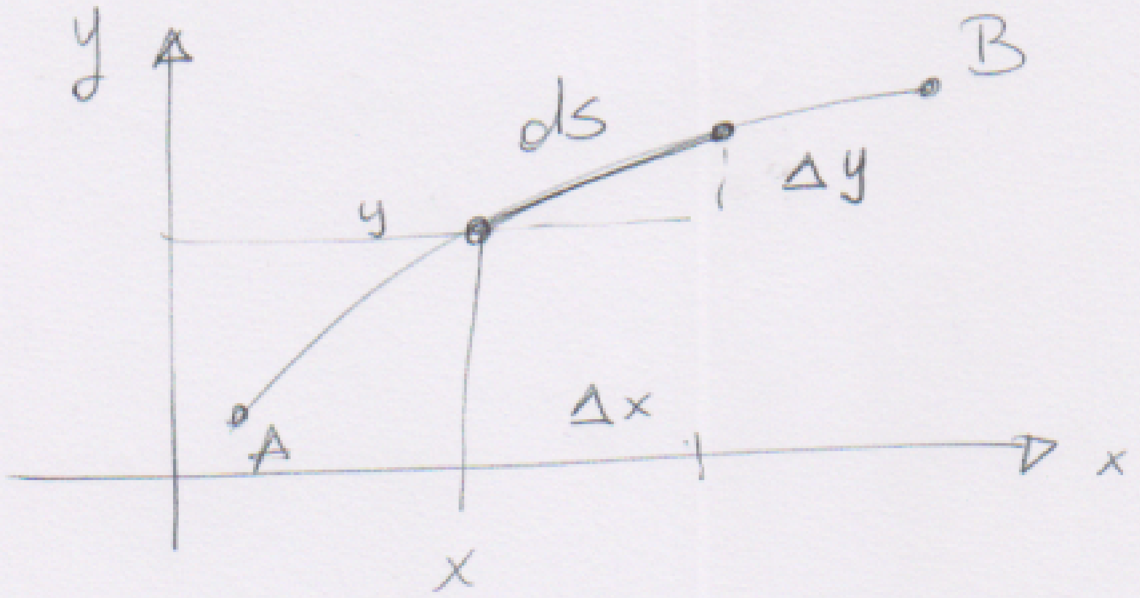
ESERCIZI: Distanza minima fra 2 punti

Distanza Euclidea fra 2 punti del piano



lunghezza infinitesima $ds = \sqrt{dx^2+dy^2}$

Lunghezza di una curva $L = \int_A^B ds$

$$L = \int_A^B ds = \int_A^B \sqrt{dx^2+dy^2} = \int_A^B \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Riporto l'integrale al formalismo sviluppato

scegliendo $f(y,\dot{y},x) = \sqrt{1+\dot{y}^2}$

$$J = \int_{x_1}^{x_2} \sqrt{1+\dot{y}^2}\, dx$$

$\delta J = 0 \Rightarrow$ curva con estremi fissati con ~~[illegible]~~ lunghezza minima

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\sqrt{1+\dot{y}^2} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\frac{\partial f}{\partial y} = 0$$

ottengo $\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$

Integrando $\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c$

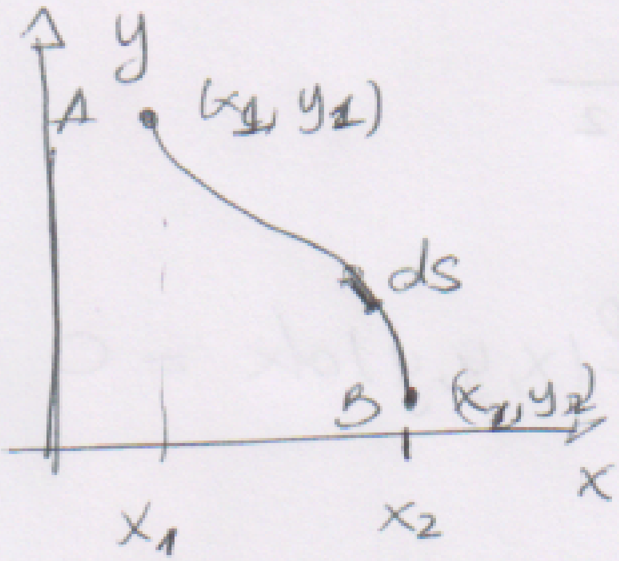
$$\dot{y} = c\sqrt{1+\dot{y}^2} \Rightarrow \dot{y}^2 = c^2(1+\dot{y}^2)$$

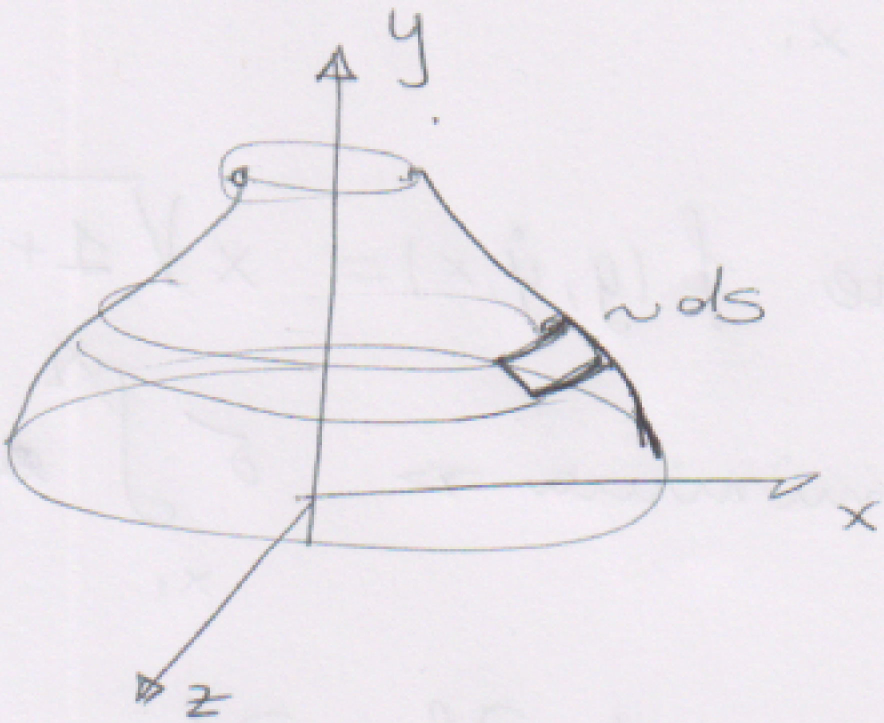
$$\dot{y}^2 = \underbrace{\frac{c^2}{1-c^2}}_{\text{costante}} = a^2 \qquad \dot{y} = a$$

Soluzione $y = ax + c \Rightarrow$ RETTA

Superficie di rivoluzione con area minima

Sup. di rivoluzione: ottenuta facendo ruotare il grafico di una funzione attorno ad un asse fisso





Fissati i punti $A=(x_1, y_1)$, $B=(x_2, y_2)$ cerchiamo la curva passante per A e B che da luogo ad una sup. di rivoluzione con area minima.

L'area di una striscolina risulta al primo ordine

$$dA = \underbrace{2\pi x}_{\text{circ. cilindro}} \, ds \quad = 2\pi x \sqrt{dx^2 + dy^2}$$

↳ altezza cilindro

$$dA = 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

L'area totale della sup. di riv.:

$$A = \int_{x_1}^{x_2} dA = \int_{x_1}^{x_2} 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx =$$

$$= 2\pi \int_{x_1}^{x_2} x \sqrt{1+\dot{y}^2}\, dx \equiv 2\pi \int_{x_1}^{x_2} f(y, \dot{y})\, dx$$

dove $f(y, \dot{y}, x) = x\sqrt{1+\dot{y}^2}$

Area minima $\Rightarrow$ $\delta \int_{x_1}^{x_2} f(x, y, \dot{y})\, dx = 0$

E-L $\Rightarrow$ $\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = \frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$= \frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0 \Rightarrow x\dot{y} = \sqrt{1+\dot{y}^2}\cdot c$$

prendendo il quadrato

$$x^2\dot{y}^2 = c^2(1+\dot{y}^2) \Rightarrow \dot{y}^2(x^2-c^2) = c^2$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{c^2}{x^2-c^2} \Rightarrow \frac{dy}{dx} = \frac{c}{\sqrt{x^2-c^2}}$$

Separazione delle variabili

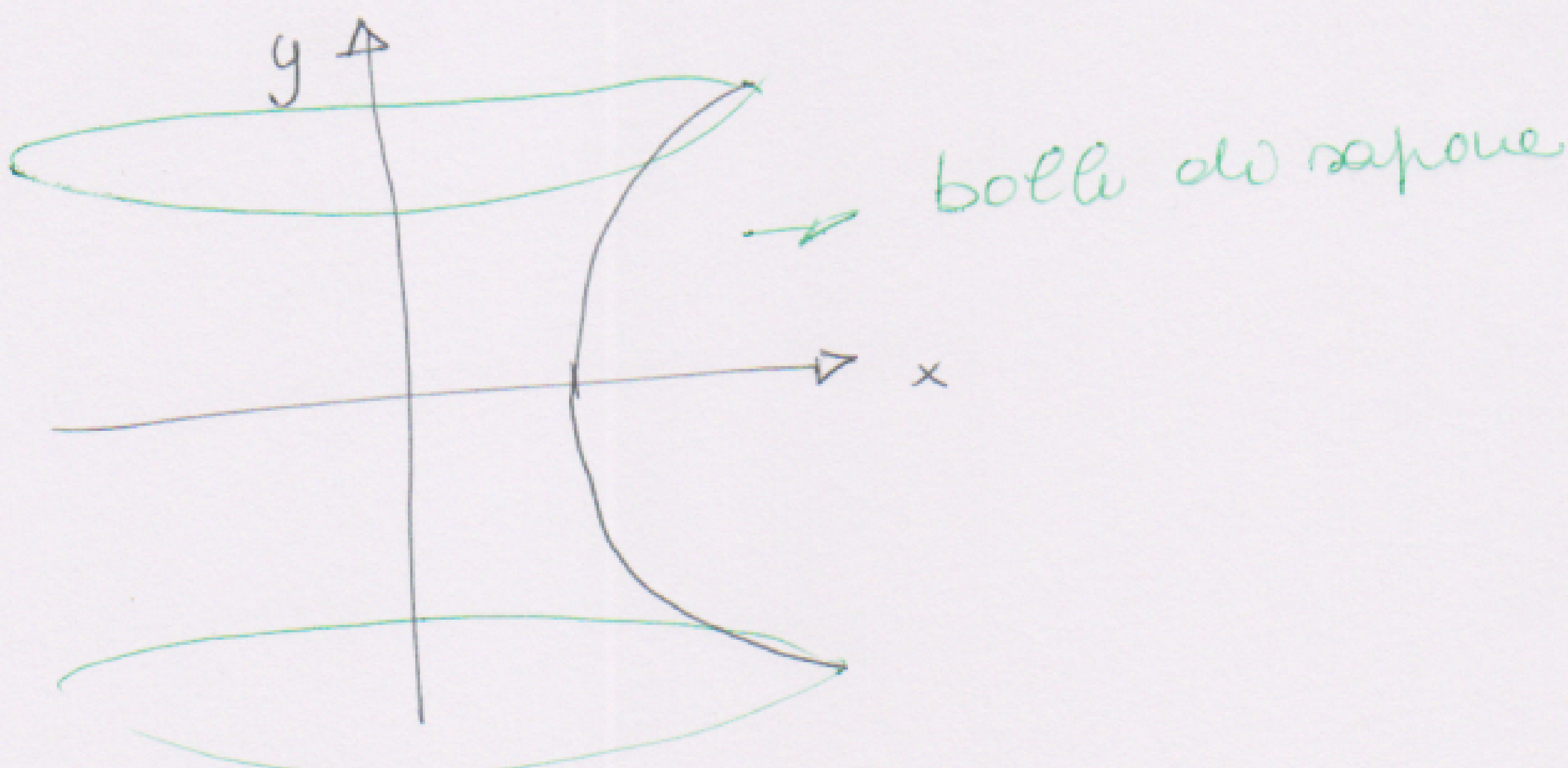
$$dy = c\frac{dx}{\sqrt{x^2-c}} \xrightarrow{\text{Integ.}} \int_{y_1}^{y_2} dy = c\int_{x_1}^{x_2} \frac{dx}{\sqrt{x^2-c^2}}$$

prendiamo per semplicità $y_1 = 0$ $x_1 = 0$

$y_2 = y$, $x_2 = x$

$$y = c\ \cancel{\cosh}\ \operatorname{arccosh}\left(\frac{x}{c}\right)$$

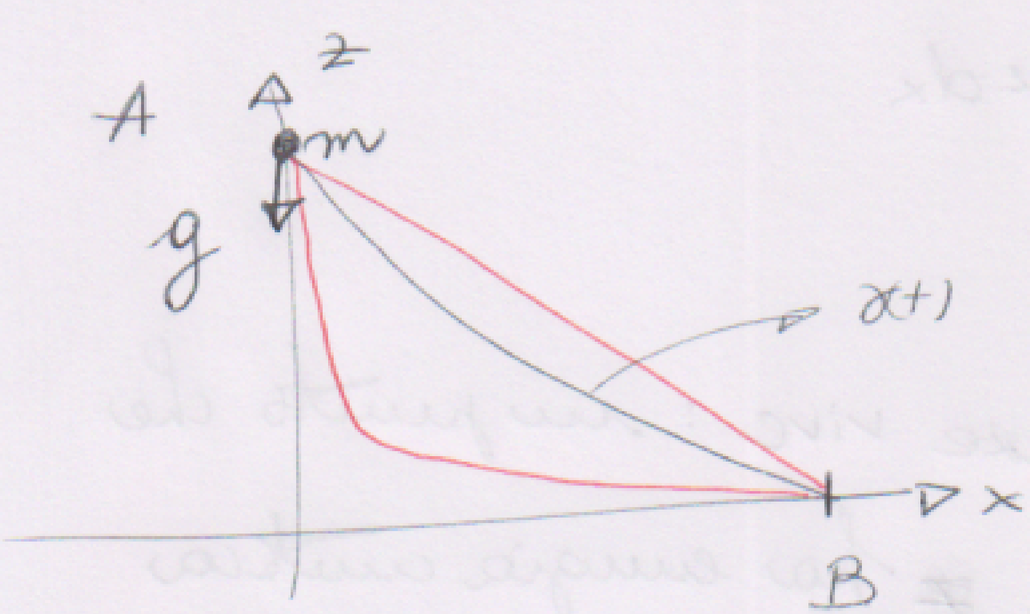
$$x = c\cosh\left(y/c\right)$$: CATENARIA



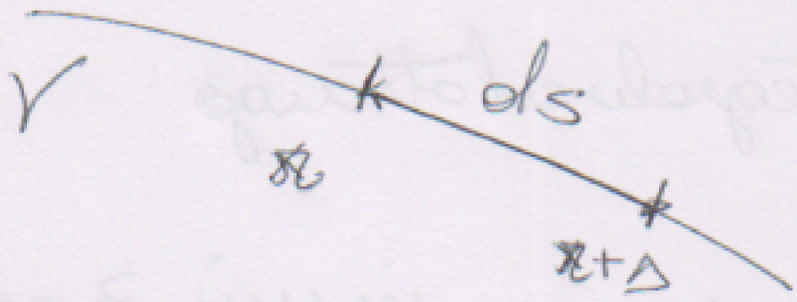
Il problema della brachistocrona

Devo trovare una curva lungo cui un punto possa scorrere senza attrito e, sottoposto alla sola forza di gravità, ~~compia~~ ~~[illegible]~~ si sposta fra 2 pt con dislivello h, nel minor tempo possibile

Problema non banale



Considero un pt che si muove con traiettoria x(t)



sia Δt il tempo impiegato da $x \to x+\Delta$

$$\Delta t = \frac{\Delta t}{\Delta s}\Delta s \simeq \frac{\Delta s}{v}$$

↓

lungh. inf. percorsa lungo la curva

tempo totale percorso fra i 2 estremi

$$t = \Sigma \Delta t \xrightarrow{\Delta t \to 0} \int_A^B dt = \int_A^B \frac{ds}{v}$$

dove, come prima $dS = \sqrt{dx^2 + dz^2} = \sqrt{1 + \left(\frac{dz}{dx}\right)^2} dx$

$$t = \int_A^B \frac{\sqrt{1 + \left(\frac{dz}{dx}\right)^2} dx}{v}$$

Teorema delle forze vive: un punto che cada da una quota z ha energia cinetica

$$\frac{1}{2} m v^2 = m g z \Rightarrow v = \sqrt{2 m g z}$$

Sostituisco nell'integrale ed ottengo

lungo la curva z

$$t = \int_A^B \frac{\sqrt{1 + (z')^2}}{\sqrt{2 m g z}} dx$$ in cui $z = z(x)$

$$z' = \frac{dz}{dx}$$

Quindi il problema si riconduce a trovare il minimo del funzionale

$$J = \int_{x_1}^{x_2} \sqrt{\frac{1 + z'^2}{z}} dx \quad : \quad f(z, z', x) = \sqrt{\frac{1 + z'^2}{z}}$$

Eq. E-L

$$\frac{\partial f}{\partial z} = \sqrt{1+z'^2}\,\frac{\partial}{\partial z} z^{-1/2} = -\frac{1}{2}\sqrt{\frac{1+z'^2}{z^3}}$$

$$\frac{\partial f}{\partial z'} = \frac{1}{\sqrt{z}}\,\frac{\partial}{\partial z'}\sqrt{1+z'^2} = \frac{1}{\sqrt{z}}\,\frac{z'}{\sqrt{1+z'^2}}$$

$$\frac{d}{dx}\frac{\partial f}{\partial z'} = \frac{d}{dx}\left(z^{-1/2}\,\frac{z'}{\sqrt{1+z'^2}}\right) =$$

$$= -\frac{1}{2} z^{-3/2}\,\frac{dz}{dx}\,\frac{z'}{\sqrt{1+z'^2}} + z^{-1/2}\,\frac{1}{(1+z'^2)}\left(\frac{d^2z}{dx^2}\sqrt{1+z'^2} - \right.$$

$$\left. - \frac{z'\cdot z'\cdot z''}{\sqrt{1+z'^2}}\right)$$

$$= -\frac{z^{-3/2}}{2}\,\frac{(z')^2}{\sqrt{1+(z')^2}} + \frac{1}{\sqrt{z}\,(1+(z')^2)}\left(z''\sqrt{1+(z')^2} - \right.$$

$$\left. - \frac{(z')^2 z''}{\sqrt{1+(z')^2}}\right)$$

$$= -\frac{z^{-3/2}(z')^2}{2\sqrt{1+(z')^2}} + \frac{1}{\sqrt{z}\,(1+(z')^2)^{3/2}}\left(z''\,(1+(z')^2) - \right.$$

$$\left. - (z')^2 z''\right)$$

$$= -\frac{z^{-3/2}(z')^2}{2\sqrt{1+(z')^2}} + \frac{1}{\sqrt{z}}\,\frac{z''}{(1+(z')^2)^{3/2}}$$

otteniamo

$$\frac{1}{\sqrt{1+(z')^2}}\left(-\frac{(z')^2}{2\cdot z^{3/2}} + \frac{z''}{\sqrt{z}\,(1+(z')^2)}\right) = -\frac{1}{2}\,\frac{\sqrt{1+(z')^2}}{z^{3/2}}$$

moltiplico per $z^{3/2}\sqrt{1+(z')^2}$

$$\cancel{-\frac{(z')^2}{2}} + \frac{z\,z''}{1+(z')^2} = -\frac{1}{2}\left(1+\cancel{(z')^2}\right)$$

$$-2zz'' = 1+(z')^2$$

Per risolvere, moltip. per z'

$$\underbrace{(z')^3 + 2zz'z''}_{} = -z'$$

$$\frac{d}{dx}\left((z')^2 z\right) = -z' \Rightarrow (z')^2 z + z = c$$

$$(z')^2 = \frac{c-z}{z}$$

$$\boxed{\begin{array}{l} z' = \sqrt{\dfrac{c-z}{z}} \quad * \\ \shortparallel \\ \dfrac{dz}{dx} \end{array}}$$

Separaz. delle variabili

$$\sqrt{\frac{z}{c+z}}\,dz = dx \xrightarrow{\text{Integ.}} \underset{x}{\overset{\shortparallel}{\int^{x} dx}} = \int^{z}\sqrt{\frac{z}{c-z}}\,dz$$

Calcoliamo $\int^z \sqrt{\frac{z}{c-z}}\,dz$

$\downarrow$

elimino c

$$\int^z \sqrt{\frac{z}{1-z/c}}\,\frac{1}{\sqrt{c}}\,dz \overset{s=z/c}{=} c\int^{z/c} \sqrt{\frac{s}{1-s}}\,\frac{\sqrt{c}}{\sqrt{c}}\,ds$$

$$= c\int^{z/c} \sqrt{\frac{s}{1-s}}\,ds =$$

$s = \sin^2\varphi$

$ds = 2\sin\varphi\cos\varphi\,d\varphi$

$$= c\int^{\arcsin\sqrt{z/c}} \frac{\sin\varphi}{\cancel{\cos\varphi}}\,2\sin\varphi\,\cancel{\cos\varphi}\,d\varphi$$

$$= 2c\int^{\arcsin\sqrt{z/c}} \sin^2\varphi\,d\varphi = c\left[\varphi - \cos\varphi\sin\varphi\Big|^{\arcsin\sqrt{z/c}}\right] + k$$

$$\int \sin^2\varphi = \frac{\varphi}{2} - \frac{\cos\varphi\sin\varphi}{2}$$

NOTA: quanto vale $\cos(\arcsin x)$

Poniamo $\cos(\arcsin x) = \psi$

$\downarrow$ quad

$$1 - \cos^2(\arcsin x) = -\psi^2 + 1$$

$$\sin^2(\arcsin x) = x^2 = 1 - \psi^2$$

Quindi $\cos(\arcsin x) = \varphi = \sqrt{1-x^2}$

$$\int^{z} \sqrt{\frac{z}{c-z}} = c\left[\arcsin\left(\sqrt{z/c}\right) - \sqrt{1-z/c}\,\sqrt{z/c}\,\right] + K$$

In conclusione

$$x = c\left[\arcsin\left(\sqrt{z/c}\right) - \sqrt{1-z/c}\,\sqrt{z/c}\,\right] + K$$

$$x - K = c \arcsin\left(\sqrt{z/c}\right) - \sqrt{\frac{z}{c}\left(1 - z/c\right)}\,)$$

$$\boxed{z = c \sin^2\left(\frac{x - K + \sqrt{z/c\,(1 - z/c)}}{c}\right)}$$

↳ eq. della curva in forma implicita

Soluzione alternativa

Torniamo a eq. *

$$z' = \sqrt{\frac{c-z}{z}} \quad \text{dove } z = z(x)$$

$$z' = \frac{dz}{dx}$$

sostituiamo $z(x) = c \sin^2\left(\frac{\psi(x)}{2}\right)$

$$z' = c \cdot 2 \sin(\psi/2) \cos(\psi/2) \frac{1}{2} \psi'$$

$$\psi' c \sin(\psi/2) \cancel{\cos(\psi/2)} = \frac{\cancel{\sqrt{c}} \cancel{\cos(\psi/2)}}{\cancel{\sqrt{c}} \sin(\psi/2)}$$

$$c \psi' \sin^2(\psi/2) = 1$$

Ricordiamo $\sin^2(\psi/2) = \frac{1}{2}(1 - \cos\psi)$

verifica : parto da $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

prendo $\alpha = \beta = \frac{\psi}{2}$

$$\cos(\psi) = \underbrace{\cos^2(\psi/2)}_{1-\sin^2(\psi/2)} - \sin^2(\psi/2)$$

$$\cos(\psi) = 1 - 2\sin^2(\psi/2) \quad ✓$$

ottenemo

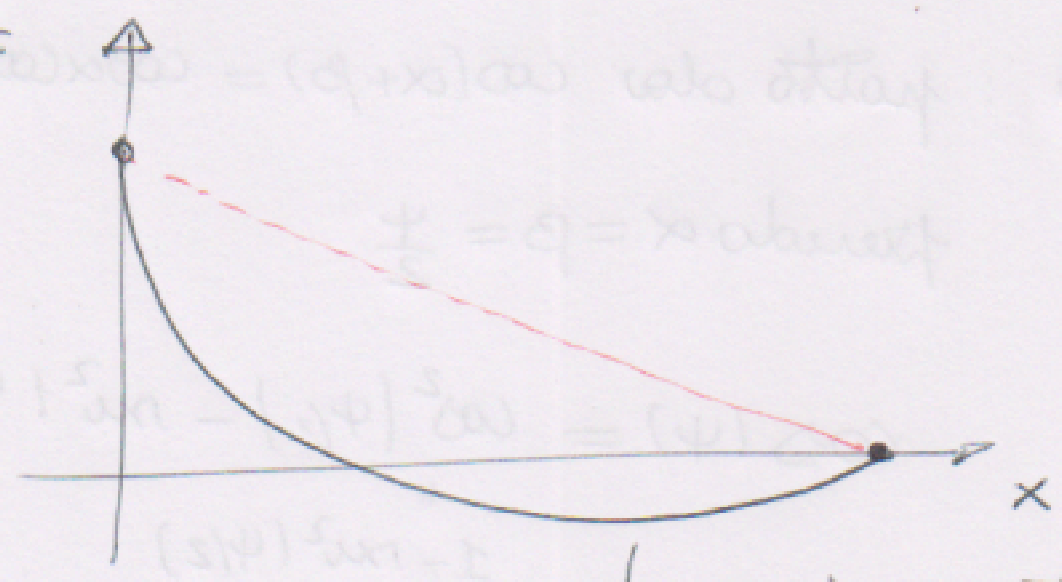
$$c\,\varphi'\,\frac{1}{2}(1-\cos\varphi)=1$$

ovvero $$\frac{c}{2}\,d\varphi\,(1-\cos\varphi)=dx$$

$$\Rightarrow x=\frac{c}{2}\int(1-\cos\varphi)\,d\varphi=\frac{c}{2}\left(\varphi-\sin\varphi\right)+k$$

Abbiamo ottenuto la soluzione espressa in forma parametrica $\varphi\in[0,2\pi)$

$$\begin{cases} x=\frac{c}{2}\left(\varphi-\sin\varphi\right)+k \\ z=c\sin^2(\varphi/2)=\frac{c}{2}\left(1-\cos\varphi\right)\end{cases}$$



↳ minimo tempo di percorrenza.