Teo Hopital : « $f(x), g(x)$ definite in un intorno di x_0

derivabili

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0} = \lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)} = \frac{\infty}{\infty}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

GERARCHIA INFINITI

$$\lim_{x \to +\infty} \frac{x^\alpha}{e^{\beta x}} = 0 \qquad \forall \beta > 0 \quad \forall \alpha \in \mathbb{R}$$

Se $\alpha \leq 0$ non c'è niente da dimostrare

$$\lim_{x \to +\infty} \left(\frac{x}{e^{\frac{\beta x}{\alpha}}} \right)^\alpha = \left(\lim_{x \to +\infty} \frac{x}{e^{\beta x/\alpha}} \right)^\alpha = \text{Hopital} \qquad \alpha > 0$$

$$= \left(\lim_{x \to +\infty} \frac{1}{\beta/\alpha \, e^{\frac{\beta x}{\alpha}}} \right)^\alpha = (0)^\alpha = 0 \qquad \text{C.V.D.}$$

ESERCIZI CON HOPITAL

- $$\lim_{x \to 1} \frac{\sqrt{x} - \sqrt[4]{x}}{x - 1} = \lim_{x \to 1} \frac{x^{1/2} - x^{1/4}}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{2} x^{-1/2} - \frac{1}{4} x^{-3/4}}{1} = \frac{1}{4}$$

- $$\lim_{x \to 0} \frac{x + \sin x}{x - \sin x} = (H) = \lim_{x \to 0} \frac{1 + \cos x}{1 - \cos x} = \frac{2}{0^+} = +\infty$$

$$\bullet \lim_{x\to 0} \frac{x\cdot\cos x - \sin x}{x^3} = \lim_{x\to 0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{3x^2} = (H) = \lim_{x\to 0} \frac{-\sin x - x\cos x}{6x}$$

$$= \lim_{x\to 0} -\frac{1}{6}\frac{\sin x}{x} - \frac{\cos x}{6} = -\frac{1}{3}$$

$$\bullet \lim_{x\to 0} \frac{\sin(e^x - 1) - x}{x \arctan x} = \lim_{x\to 0} \frac{\cos(e^x-1)e^x - 1}{\arctan x + x\cdot\frac{1}{1+x^2}} = \lim_{x\to 0} \frac{e^x\cos(e^x-1)-1}{\arctan x + \frac{x}{1+x^2}}$$

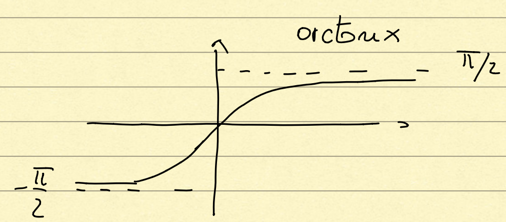
$$= \lim_{x\to 0} \frac{e^x\cos(e^x-1) - e^x\sin\cancel{(e^x-1)}\cdot e^x}{\frac{1}{1+x^2} + \frac{(1+x^2)-2x^2}{(1+x^2)^2}} = \boxed{\frac{1}{2}}$$

$$\bullet \lim_{x\to 1} \frac{x^x - x}{1 - x + \ln x} = \lim_{x\to 1} \frac{e^{\ln x\cdot x} - x}{1 - x + \ln x} = (\text{Hopital}) =$$

$$= \lim_{x\to 1} \frac{e^{\ln x\cdot x}(\ln x\cdot x)' - 1}{\frac{1}{x} - 1} = \lim_{x\to 1} \frac{e^{\ln x\cdot x}(\ln x + 1) - 1}{\frac{1}{x} - 1} = \text{Hop.}$$

$$= \lim_{x\to 1} \frac{e^{\ln x\cdot x}(1+\ln x)^2 + \frac{1}{x}e^{\ln x\cdot x}}{-\frac{1}{x^2}} = -2$$

$$\bullet \lim_{x\to +\infty} \frac{\cos\left(\frac{x}{x^2-1}\right) - 1}{\frac{\pi}{2} - \arctan x} = \frac{0}{0}$$



$$\lim_{x\to +\infty} \left[\frac{-\sin\left(\frac{x}{x^2-1}\right)\cdot\left(\frac{x}{x^2-1}\right)'}{-\frac{1}{1+x^2}}\right] = \lim_{x\to +\infty} \frac{-\sin\left(\frac{x}{x^2-1}\right)\cdot\frac{(x^2-1)-2x^2}{(x^2-1)^2}}{-\frac{1}{1+x^2}}$$

$$= \lim_{x \to +\infty} \frac{-\sin\left(\frac{x}{x^2-1}\right) \cdot \frac{-x^2-1}{(x^2-1)^2}}{-\frac{1}{1+x^2}} =$$

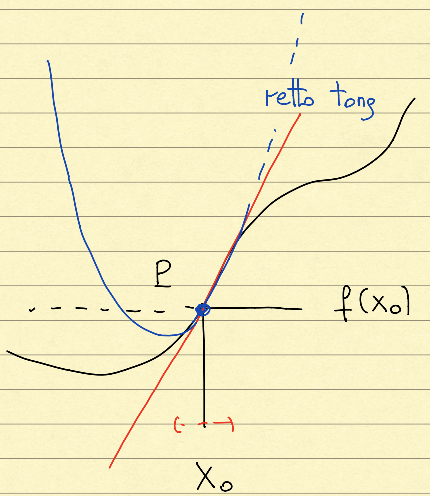
$$\frac{x}{x^2-1} \sim \frac{1}{x} \qquad \text{per } x \gg 1$$

$$\frac{-(x^2+1)}{(x^2-1)^2} \sim -\frac{1}{x^2}$$

$$-\frac{1}{1+x^2} \sim -\frac{1}{x^2}$$

$$= \lim_{x \to +\infty} \frac{-\sin\left(\frac{1}{x}\right) \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = 0$$

Polinomi di Taylor



La prima approssimazione della funzione $f(x)$ in un intorno di x_0 a cui posso pensare è la retta tangente

$$\boxed{\begin{cases} y = f(x) \\ y = f(x_0) + f'(x_0)(x - x_0) \end{cases}} \quad \text{App. di grado 1} \quad \begin{cases} y'(x_0) = f'(x_0) \\ y'(x_0) = f'(x_0) \end{cases}$$

Approx di grado 2

$n! = n \cdot (n-1)(n-2) \cdot \ldots \cdot 1$

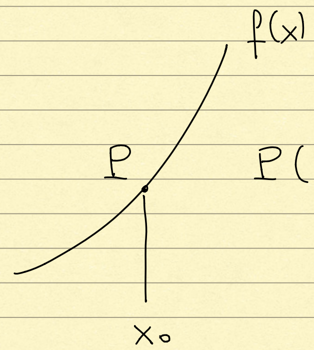
$2! = 2 \cdot 1 = 2$

$$\boxed{\begin{array}{l} y = f(x) \\ y = f(x_0) + f'(x_0)(x - x_0) + \dfrac{f''(x_0)}{2!}(x - x_0)^2 \end{array}}$$

$y(x_0) = f(x_0)$ $\qquad y''(x_0) = f''(x_0)$

$y'(x_0) = f'(x_0)$

In generale posso costruirmi un polinomio di grado n che approssima la funzione $f(x)$ in un intorno di x_0



$P(x_0, f(x_0))$

L'appx di grado n è dato da

$$\boxed{y = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$$

DEF. Data $f(x)$ derivabile quanto volte voglio in un intorno di x_0

Si definisce il polinomio di Taylor di ordine n (nel p.to x_0)

$$P_n(x_0; f) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

Se $x_0 = 0$ si chiamano polin. di MACLAURIN

Es. $f(x) = e^x$ in $x_0 = 0$

$$P_n(0; f) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$f^{IV} = f$

Es. $\sin x = f(x)$ in $x_0 = 0$

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x \quad f^{IV}(x) = \sin x$

$f'(0) = 1 \ ; \ f''(0) = 0 \ ; \ f'''(0) = -1 \ ; \ f^{IV}(0) = 0 \ ; \ f^{V}(0) = 1 \ ; \ f^{VI}(0) = 0 \dots$

$$\boxed{P_{2n+1}(0; \sin x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{j=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

$2n+1$ se n è pari devo prendere $-$

se n è dispari devo prendere $+$

Es $\quad f(x) = \cos x \quad$ in $x_0 = 0$

$f'(x) = -\sin x$; $f''(x) = -\cos x$; $f'''(x) = \sin x$; $f^{IV}(x) = \cos x$

$f(0) = 1$; $f'(0) = 0$; $f''(0) = -1$; $f'''(0) = 0$; $f^{IV}(0) = 1 \ldots$

$$\boxed{P_{2n}(0; \cos x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots + (-1)^n \frac{x^{2n}}{2n!} = \sum_{j=0}^{n} (-1)^n \frac{x^{2n}}{(2n)!}}$$

$2n$ — n pari $+$; n dispari $-$

Sviluppi di Maclaurin

$$\tan x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} x^3 + \ldots$$

Es. Calcolare lo sviluppo di Maclaurin all'ordine $n = 3$ di

$$f(x) = \ln(1+3x)$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

Sviluppo all'ordine 3

$$\ln(1+3x) \simeq (3x) - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} = \boxed{3x - \frac{9x^2}{2} + 9x^3}$$

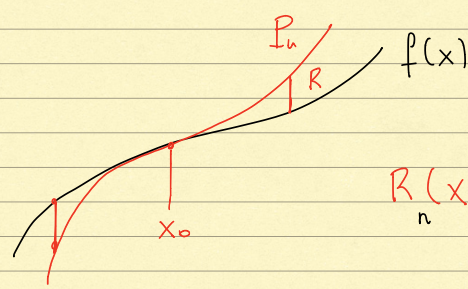
ESERCIZIO Calcolare fino all'ordine 10 lo sviluppo di Maclaurin

$f(x) = \cos(x^2)$ $\qquad \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \ldots\ldots$

$z = x^2 \qquad 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} \qquad P_{10}(0; \cos(x^2)) = \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!}\right)$

Resto nella forma di Peano

$$f(x) \sim P_n(x_0; f(x)) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$



$$R_n(x; f) = f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

$R_n(x_0; f) = 0 \implies R_n(x; f)$ è infinitesimo per $x \to x_0$

Teo: « $R_n(x; f)$ è un infinitesimo di ordine superiore a $(x-x_0)^n$

$$\lim_{x \to x_0} \frac{R_n(x; f)}{(x-x_0)^n} = 0 \quad »$$

$R_n(x; f) = o((x-x_0)^n)$

Dim

$$\frac{f(x) - f(x_0) - f'(x_0)(x-x_0) \ldots - \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}}{(x-x_0)^n} = \frac{R}{(x-x_0)^n}$$

$$\lim_{x \to x_0} \frac{R}{(x-x_0)^n} = \frac{0}{0} = \text{(Applico l'Hopital } n-1 \text{ volte)} = 0$$

$$\lim_{x \to x_0} \frac{\left[f^{(n-1)}(x) - f^{(n-1)}(x_0)\right] - f^{(n)}(x_0)(x-x_0)}{n!\,(x-x_0)} =$$

$$= \lim_{x \to x_0} \frac{1}{n!} \left[\underbrace{\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{(x-x_0)}}_{\to f^{(n)}(x_0)} - f^{(n)}(x_0) \right] = 0 \,!$$

C.V.D.

Quindi

$$f(x) = \left[\sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j \right] + \overbrace{o((x-x_0)^n)}^{R_n(x, x_0; f)} \qquad (*)$$

Sfruttando (*)

$$\sin x = x + o(x)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$$

Con Taylor si possono risolvere anche i limiti

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x + o(x)}{x} = \lim_{x \to 0} \left[1 + \frac{o(x)}{x} \right] = 1$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \left\{ \begin{array}{l} \text{Sviluppo di Maclaurin di } e^x \\ e^x = 1 + x + \dots = 1 + x + o(x) \end{array} \right\} =$$

$$= \lim_{x\to 0} \frac{x + o(x)}{x} = \lim_{x\to 0} \left[1 + \frac{o(x)}{x} \right] = 1$$

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \left\{ \begin{array}{l} \cos x = 1 - \frac{x^2}{2} + o(x^2) \\ 1 - \cos x = \frac{x^2}{2} + o(x^2) \end{array} \right\} =$$

$$= \lim_{x\to 0} \frac{x^2/2 + o(x^2)}{x^2} = \lim_{x\to 0} \frac{1}{2} + \frac{o(x^2)}{x^2} = \frac{1}{2}$$

Es (Risolvere con Taylor il seguente limite)

$$\lim_{x\to 0} \frac{e^x - 1 + \ln(1-x)}{\tan x - x} = * \left\{ \begin{array}{l} e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \dots \\ \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \dots \\ \tan(x) = x + x^3/3 + \dots \end{array} \right.$$

$$\tan x - x = \frac{x^3}{3} + o(x^3)$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$$

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3)$$

$$* = \lim_{x \to 0} \frac{\cancel{x} + \cancel{\frac{x^2}{2}} + \frac{x^3}{3!} + \left(\cancel{-x} - \cancel{\frac{x^2}{2}} - \frac{x^3}{3}\right) + \cancel{o(x^3)}}{x^3/3 + \cancel{o(x^3)}} = \lim_{x \to 0} \frac{\cancel{x^3}\left(\frac{1}{6} - \frac{1}{3}\right)}{\frac{\cancel{x^3}}{3}} =$$

$$= 3\left(\frac{1}{6} - \frac{1}{3}\right) = \left(\frac{1}{2} - 1\right) = -1/2$$