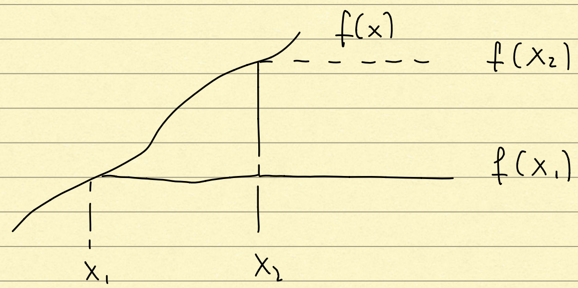
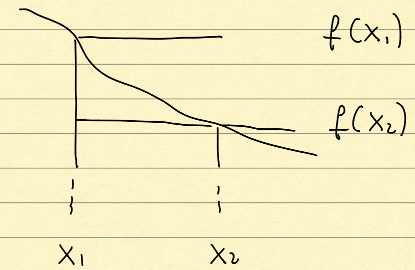
$f : (a,b) \longrightarrow \mathbb{R}$ sia derivabile

La f è crescente (decrescente) in (a,b) $\Longleftrightarrow$ $f'(x) \geq 0$ $\left(f'(x) \leq 0\right)$





CRESCENTE

$$R = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} \qquad f(x_2) - f(x_1) > 0 \qquad (x_2 - x_1) > 0$$

$$\boxed{f'(x_1) = \lim_{x_2 \to x_1} R \geq 0}$$

DECRESCENTE

$$R = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} \qquad f(x_2) - f(x_1) < 0 \qquad (x_2 - x_1) > 0$$

$$\boxed{f'(x_1) = \lim_{x_2 \to x_1} R \leq 0}$$

Es: Studiare la funzione $f(x) = x^3 e^{-x}$ e disegnare un grafico qualitativo della $f(x)$

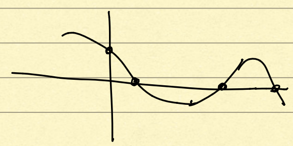
$\text{dom}(f) = \mathbb{R}$

LIMITI

$$\lim_{x \to +\infty} x^3 e^{-x} = \lim_{x \to +\infty} \frac{x^3}{e^x} = 0$$

$$\lim_{x \to -\infty} x^3 e^{-x} = -(\infty)^3 \cdot \infty = -\infty$$

INTERSEZIONE CON ASSI

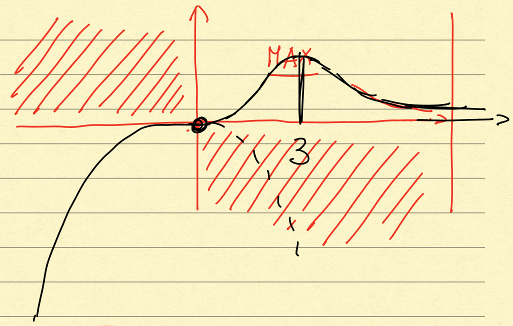


ASSE y $\begin{cases} x = 0 \\ y = f(0) \end{cases}$ $\exists$!? $\begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix}$

ASSE X $\begin{cases} f(x) = 0 \\ y = 0 \end{cases}$ $\exists$!? $\Rightarrow \begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix}$

segno di $f(x)$ $\begin{cases} f(x) > 0 & x > 0 \\ f(x) < 0 & < 0 \end{cases}$

$f(x) = 0$



Derivata 1° $f(x) = x^3 e^{-x}$

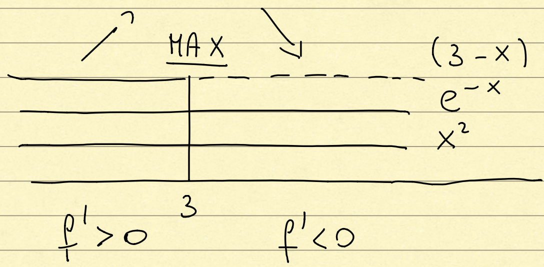
$$f'(x) = \left(x^3 e^{-x}\right)' = 3x^2 \cdot e^{-x} + \left(e^{-x}\right)' \cdot x^3 = 3x^2 e^{-x} - x^3 e^{-x}$$

$$f'(x) = (3x^2 - x^3)e^{-x} = x^2(3-x)e^{-x}$$

$$\boxed{f'(x) = (3-x) \cdot x^2 e^{-x}}$$

Studio il segno di $f'(x)$

$3 - x > 0 \quad x < 3$



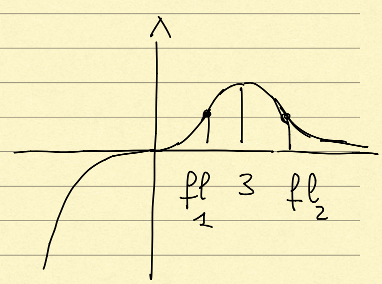
$f'(0) = ??$

In $x = 0$ la $f(x)$ ha tangente orizzontale $(f'(0) = 0)$

$\Rightarrow$ necessariamente $x = 0$ è un p.to di flesso

(la funzione cambia la sua concavità, $f''(0) = 0$)

Verifichiamo che effettivamente $f''(0) = 0$



$$f''(x) = \left[(3-x)x^2 e^{-x}\right]' = \left[(3x^2 - x^3)e^{-x}\right]'$$

$$f''(x) = (6x - 3x^2)e^{-x} - e^{-x}(3x^2 - x^3)$$

$$f''(x) = e^{-x}\left[x^3 - 6x^2 + 6x\right] = xe^{-x}(x^2 - 6x + 6)$$

$$f''(0) = 0$$

$f''(x) = 0$ (Cerco gli altri p.ti di flesso)

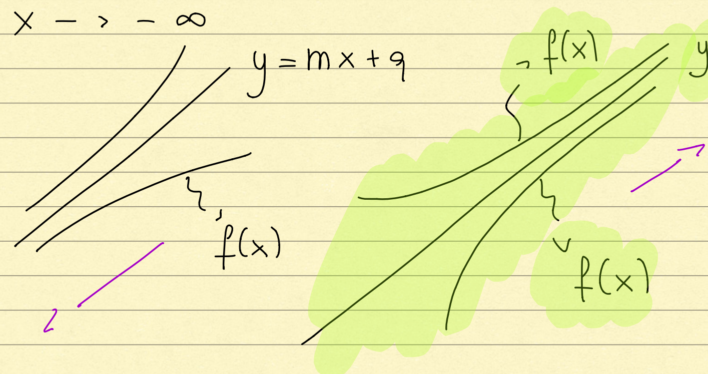
devo risolvere $x^2 - 6x + 6 = 0$ $\quad x_{1,2} = \dfrac{6 \pm \sqrt{36-24}}{2}$

$$x_{1,2} = 3 \pm \frac{\sqrt{12}}{2} = 3 \pm \sqrt{3} \begin{cases} 3+\sqrt{3} > 3 \rightsquigarrow fl_2 \\ 3-\sqrt{3} \in (0,3) \rightsquigarrow fl_1 \end{cases}$$

ASINTOTI OBLIQUI

$x \to +\infty$

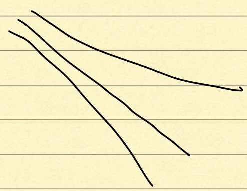
$x \to -\infty$

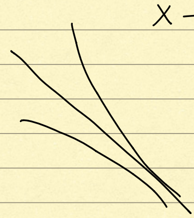


$m \neq 0$

$$\boxed{f(x) \sim mx + q}$$

$x \to +\infty$





$x \to -\infty$

Per determinare i valori m e q dell'asintoto obliquo vedo che

$$\lim_{x \to +\infty} \frac{f(x)}{mx+q} = 1 = \lim_{x \to +\infty} \frac{f(x)/x}{m + \cancel{q/x}} = 1$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = m \quad 1^\circ$$

Per trovare q osservo che $\lim\limits_{x \to +\infty} \frac{f(x)}{mx+q} = 1$

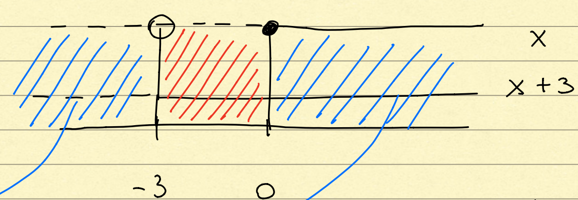
$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} mx+q \Rightarrow \lim_{x \to +\infty} [f(x) - mx] = q$$ 2°

Se 1° e 2° forniscono $m \neq 0$ $q \in \mathbb{R}$ $\Rightarrow$ $\exists$ l'asintoto obliquo per $x \to +\infty$ (lo stesso discorso vale per $x \to -\infty$)

STUDIO DI F.NE

$$f(x) = \sqrt{\frac{x^3}{x+3}} = \sqrt{\frac{x^2 \cdot x}{x+3}} = |x| \sqrt{\frac{x}{x+3}} \qquad \left(\sqrt{x^2} = |x| \right)$$

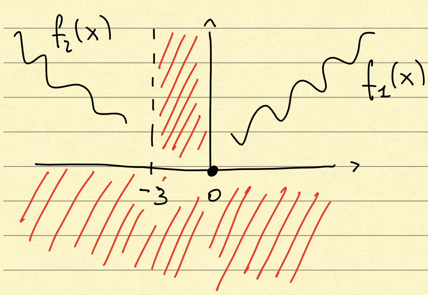
$$\mathrm{dom}(f) = \left\{ \frac{x}{x+3} \geq 0 \right\}$$



f(x) nel suo dom è sempre ≥ 0

$$\mathrm{dom}(f) = (-\infty, -3) \cup [0, +\infty)$$

$$f(x) = 0 \iff x = 0$$



Per $x \geq 0$ $\quad f_1(x) = x \sqrt{\frac{x}{x+3}}$: $[0, +\infty)$

Per $x < -3$ $\quad f_2(x) = -x \sqrt{\frac{x}{3+x}}$: $(-\infty, -3)$

LIMITI $\quad \lim\limits_{x \to +\infty} f_1(x) = \lim\limits_{x \to +\infty} x \sqrt{1 - \frac{3}{x+3}} = +\infty$

$\lim\limits_{x \to -\infty} f_2(x) = \lim\limits_{x \to -\infty} -x \sqrt{1 - \frac{3}{x+3}} = +\infty$

ASINTOTI OBLIQUI (Ho senso cercarli poiché $\lim_{x \to \pm\infty} f(x) = +\infty$)

ASINT. OBLIQUO per $x \to +\infty$ ($f_1(x)$)

$$N = \frac{-3x}{x+3} = \frac{\cancel{x}(-3)}{\cancel{x}(1+3/x)}$$

$$\lim_{x \to +\infty} \frac{f_1(x)}{x} = \lim_{x \to +\infty} \sqrt{1 - \frac{3}{x+3}} = 1 = m$$

$$q = \lim_{x \to +\infty} f_1(x) - x = \lim_{x \to +\infty} \left(x\sqrt{1 - \frac{3}{x+3}} - x \right) = \lim_{x \to +\infty} x\left(\sqrt{1 - \frac{3}{3+x}} - 1 \right) =$$

$$= \lim_{x \to +\infty} x \cdot \frac{\left(\sqrt{1 - \frac{3}{3+x}} - 1\right)\left(\sqrt{\ldots} + 1\right)}{\left(\sqrt{\ldots} + 1\right)} = \lim_{x \to +\infty} \frac{x \cdot \left(\cancel{1} - \frac{3}{x+3} - \cancel{1} \right)}{\left(\sqrt{1 - \frac{3}{x+3}} + 1 \right)} = \frac{-3}{2}$$

$\Rightarrow$ l'asintoto obliquo per $x \to +\infty$ è $y = x - 3/2$

ASINT. OBLIQUO per $x \to -\infty$ ($f_2(x)$)

$$\lim_{x \to -\infty} \frac{f_2(x)}{x} = \lim_{x \to -\infty} -\sqrt{1 - \frac{3}{x+3}} = -1 = m$$

$$q = \lim_{x \to -\infty} f_2(x) + x = \lim_{x \to -\infty} -x\sqrt{1 - \frac{3}{3+x}} + x =$$

$$= \lim_{x \to -\infty} -x\left(\sqrt{1 - \frac{3}{3+x}} - 1 \right) = \lim_{x \to -\infty} \frac{-x\left(-\frac{3}{3+x} \right)}{\sqrt{1 - \frac{3}{x+3}} + 1} = 3/2$$

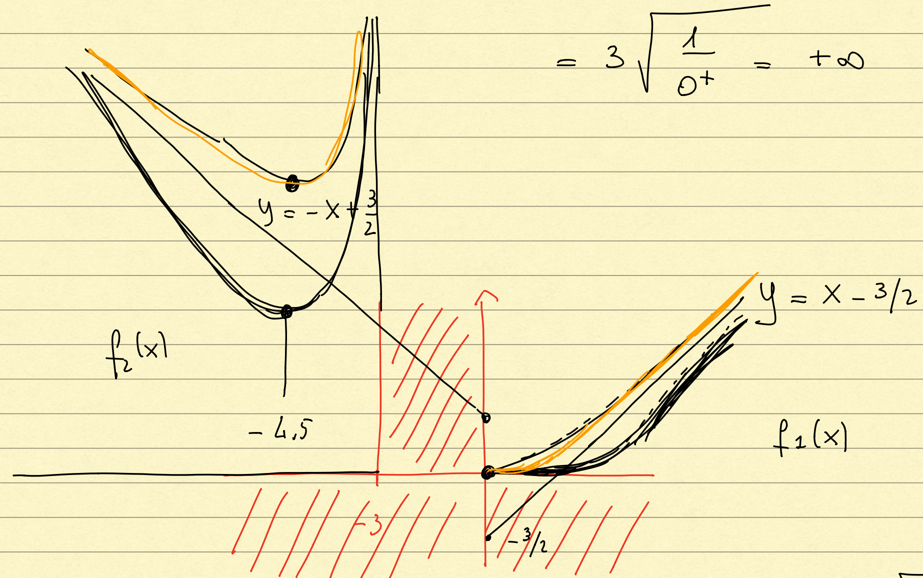
In conclusione

per $x \to +\infty$
$y = x - 3/2$
per $x \to -\infty$
$y = -x + 3/2$

ASINTOTI OBL.

$$\lim_{x \to -3^-} f_2(x) = \lim_{x \to -3^-} -x\sqrt{1 - \frac{3}{x+3}}$$

$$= 3\sqrt{\frac{1}{0^+}} = +\infty$$



Derivata 1°

$$f(x) = |x|\sqrt{1 - \frac{3}{3+x}} = |x|\sqrt{\frac{x}{x+3}} \quad \begin{cases} f_1 = x\sqrt{\dots} \\ f_2 = -x\sqrt{\dots} \end{cases}$$

$$f_1'(x) = -f_2'(x) \qquad f_1'(x) = \left(x\sqrt{\frac{x}{x+3}}\right)' = \sqrt{\dots\dots} + x\,\frac{\left(\frac{x}{x+3}\right)'}{2\sqrt{\dots}}$$

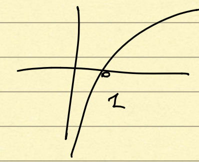
$$f_1'(x) = \frac{2\left(\sqrt{\ }\right)^2 + x\left(\frac{x+3-x}{(x+3)^2}\right)}{2\sqrt{\dots\dots}} = \frac{2\left(\frac{x}{x+3}\right) + \frac{3x}{(x+3)^2}}{2\sqrt{\frac{x}{x+3}}}$$

$$f_1'(x) = \frac{2x(x+3) + 3x}{2\sqrt{\frac{x}{x+3}}\,(x+3)^2} = \frac{2x^2 + 6x + 3x}{2\sqrt{\dots}\,(x+3)^2} = \frac{x(2x+9)}{\dots\dots\dots > 0}$$

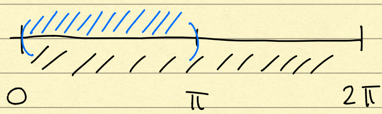
$$f_2'(x) = -f_1'(x) = -\frac{x(2x+9)}{\dots\dots > 0} \qquad f' = 0 \begin{cases} x = 0 \\ x = -9/2 \end{cases}$$

STUDIO DI FUNZIONE

$$f(x) = -\ln(\sin x)$$



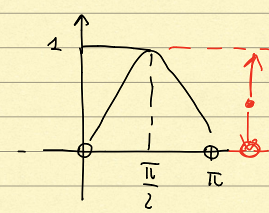
$f(x)$ è periodica di periodo 2π per cui basta studiarla in un intervallo di ampiezza 2π, ad esempio $[0,2\pi]$



Qual'è il dominio di f relativamente all'intervallo $[0,2\pi]$?

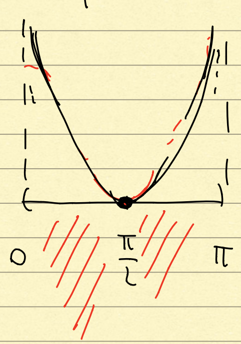
$\mathrm{dom}(f) = \{\sin x > 0\} \cap [0,2\pi] = (0,\pi)$ Studio $f(x)$ in $(0,\pi)$

$f(x) : (0,\pi) \to \mathbb{R}$ $\ln$ $(0,\pi)$ $0 < \sin x \leq 1$



$\Rightarrow \ln(\sin x) \leq 0$ in $(0,\pi)$

$\Rightarrow f(x) = -\ln(\sin x) \geq 0$ in $(0,\pi)$ $\begin{cases} f(x) > 0 & \text{in } (0,\pi)\setminus\{\pi/2\} \\ f(x) = 0 & \text{se } x = \pi/2 \end{cases}$



LIMITI ALL'ESTREMO DEL DOMINIO

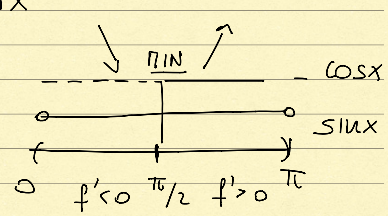
$$\lim_{x\to 0^+} -\ln(\sin x) = -\ln(0^+) = +\infty$$

$$\lim_{x\to \pi^-} -\ln(\sin x) = -\ln(0^+) = +\infty$$

DERIVATA 1ª

$$f'(x) = \left[-\ln(\sin x)\right]' = -\frac{1}{\sin x}\cdot \cos x = -\frac{\cos x}{\sin x} = -\cot x$$

$f'(x) = -\dfrac{\cos x}{\sin x}$ Studio il segno della $f'(x)$



$$f\left(\frac{\pi}{2}\right) = -\ln\left(\sin\left(\frac{\pi}{2}\right)\right) = -\ln(1) = 0$$

DERIVATA 2°

$$f''(x) = \left[-\frac{\cos x}{\sin x} \right]' = - \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} =$$

$$f''(x) = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = 1 + \cot^2 x = 1 + \frac{\cos^2 x}{\sin^2 x} > 0 \qquad (0, \pi)$$

Quindi la funzione è convessa $f''(x) > 0$ ovunque nel dominio