$$\boxed{\begin{array}{l} \dfrac{\ln(1+f(x))}{f(x)} \to 1 \quad \text{se} \quad f(x) \to 0 \\[2ex] \dfrac{e^{f(x)}-1}{f(x)} \to 1 \quad \text{"} \\[2ex] \dfrac{\sin(f(x))}{f(x)} \to 1 \quad \text{"} \end{array}}$$

$\sin x \sim x$ → $\sin^4 x \sim x^4$

$\tan x \sim x$ → $\tan^4 x \sim x^4$ se $x \to 0$

$\ln(1+x) \sim x$ se $x \to 0$

$\sim>$

$\ln(1+f(x)) \sim f(x)$

$e^{f(x)}-1 \sim f(x)$

Es
$$\lim_{x\to 0}\left\{\frac{\ln(1+\operatorname{tg}^4 x)}{e^{2\sin^4 x}-1}\right\} = \lim_{x\to 0}\frac{\operatorname{tg}^4 x}{2\sin^4 x} = \lim_{x\to 0} = \lim_{x\to 0}\frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2}$$

$\cancel{\sin(\pi\cos x) \sim \cos x \cdot \pi}$

Es
$$\lim_{x\to 0}\frac{\sin(\pi\cos x)}{x\sin x} =$$

$1 = \sin^2 x + \cos^2 x$ $\qquad \cos x = \sqrt{1-\sin^2 x} = (1 \underbrace{-\sin^2 x}_{f(x)})^{1/2}$

$$\lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \lim_{x\to 0}\frac{(1+f(x))^\alpha - 1}{f(x)} = \alpha \quad \text{se} \quad \underset{x\to 0}{f(x) \to 0}$$

$(1+f(x))^\alpha - 1 \sim \alpha f(x)$ se $f(x) \to 0$

$(1+f(x))^\alpha \sim 1 + \alpha f(x)$ $\qquad (1-\sin^2 x)^{1/2} \sim 1 - \dfrac{\sin^2 x}{2}$

$$\boxed{\sin(\pi\cos x)} = \sin\left(\pi\sqrt{1-\sin^2 x}\right) = \sin\left(\pi\left(1-\frac{\sin^2 x}{2}\right)\right) =$$

$$= \sin\left(\pi - \frac{\pi}{2}\sin^2 x\right) = \cancel{\sin(\pi)\cos\left(\frac{\pi}{2}\sin^2 x\right)} \boxed{- \sin\left(\frac{\pi}{2}\sin^2 x\right)\cdot\cos(\pi)}$$

Allora il limite lo riscrivo come

$$\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}\sin^2 x\right)}{x\sin x} = \lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}x^2\right)}{x^2} = \lim_{x\to 0}\frac{\frac{\pi}{2}\cancel{x^2}}{\cancel{x^2}} = \frac{\pi}{2}$$

$\sin x \sim x$

Es
$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} = \begin{cases} \ln(1+f(x)) \sim f(x) \\ f(x) \text{ infinit.mo} \end{cases}$$

$$\cos x = \sqrt{1-\sin^2 x} = (1-\sin^2 x)^{1/2} \sim 1 - \frac{\sin^2 x}{2} \sim 1 - \frac{x^2}{2}$$

$$\boxed{\ln(\cos x)} \sim \ln\left(1 - \frac{x^2}{2}\right) \sim \boxed{-\frac{x^2}{2}}$$

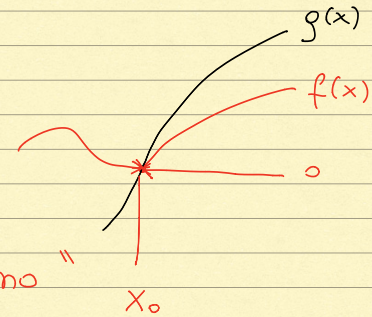
$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} = \lim_{x\to 0}\frac{-x^2/2}{x^2} = -\frac{1}{2}$$

Es
$$\lim_{x\to 0}\frac{\sin(2x)}{\tan(3x)} = \lim_{x\to 0}\frac{2x}{3x} = \frac{2}{3}$$

Vanno a zero più veloc. di x^2

Es
$$\lim_{x\to 0}\boxed{\frac{x^3 + x^2\sin x + \sin^2 x}{x^4 + x^3 + x\sin x}} = \lim_{x\to 0}\frac{\cancel{x^3} + \cancel{x^3} + x^2}{\cancel{x^4} + \cancel{x^3} + x^2}$$

$$= \lim_{x \to 0} \frac{x^2}{x^2} = 1$$



"o - piccoli di un infinitesimo"

$f(x), g(x)$ infinitesime $x \to x_0$ $\begin{cases} \lim_{x \to x_0} f(x) = 0 \\ \lim_{x \to x_0} g(x) = 0 \end{cases}$

Def. : " $f(x)$ è un o-piccolo di $g(x)$ $\quad f(x) = o(g(x))$

se $\displaystyle \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

Se $f(x) = o(g(x))$ $\quad f(x)$ va a zero più rapidamente di $g(x)$

($f(x)$ è un infinitesimo di ordine superiore a $g(x)$

ESEMPIO

$f(x) = \sin(x^2)$

$g(x) = x$

$x \to 0$, f, g infinitesime

$\Rightarrow$ $\boxed{f(x) = o(g(x)) \text{ per } x \to 0}$

Infatti $\displaystyle \lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{x \to 0} \frac{x^2}{x} = 0 = \lim_{x \to 0} \frac{f(x)}{g(x)}$

Esempio

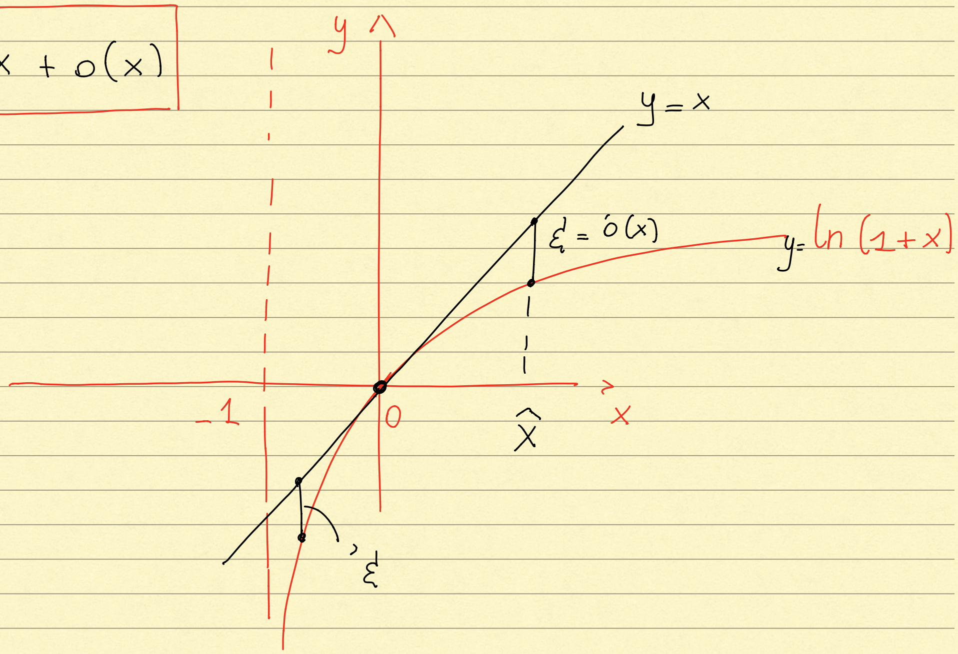
$$\begin{cases} f(x) = \ln(1+x) - x \\ g(x) = x \end{cases} \qquad \text{Infinitesimi per } x \to 0$$

$$\frac{f(x)}{g(x)} = \frac{\ln(1+x) - x}{x} = \frac{\ln(1+x)}{x} - 1 \;\; \boxed{(x \to 0)} = 0$$

$$\Rightarrow \quad f(x) = o(x)$$

$$\ln(1+x) - x = o(x) \qquad \Rightarrow \quad \ln(1+x) \sim x$$

$$\boxed{\ln(1+x) = x + o(x)}$$



Se considero

$$f(x) = \sin x - x$$

$$g(x) = x$$

anche in questo caso $f(x) = o(g(x)) = o(x)$

E allora $\boxed{\sin x = x + o(x)}$

In generale

$$\boxed{\begin{aligned} \sin(h(x)) &= h(x) + o(h(x)) \\ \ln(1+h(x)) &= h(x) + o(h(x)) \\ e^{h(x)} - 1 &= h(x) + o(h(x)) \\ (1+h(x))^{\alpha} &= 1 + \alpha h(x) + o(h(x)) \end{aligned}}$$

$$\boxed{\begin{aligned} h(x) &\to 0 \\ x &\to x_0 \end{aligned}}$$

PRINCIPIO DI SOST. DEGLI INFINITESIMI

$f(x), f_1(x)$ con f, f_1, g, g_1 infinitesimi per $x \to x_0$

$g(x), g_1(x)$

e supponiamo che $f_1(x) = o(f(x))$ $g_1(x) = o(g(x))$

ossia f_1 e g_1 infinitesimi di ordine superiore a f, g risp.nte.

$$\lim_{x \to x_0} \frac{f_1}{f} = \lim_{x \to x_0} \frac{g_1}{g} = 0$$

$\Rightarrow$ (Principio di sost.) $\displaystyle \lim_{x \to x_0} \frac{f(x) + \cancel{f_1(x)}}{g(x) + \cancel{g_1(x)}} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$

$$\lim_{x \to x_0} \frac{f(x) + o(\cancel{f(x)})}{g(x) + o(\cancel{g(x)})} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$$

Riprendiamo l'esercizio di prima $\displaystyle \lim_{x \to 0} \frac{x^3 + x^2 \sin x + \sin^2 x}{x^4 + x^3 + x \sin x}$

$$\sin x \cdot x^2 = o(x^2) \qquad \lim_{x \to 0} \frac{\sin x \cdot x^2}{x^2} = 0$$

$$x^3 = o(x^2) \qquad \lim_{x \to 0} \frac{x^3}{x^2} = 0$$

$$x^4 = o(x^2) \qquad \lim_{x \to 0} \frac{x^4}{x^2} = 0$$

$$\lim_{x \to 0} \frac{\sin^2 x + o(x^2) + o(x^2)}{x \cdot \sin x + o(x^2) + o(x^2)} = *$$

$$\sin x = x + o(x) \qquad x \cdot \sin x = x^2 + \underbrace{x \cdot o(x)}$$

$$\underbrace{x \cdot o(x) = o(x^2)}_{\text{Vera !!}} \quad \leftrightsquigarrow \quad \lim_{x \to 0} \frac{\cancel{x} \cdot o(x)}{x^{\cancel{2}}} = \lim_{x \to 0} \frac{o(x)}{x} = 0$$

$$\sin^2 x = (x + o(x))^2 = x^2 + o(x) \cdot o(x) + \underbrace{2\, x \cdot o(x)}$$

$$= x^2 + \underbrace{o(x) \cdot o(x)} + \underbrace{2\, o(x^2)} = x^2 + o(x^2) + o(x^2)$$

$$\underbrace{o(x) \cdot o(x) = o(x^2)}_{\text{Vera}} \quad \leftrightsquigarrow \quad \lim_{x \to 0} \frac{o(x) \cdot o(x)}{x \cdot x} = 0$$

$$2 \cdot o(x^2) = o(x^2) \quad \leftrightsquigarrow \quad \lim_{x \to 0} \frac{2 \cdot o(x^2)}{x^2} = 0$$

$$\lim_{x \to 0} \frac{x^2 + 4o(x^2)}{x^2 + 3o(x^2)} = \lim_{x \to 0} \frac{x^2 + \cancel{o(x^2)}}{x^2 + \cancel{o(x^2)}} = \lim_{x \to 0} \frac{x^2}{x^2} = 1$$

ALGEBRA DEGLI o-Piccoli

$$\boxed{\begin{array}{ll}
i) & o(f(x)) \pm o(f(x)) = o(f(x)) \\
ii) & o(f(x)) \cdot o(g(x)) = o(f(x) \cdot g(x)) \\
iii) & c \cdot o(f(x)) = o(f(x)) \qquad c \neq 0 \\
iv) & \text{se } f(x) = x^m \quad g(x) = x^n \quad m,n > 0 \\
 & \text{infinitesimi per } x \to 0 \\
 & \boxed{o(x^m) + o(x^n) = o(x^p)} \qquad p = \min\{m,n\}
\end{array}}$$

Prop. iv (dimostrazione)

$$\lim_{x \to 0} \frac{o(x^m) + o(x^n)}{x^p} = \lim_{x \to 0} \frac{o(x^m)}{x^p} + \frac{o(x^n)}{x^p} =$$

$$= \begin{cases} \text{Se } p = m \quad m < n & \displaystyle\lim_{x \to 0} \cancel{\frac{o(x^m)}{x^m}} + \frac{o(x^n)}{x^m} = \\ & \displaystyle = \lim_{x \to 0} \cancel{\frac{o(x^n)}{x^n}} \cdot x^{\overset{>0}{\cancel{(n-m)}}} = 0 \\ p = n \text{ si dimostra in maniera analoga!} & \end{cases}$$

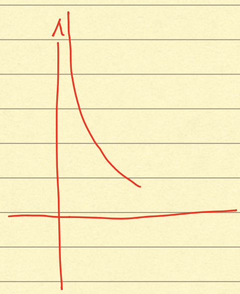
ESERCIZI

$$\lim_{x \to 0^+} \left(\log_3 x + \frac{1}{x} \right) = \infty - \infty = \lim_{x \to 0} \left(\frac{\ln x}{\ln 3} + \frac{1}{x} \right) =$$

$$\left. \log_3 x = \left(\frac{\ln x}{\ln 3} \right) \right\} \quad = \lim_{x \to 0^+} \left(\frac{\overbrace{x \cancel{\ln x}}^{\to 0} + \ln 3}{x \ln 3} \right) = * = \frac{\cancel{\ln 3}}{0^+ \cdot \cancel{\ln 3}} = \frac{1}{0^+} = +\infty$$

Dobbiamo vedere dove tende $x \ln x$ se $x \to 0^+$

$$\lim_{x \to 0^+} x \cdot \ln x = \left\{ \begin{array}{l} x = \frac{1}{t} \\ x \to 0^+ \\ t \to +\infty \\ t = \frac{1}{x} \end{array} \right\} = \lim_{t \to +\infty} \frac{\ln\left(\frac{1}{t}\right)}{t} = \lim_{t \to +\infty} - \frac{\ln t}{t} = 0^-$$



Es. $\lim_{x \to +\infty} x e^{x} \cdot \sin \left[e^{-x} \cdot \sin\left(\frac{2}{x}\right) \right] = \infty \cdot 0 = ??$

$$\underbrace{\left[e^{-x} \cdot \sin\left(\frac{2}{x}\right) \right]}_{\to 0}$$

$[\ldots]$ è infinitesimo

per $x \to +\infty$

$$\sin(f(x)) \sim f(x)$$

$$\sin\left(e^{-x} \cdot \sin\left(\frac{2}{x}\right)\right) \sim e^{-x} \sin\left(\frac{2}{x}\right)$$

$$\Rightarrow \lim_{x \to +\infty} x e^{x} \sin(\ldots) = \lim_{x \to +\infty} x \cancel{e^{x}} \cdot \cancel{e^{-x}} \cdot \sin\left(2/x\right) = \lim_{x \to +\infty} x \cdot \sin\left(\frac{2}{x}\right)$$

$$= \lim_{x \to +\infty} \underbrace{\frac{\sin(2/x)}{2/x}}_{\to 1} \cdot 2 = 2$$

$$2 \cdot a = \frac{a}{\left(\frac{1}{2}\right)}$$

$$x \sin\left(\frac{2}{x}\right) = \frac{\sin\left(\frac{2}{x}\right)}{\left(\frac{1}{x}\right) \cdot 2} \cdot 2$$

$$x \sin\left(\frac{2}{x}\right) \sim x \cdot \frac{2}{x} = 2$$

Es

$$\lim_{x \to \frac{\pi}{2}} \tan x \left(e^{\cos x} - 1\right) = \pm\infty \cdot 0$$

$e^{f(x)} - 1 \sim f(x)$ se $f(x)$ è infinitesimo

$e^{\cos x} - 1 \sim \cos x$ se $\cos x \to 0$

$$\lim_{x \to \frac{\pi}{2}} \tan x \cdot \cos x = \lim_{x \to \frac{\pi}{2}} \sin x = 1$$

Es

$$\lim_{x \to 0} x \tan\left(xa + \arctan\left(\frac{b}{x}\right)\right) = \pm\infty \cdot 0$$

$$\boxed{\tan(A+B)} = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B} =$$

$$= \frac{\cancel{\cos A\cos B}\left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right)}{\cancel{\cos A\cos B}\left(1 - \frac{\sin A\sin B}{\cos A\cos B}\right)} = \boxed{\left(\frac{\tan A + \tan B}{1 - \tan A\tan B}\right)}$$

$$\tan\left(xa + \arctan\left(\frac{b}{x}\right)\right) = \frac{\tan(xa) + \boxed{\tan\left(\arctan\left(\frac{b}{x}\right)\right)}}{1 - \tan(xa)\underbrace{\boxed{\tan\left(\arctan\left(\frac{b}{x}\right)\right)}}_{= b/x}}$$

$$\tan\left(ax + \arctan\left(\frac{b}{x}\right)\right) = \frac{\tan(ax) + b/x}{1 - \tan(ax)\cdot\frac{b}{x}}$$

$a, b \in \mathbb{R}$

$$\lim_{x\to 0} x\tan\left(ax + \arctan\left(\frac{b}{x}\right)\right) = \lim_{x\to 0} x\cdot\left[\frac{\tan(ax) + b/x}{1 - \tan(ax)\cdot\frac{b}{x}}\right] =$$

$$= \lim_{x\to 0} \frac{x\cancel{\tan}(ax) + b}{1 - \frac{b\cdot\tan(ax)}{x}} =$$

Sappiamo che

$\tan(f(x)) \sim f(x)$

$f(x)$ infinitesimo

$$= \lim_{x\to 0} \frac{b}{1 - b\cdot\frac{ax}{x}} = \left(\frac{b}{1 - b\cdot a}\right)$$

Es

$$\lim_{x \to 0} \frac{x \sin x}{|x|} = \lim_{x \to 0} \frac{x^2}{|x|} \cdot \underbrace{\left(\frac{\sin x}{x}\right)}_{1} = 0$$

$$\lim_{x \to 0} \frac{x \cos x}{|x|} = \begin{cases} \lim_{x \to 0^+} \dfrac{x \cos x}{x} = 1 \\ \lim_{x \to 0^-} \dfrac{x \cos x}{-x} = -1 \end{cases}$$

$$\lim_{x \to 0^{\pm}} \frac{x \cos x}{|x|} = \pm 1$$