2 + 2)} =$$

$$= \lim_{x \to \pm\infty} |x| \log\left(1 + \frac{x-1}{x^2+2}\right)$$

$$= \lim_{x \to \pm\infty} \log\left(1 + \frac{x-1}{x^2+2}\right)^{|x|} =$$

$$\underbrace{\frac{x-1}{x^2+2}}_{\sim\, 1/x} = \frac{1}{t}$$

$x \to \pm\infty$

$t \to \pm\infty$

$|x|$ se $x \to +\infty$ e x

$|x|$ se $x \to -\infty$ $-x$

$$= \lim_{t \to \pm\infty} \log\left(1 + \frac{1}{t}\right)^{|x|} = (*)$$

$$t(x - 1) = x^2 + 2$$

$$x^2 - tx + (2 + t) = 0$$

$$x_1 = \frac{t + \sqrt{t^2 - 4(2+t)}}{2}$$

$$x_2 = \frac{t - \sqrt{t^2 - 4(2+t)}}{2}$$

$$x = \frac{t \pm \sqrt{t^2 - 4(2+t)}}{2}$$

$$\frac{\left|t \pm \sqrt{t^2 - 4(2+t)}\right|}{2t}$$

$$* = \lim_{t \to \pm\infty} \left[\log\left(1 + \frac{1}{t}\right)^{t}\right]^{\frac{\left|t \pm \sqrt{t^2 - 4(2+t)}\right|}{2t}}$$

$$* = \log\left[e^{\lim_{t \to \pm\infty} \frac{\left| t \pm \sqrt{t^2 - 4(2+t)} \right|}{2t}} \right] = (*)$$

?

Devo risolvere

$$\lim_{t \to \pm\infty} \frac{\left| t + \sqrt{t^2 - 4(2+t)} \right|}{2t}$$

$$\lim_{t \to \pm\infty} \frac{\left| t + \sqrt{t^2 - 4(2+t)} \right|}{2t} = \checkmark$$ Questo mi da il risultato del limite

$$\lim_{t \to \pm\infty} \frac{|t| \cdot \left| 1 + \sqrt{1 - \frac{4(2+t)}{t^2}} \right|}{2t}$$

1 se $t \to +\infty$

-1 se $t \to -\infty$

$$\lim_{x \to \pm\infty} |x| \log\left(1 + \frac{x-1}{x^2+2}\right) =$$

$$= \lim_{x \to \pm\infty} |x| \cdot \left(\frac{x^2+2}{x-1}\right) \cdot \left(\frac{x-1}{x^2+2}\right) \log\left(1 + \frac{x-1}{x^2+2}\right) =$$

$$= \lim_{x \to \pm\infty} \frac{|x|(x-1)}{x^2+2} \underbrace{\log \underbrace{\left[\left(1 + \frac{x-1}{x^2+2} \right)^{\frac{x^2+2}{x-1}} \right]}_{\to e}}_{\to 1}$$

$$= \lim_{x \to \pm\infty} \frac{|x|(x-1)}{x^2+2} \begin{cases} +\infty & = \lim_{x \to +\infty} \dfrac{x^2 - x}{x^2+2} = 1 \\ -\infty & \lim_{x \to -\infty} \dfrac{-x^2+x}{x^2+2} = -1 \end{cases}$$