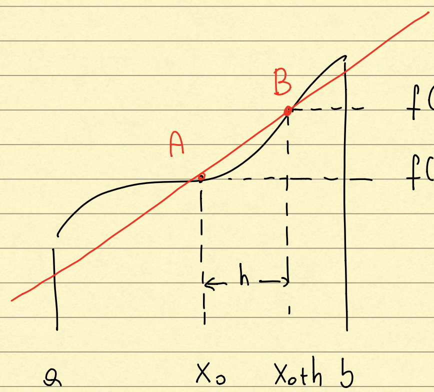
$f(x) : (a,b) \to \mathbb{R}$ $\qquad x_0 \in (a,b)$




$$R = \frac{f(x_0+h) - f(x_0)}{h}$$

Coeff. ang. della retta per

$A \equiv (x_0, f(x_0))$

$B = (x_0+h, f(x_0+h))$

Def. Se $\exists$ il limite

$$\lim_{h \to 0} R = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) \in \mathbb{R}$$

questo limite lo chiamo derivata 1° di $f$ in $x_0$ e lo indico con uno dei seguenti modi $\quad f'(x_0) \quad \frac{df}{dx}(x_0) \quad Df(x_0) \ldots$

Posso introdurre la derivata dx e sx

$$f'_+(x_0) = \lim_{h \to 0^+} R \qquad f'_-(x_0) = \lim_{h \to 0^-} R \qquad (*)$$

Def. Una funzione $f$ si dice derivabile in $x_0$ se $\exists$ "finiti" i limiti $(*)$ e $\quad f'_+(x_0) = f'_-(x_0)$

Es. La funzione $|x|$ non è derivabile in $x_0 = 0$

$f(x) = |x|$

$x_0 = 0$

$$R_+ = \frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \frac{h}{h} = 1 \qquad h > 0$$

$$R_- = \frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} = -\frac{h}{h} = -1 \qquad h<0$$

$$\lim_{h\to 0^+} R_+ = 1 \neq -1 = \lim_{h\to 0^-} R_-$$

## ESEMPI DI DERIVATE DI FUNZ. ELEMENTARI

$$f(x) = x^n \qquad n \in \mathbb{R}$$

Supponiamo $x_0 \in dom(f)$ e supp.mo di voler calcolare $f'(x_0)$

$$R = \frac{f(x_0+h)-f(x_0)}{h} = \frac{(x_0+h)^n - x_0^n}{h}$$

$$\lim_{h\to 0} \frac{(x_0+h)^n - x_0^n}{h} = \lim_{h\to 0} \frac{x_0^n\left[\left(1+\frac{h}{x_0}\right)^n - 1\right]}{x_0 \cdot \frac{h}{x_0}} = n\, x_0^{n-1}$$

Dal momento che $x_0$ è arbitrario lo posso sostituire con $x$

$$f(x) = x^n \qquad f'(x) = n x^{n-1}$$

$$n=0 \qquad f(x) = 1 \qquad f'(x) = 0$$

$$n=1 \qquad f(x) = x \qquad f'(x) = 1$$

ESEMPIO $f(x) = \ln x : (0, +\infty) \to \mathbb{R}$

$$f'(x) = \left(\text{se esiste}\right) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \underbrace{\boxed{\frac{\ln(1 + h/x)}{h/x}}}_{\downarrow 1} \cdot \frac{1}{x} = \frac{1}{x}$$

$$f(x) = \ln x \implies f'(x) = \frac{1}{x}$$

ESEMPIO $f(x) = \sin x \qquad f'(x) = \cos x$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \left\{ \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} \right\} =$$

$$\lim_{h \to 0} \left(\frac{\sin h}{h}\right) \cos x + \sin x \left(\frac{\cos h - 1}{h}\right) =$$

$$= \lim_{h \to 0} \left(\frac{\sin h}{h}\right) \cos x - \sin x \overbrace{\boxed{\frac{(1 - \cos h)}{h^2}}}^{1/2} \cdot \overbrace{h}^{0} = \cos x$$

TABELLA DERIVATE FUNZIONI ELEMENTARI

REGOLE DI DERIVAZIONE

$f(x) = 1 \qquad f' = 0$

$f(x) = x^n \qquad f'(x) = n x^{n-1}$

$f(x) = e^x \qquad f'(x) = e^x$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$f(x) = \sin x \qquad f'(x) = \cos x$

$f(x) = \cos x \qquad f'(x) = -\sin x$

$f(x) = \tan x \qquad f'(x) = \frac{1}{\cos^2 x}$

$$\boxed{\begin{gathered} \left( f(x) \pm g(x) \right)' = f'(x) \pm g'(x) \\ (f \cdot g)' = f'g + g'f \\ \left( \frac{f}{g} \right)' = \frac{f'g - g'f}{g^2} \quad \text{(D1)} \\ \left[ f(g(x)) \right]' = f'(g(x)) \cdot g'(x). \\ \text{(D2)} \end{gathered}}$$

Dimostriamo (D1) e (D2)

(D1)

$$\left( \frac{f(x)}{g(x)} \right)' = \lim_{h \to 0} \frac{\left[ \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \right]}{h} = \lim \frac{f(x+h)\, g(x) - f(x)\, g(x+h)}{g(x+h)\, g(x)\, h}$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left[ \frac{f(x+h)\, g(x) - f(x)\, g(x) + f(x)\, g(x) - f(x)\, g(x+h)}{h} \right]$$

$$= \lim_{h \to 0} \underbrace{\frac{1}{g(x+h)g(x)}}_{\to g^2(x)} \left[ g(x) \overbrace{\frac{f(x+h) - f(x)}{h}}^{\to f'(x)} - f(x) \overbrace{\frac{g(x+h) - g(x)}{h}}^{\to g'(x)} \right]$$

$$= \frac{f'(x)\,g(x) - g'(x)\,f(x)}{[g(x)]^2} = \left(\frac{f(x)}{g(x)}\right)'$$

Esempio

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{(\cos x)^2} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \begin{cases} 1 + \tan^2 x \\ \dfrac{1}{\cos^2 x} \end{cases}$$

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Dimostriamo (D2)

$$\boxed{\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)}$$

$f(y)$ $\quad f'(y)$

$g(x)$ $\quad g'(x)$

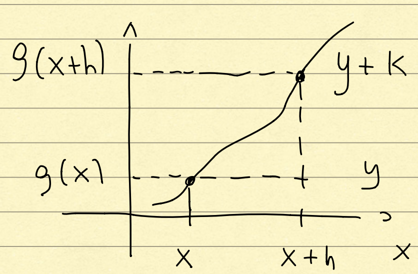
Esempio

$$\left(\cos(x^2)\right)' = -\sin(x^2) \cdot 2x$$

$f(y) = \cos y \quad f'(y) = -\sin y$

$g(x) = x^2 \quad g'(x) = 2x$

dimostrazione




$$\boxed{\begin{array}{l} y = g(x) \\ y + k = g(x+h) \end{array}}$$

Se $h \to 0$ anche $k \to 0$

$$\left[f(g(x))\right]' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{k \to 0} \frac{f(y+k) - f(y)}{k} \cdot \frac{k}{h}$$

$$= \lim_{\substack{k \to 0 \\ h \to 0}} \underbrace{\frac{f(y+k) - f(y)}{k}}_{f'(y)} \cdot \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)} = f'(g(x)) \cdot g'(x)$$

C.V.D

ESERCIZI

$$\left[-\ln(\cos x)\right]' = \left(-\frac{1}{\cos x}\right) \cdot (-\sin x) = \tan x$$

$$\left[e^{\sqrt{x}}\right]' = e^{\sqrt{x}} \cdot (\sqrt{x})' = \left(\frac{e^{\sqrt{x}}}{2\sqrt{x}}\right)$$

$$\sqrt{x} = x^{1/2}$$

$$\left[e^{x}\right] = e^{x} \cdot (x)' = e^{x}$$

$$f(x) \longrightarrow f'(x) \longrightarrow f''(x) \longrightarrow f'''(x) \cdots \longrightarrow f^{(n)}(x)$$

$f(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$

$f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$

$f'''(x) = 0$

$\vdots$

$(2x)' = \cancel{2'x} + \underset{1}{x'} \cdot 2 = 2$

ESERCIZI Calcolare le derivate $1^a$, $2^a$ e $3^a$ di

$f(x) = x^2 \cdot \sin x$

$$\boxed{f'(x) = 2x \cdot \sin x + x^2 \cos x}$$

$f''(x) = (2x \cdot \sin x)' + (x^2 \cos x)' = 2(\sin x + x\cos x) + (2x\cos x - x^2 \sin x)$

$$\boxed{f''(x) = \sin x\,(2 - x^2) + \cos x \cdot (4x)}$$

$f'''(x) = [\sin x \cdot (2 - x^2)]' + [\cos x \cdot 4x]'$

$f'''(x) = \cos x\,(2 - x^2) + \sin x \cdot (-2x) + (-\sin x \cdot 4x) + 4\cos x$

$$\boxed{f'''(x) = \sin x \cdot (-6x) + \cos x\,(6 - x^2)}$$

$f(x) = \ln(x + x^2)$ $\qquad f'(x) = \dfrac{1}{x + x^2} \cdot (1 + 2x)$

$$\boxed{f'(x) = \frac{1 + 2x}{x + x^2}}$$

$$\left(\frac{h}{g}\right)' = \frac{h'g - g'h}{g^2}$$

$$\boxed{f''(x) = \frac{2(x+x^2) - (1+2x)^2}{(x+x^2)^2} = \frac{-2x^2-2x-1}{(x+x^2)^2} = -\frac{(2x^2+2x+1)}{(x+x^2)^2}}$$

$$f'''(x) = -\cdot\frac{(2x^2+2x+1)'\cdot(x+x^2)^{\cancel{2}\,1} - 2\,(\cancel{x^2+x})(1+2x)(2x^2+2x+1)}{(x+x^2)^{\cancel{4}\,3}}$$

$$f'''(x) = -\;\frac{(4x+2)(x+x^2) - 2(1+2x)(2x^2+2x+1)}{(x+x^2)^3}$$

$$f'''(x) = -\;\frac{4x^3+6x^2+2x - 2\,(4x^3+6x^2+4x+1)}{(x+x^2)^3} =$$

$$\boxed{f'''(x) = \frac{4x^3+6x^2+6x-2}{(x+x^2)^3}}$$