ES.

$$\lim_{x\to 0} \frac{x\sin x + \ln(1-x^2)}{x^2(2x+x^2)^2} = \lim_{x\to 0} \frac{\cdots\cdots}{x^2(2x+o(x))^2} = \lim_{x\to 0} \frac{x\sin x + \ln(1-x^2)}{4x^4+o(x^4)} =$$

$$\sin x = x - \frac{x^3}{3!} + \cdots \qquad \boxed{x\sin x = x^2 - \frac{x^4}{3!} + o(x^4)}$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

$$\ln(1-x^2) = -x^2 - \frac{(-x^2)^2}{2} + \cdots \qquad \boxed{\ln(1-x^2) = -x^2 - \frac{x^4}{2} + o(x^4)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2} - \frac{x^4}{6} \cancel{-x^2} - \frac{x^4}{2} + o(\cancel{x^4})}{4x^4 + o(\cancel{x^4})} = \frac{-\frac{1}{6} - \frac{1}{2}}{4} = -\frac{4}{4\cdot 6} = -\frac{1}{6}$$

Es

$$\lim_{x\to 0} \frac{\sin(x^4)\left[\sin(x^2) - (\sin x)^2\right]}{\left[1-\cos(x^4)\right]} = (*)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots$$

$$1-\cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \cdots$$

$$\sin(z) = z - \frac{z^3}{3!} + \cdots$$

$$\boxed{1-\cos(x^4) = \frac{(x^4)^2}{2!} - \frac{(x^4)^4}{4!} = \frac{x^8}{2} + o(x^8)}$$

6

$$\boxed{\sin(x^4) = x^4 - \frac{x^{12}}{3!} + \cdots = x^4 + o(x^4)}$$

$$\boxed{\sin(x^2) = \left(x^2 - \frac{x^6}{3!} + \cdots\right)}$$

$$(\sin x)^2 = \left[x - \frac{x^3}{3!} + \cdots\right]^2 \cong \left(x - \frac{x^3}{3!}\right)^2 = \left(x^2 + \frac{x^6}{36} - \frac{2x^4}{6}\right)$$

$$\sin(x^2) - (\sin x)^2 = \left(\cancel{x^2} - \frac{x^6}{3!} + o(x^6)\right) - \left(\cancel{x^2} + \frac{x^6}{36} - \frac{x^4}{3} + o(x^6)\right)$$

$$\sin(x^2) - (\sin x)^2 = \frac{x^4}{3} + o(x^4)$$

$$* = \lim_{x \to 0} \boxed{\frac{[x^4 + o(\cancel{x^4})][\frac{x^4}{3} + o(\cancel{x^4})]}{x^8/2 + o(\cancel{x^8})}} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$\frac{[x^4 + o(\cancel{x^4})][x^5 + o(\cancel{x^5})]}{x^8/2 + o(\cancel{x^8})} =$$

$$\boxed{x^6 + o(x^6) = o(x^4)}$$

$$\frac{x^6 + o(x^6)}{x^4} = \frac{o(x^4)}{x^4} \Rightarrow 0$$

$$x^2 + o(x^2) \longrightarrow 0$$

$$\lim_{x \to 0} \frac{\frac{x^8}{3} + o(\cancel{x^8})}{x^8/2 + o(\cancel{x^8})} = \frac{2}{3}$$

$$o(x^8) = o(x^7) = o(x^6)$$

$$o(x^2) = \frac{o(x^6)}{x^6} \longrightarrow 0$$

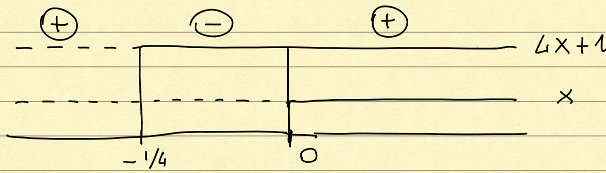
$$o(x^n) = o(x^m) \qquad n \geq m$$

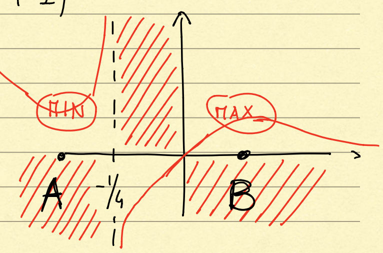
STUDIO DI FUNZIONE

$$f(x) = \left(\frac{x}{4x+1}\right) \cdot e^{-x}$$

$$\text{dom}(f) = \left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, +\infty\right)$$

segno di f





LIMITI

$$\lim_{x \to -\infty} f(x) = \frac{1}{4} e^{+\infty} = +\infty \qquad \lim_{x \to -\frac{1}{4}^+} f(x) = \frac{-1/4}{0^+} e^{\frac{1}{4}} = -\infty$$

$$\lim_{x \to +\infty} f(x) = \frac{1}{4} e^{-\infty} = 0 \qquad \lim_{x \to -\frac{1}{4}^-} f(x) = \frac{-1/4}{0^-} e^{1/4} = +\infty$$

DERIVATA 1°

$$f'(x) = \left(\frac{x}{4x+1}\right)' \cdot e^{-x} - e^{-x} \cdot \left(\frac{x}{4x+1}\right) = e^{-x}\left[\left(\frac{x}{4x+1}\right)' - \frac{x}{(4x+1)}\right]$$

$$f'(x) = e^{-x}\left[\frac{\cancel{4x}+1-\cancel{4x}}{(4x+1)^2} - \frac{x}{(4x+1)}\right] = e^{-x}\left[\frac{1-x(4x+1)}{(4x+1)^2}\right]$$

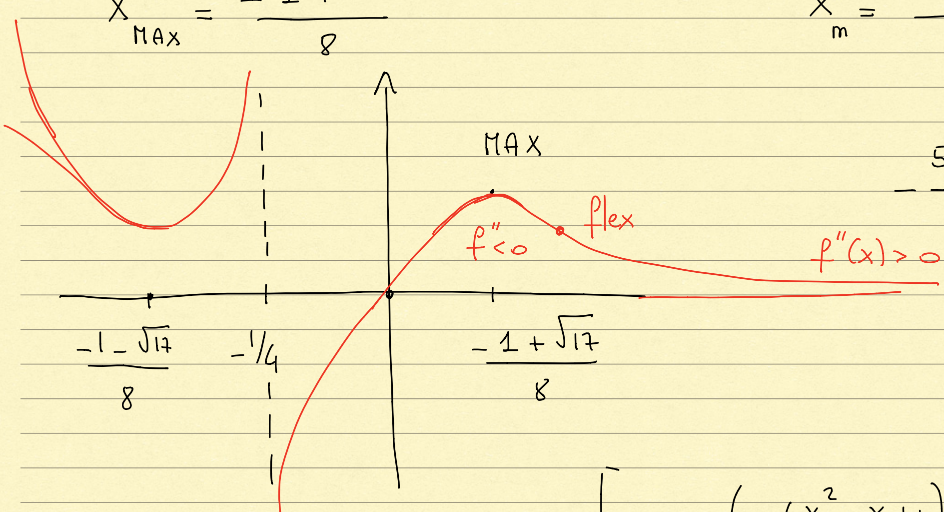
$$f'(x) = e^{-x}\left[\frac{-4x^2-x+1}{(4x+1)^2}\right]$$

$f'(x) = 0 \iff -4x^2 - x + 1 = 0$

$$x_{1,2} = \frac{1 \pm \sqrt{1+16}}{-8} = \frac{-1 \pm \sqrt{17}}{8}$$

$\Rightarrow$ MAX ho come cordinata x — min ho come coord. x

$$x_{MAX} = \frac{-1+\sqrt{17}}{8} \qquad x_m = \frac{-1-\sqrt{17}}{8}$$



$$-\frac{5,\ldots}{8} = -\frac{4+0,\ldots}{8} = -\frac{1}{2} - \frac{0,\ldots}{8}$$

DERIVATA 2°

$$f''(x) = \left[e^{-x}\frac{(-4x^2-x+1)}{(4x+1)^2}\right]'$$

$$f''(x) = -e^{-x}\left[\frac{-4x^2-x+1}{(4x+1)^2}\right] +$$

$$+\, e^{-x}\left[\frac{(-8x-1)(4x+1)^{\cancel{2}} - 2\cancel{(4x+1)}\cdot 4(-4x^2-x+1)}{(4x+1)^{\cancel{4}\,3}}\right]$$

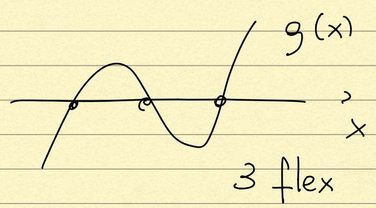
$$f''(x) = e^{-x}\left[\frac{4x^2+x-1}{(4x+1)^2}\right] + e^{-x}\left[\frac{-8x^2-12x-1-8(-4x^2-x+1)}{(4x+1)^3}\right]$$

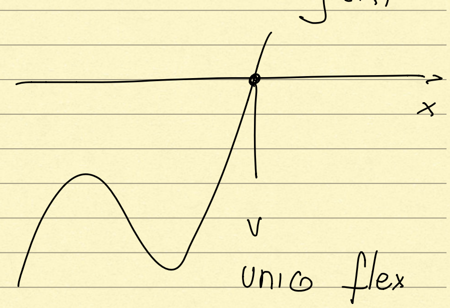
$$f''(x) = e^{-x}\left[\frac{4x^2+x-1}{(4x+1)^2}\right] + e^{-x}\left[\frac{24x^2-4x-9}{(4x+1)^3}\right]$$

$$f''(x) = e^{-x}\left[\frac{(4x^2+x-1)(4x+1)+24x^2-4x-9}{(4x+1)^3}\right]$$

$$f''(x) = e^{-x}\left[\frac{16x^3+32x^2-7x-10}{(4x+1)^3}\right]$$

$$g(x) = 16x^3+32x^2-7x-10$$





ESERCIZIO: Risolvere l'equazione complessa $\boxed{z^6 + i\,\bar{z}^3 = 0}$

$z \in \mathbb{C}$ $\quad \begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \quad \begin{cases} z = \rho e^{i\theta} \\ \bar{z} = \rho e^{-i\theta} \end{cases} \quad \rho = |z|$

$|z| = \sqrt{x^2 + y^2}$

$z^6 = -i\,\bar{z}^3 \qquad |z^6| = |-i\,\bar{z}^3| \Rightarrow |z|^6 = |-i| \cdot |\bar{z}|^3$

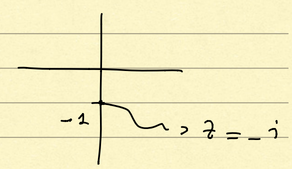
$|-i| = 1 \qquad |\bar{z}| = |z| \qquad \rightarrow \boxed{\rho^6 = \rho^3} \qquad \rho^3(\rho^3 - 1) = 0$

$\begin{cases} \rho = 0 \\ \rho = 1 \end{cases}$

Allora le soluzioni di $z^6 + i\,\bar{z}^3 = 0$

hanno modulo ρ $\begin{cases} \rho = 0 \\ \rho = 1 \end{cases} \Rightarrow \begin{cases} z = 0 \\ z = e^{i\theta} \end{cases}$

Quindi a parte $z = 0$ le altre soluzioni hanno necessariamente modulo 1.



$z^6 = -i\,\bar{z}^3 \qquad z = e^{i\theta} \qquad \bar{z} = e^{-i\theta}$

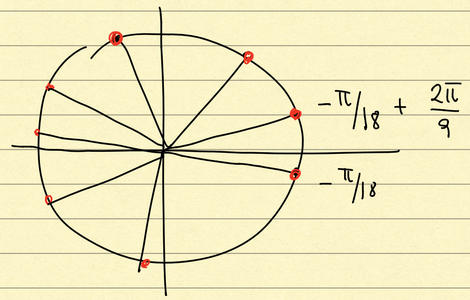
$$e^{6i\theta} = -i\,e^{-i3\theta} = e^{-i\frac{\pi}{2}} \cdot e^{-i3\theta}$$

$$e^{6i\theta} = e^{-i3\theta - i\pi/2} \Rightarrow \boxed{6\theta = -3\theta - \pi/2 + 2k\pi}$$

$9\theta = -\dfrac{\pi}{2} + 2k\pi \qquad \boxed{\theta = -\dfrac{\pi}{18} + \dfrac{2k\pi}{9}} \qquad \underbrace{k = 0, 1, 2, \dots, 8}_{\text{Sono 9 soluz}}$

Le soluzioni (eccetto $z = 0$) stanno tutte sul cerchio di raggio 1

In generale ho 10 soluzioni

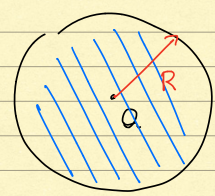


Es. Risolvere la disequazione $\boxed{|z+i| > |z-i|}$

Se $a \in \mathbb{C}$

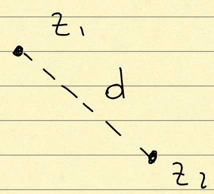
$|z-a| < R$ indica l'interno del cerchio di raggio R e centro in a

$R \in \mathbb{R}$ e $R > 0$

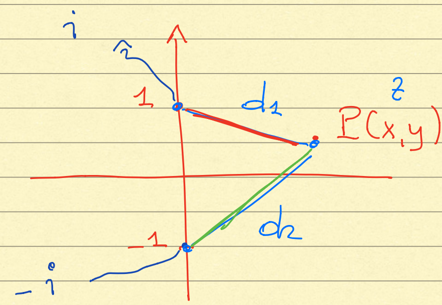


Se z_1, z_2

$|z_1 - z_2|$ rappresenta la distanza fra i p.ti z_1 e z_2



$d = |z_1 - z_2| = |z_2 - z_1|$



P identifico un p.to $z \in \mathbb{C}$

$d_1 = |z - i|$

$d_2 = |z - (-i)| = |z + i|$

Io voglio determinare i p.ti z t.c. $d_2 > d_1$

Allora "intuitivamente" vedo che per valere $d_2 > d_1$
P deve stare nel semipiano $y > 0$. Allora $\Im(z) > 0$

$z = x + iy$ $y > 0$. La disequazione è risolta da tutti i numeri z con parte immaginaria positiva ($\Im(z) > 0$)

Risolviamo invece per via analitica

$|z+i| > |z-i| \qquad z = x+iy$

$|x+iy+i| > |x+iy-i|$

$|x+i(y+1)|^2 > |x+i(y-1)|^2$

$x^2+(y+1)^2 > x^2+(y-1)^2$

$\cancel{x^2}+\cancel{y^2}+2y+\cancel{1} > \cancel{x^2}+\cancel{y^2}-2y+\cancel{1}$

$2y > 0 \implies y > 0$

Determinare le soluzioni di

C.E.

$z \neq i$

$$\frac{|z+1|}{|z-i|} = 2 \qquad z \in \mathbb{C} \qquad z = x+iy$$

$|z+1| = 2|z-i| \qquad |x+1+iy| = 2|x+iy-i|$

$|(x+1)+iy| = 2|x+i(y-1)|$

$|(x+1)+iy|^2 = 4|x+i(y-1)|^2$

$(x+1)^2+y^2 = 4(x^2+(y-1)^2)$

$x^2+1+2x+y^2 = 4(x^2+y^2+1-2y)$

$x^2+y^2+1+2x = 4x^2+4y^2+4-8y$

$3x^2+3y^2-2x-8y+3=0$

CERCHIO

$x^2+y^2+Ax+By+C=0$

$$\begin{cases} R = \sqrt{\dfrac{A^2}{4}+\dfrac{B^2}{4}-C} > 0 \\ \text{Centro} \equiv \left(-\dfrac{A}{2}; -\dfrac{B}{2}\right) \end{cases}$$

$$x^2+y^2-\left(\frac{2}{3}\right)x-\left(\frac{8}{3}\right)y+1=0$$ (È un cerchio)

$$\text{Centro} \equiv \left(\frac{1}{3};\frac{4}{3}\right) \qquad R=\sqrt{\frac{1}{9}+\frac{\cancel{64}^{16}}{9\cdot\cancel{4}}-1}=\sqrt{\frac{17-9}{9}}=\left(\frac{2\sqrt{2}}{3}\right)$$

=> Le soluzioni della mia equazione sono ∞ e sono tutti i p.ti del piano che stanno sul cerchio di raggio $\frac{2\sqrt{2}}{3}$ e centro

$$\mathcal{C}\equiv\left(\frac{1}{3};\frac{4}{3}\right)$$

Es.

Calcolare il limite $\lim_{x\to 1^+}(x-1)^{\frac{1}{(1-x)}}$

$x-1=t \qquad x\to 1^+ \ ; \ t\to 0^+ \qquad 1-x=-t$

$$\lim_{t\to 0^+} t^{-\frac{1}{t}}=\lim_{t\to 0}\left(e^{\ln t}\right)^{-1/t}=\lim_{t\to 0^+} e^{-\frac{\ln t}{t}}=$$

$$=\text{Hop.}=\lim_{t\to 0^+} e^{-\frac{1}{t}}=e^{-\frac{1}{0^+}}=e^{-\infty}=0$$

Es.

$$\lim_{x\to+\infty}\frac{(x+1)\ln\left(1+\frac{1}{x}\right)}{x}=\lim_{x\to+\infty}\left(\frac{x+1}{x}\right)\ln\left(1+\frac{1}{x}\right)$$

$$= \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right) \ln\left(1 + \frac{1}{x}\right) = 1 \cdot \ln(1) = 0$$

Es

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^2} = \lim_{x \to 0} \frac{1}{x^2} \cdot e^{-1/x^2} = \left\{ \begin{array}{l} \frac{1}{x^2} = t \\ x \to 0 \\ t \to +\infty \end{array} \right\} =$$

$$= \lim_{t \to +\infty} t e^{-t} = \lim_{t \to +\infty} \frac{t}{e^t} = \text{Hop.} = \lim_{t \to +\infty} \frac{1}{e^t} = \frac{1}{+\infty} = 0$$

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