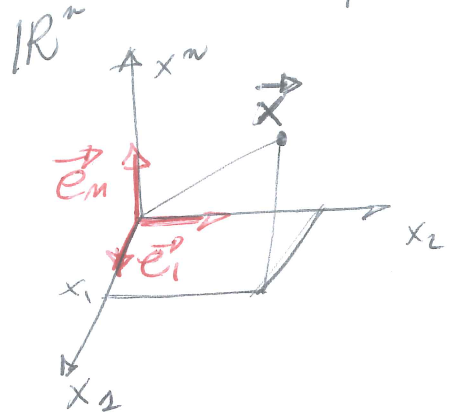
Notazioni

Vettori in $\mathbb{R}^n$

$$\vec{x} = \underbrace{(x_1, x_2 \dots x_n)}_{\text{componente del vettore}} \in \mathbb{R}^n$$



Versori canonici : $\vec{e}_i = (0, 0 \dots \underset{i\text{-esima pos.}}{1}, \dots 0)$

$$\langle \vec{e}_i \cdot \vec{e}_j \rangle = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

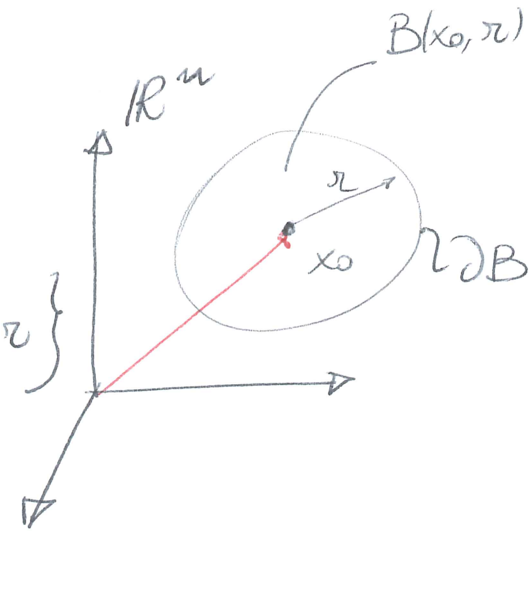
$$\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i$$

Sottoinsiemi di $\mathbb{R}^n$ $\qquad U \subseteq \mathbb{R}^n$

tipicamente aperto e connessi

$\bar{U}$: chiusura di U

∂U : bordo di U



Palla di centro x_0 e raggio r

$$B(x_0, r) : \left\{ \vec{x} \in \mathbb{R}^n \,\middle|\, \sqrt{\sum (x_i - x_{0,i})^2} \leq r \right\}$$

Misura della palla unitaria in $\mathbb{R}^n$

$\alpha(n) = |B(0,1)|$ ⟶ misura

Si ha che

$\alpha(2) = \pi$

$\alpha(3) = \frac{4}{3}\pi$

$\vdots$

$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ ⟶ $\alpha(2n) = \frac{\pi^n}{n!}$

⟶ $\alpha(2n+1) = \frac{2^{n+1}\pi^n}{(2n+1)!!}$

⟶ f. di Eulero

$|B(x_0, r)| = r^n \alpha(n)$

$|\partial B(x_0, r)| = r^{n-1} n\, \alpha(n)$

Derivate $u(\vec{x}) = u(x_1 \dots x_n) : \mathbb{R}^n \to \mathbb{R}$

deriv. parziale $\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u$

gradiente $\vec{\nabla} u \doteq \sum \vec{e}_i \frac{\partial u}{\partial x_i} \equiv \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \dots \frac{\partial u}{\partial x_n}\right)$

<u>NOTA</u> $u(x)$ è uno scalare

$\vec{\nabla} u$ è un vettore $\in \mathbb{R}^n$

Derivate seconde

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \doteq \partial^2_{x_i x_j} u \qquad \frac{\partial^2 u}{\partial x_i^2} \doteq \partial^2_{x_i^2} u$$

Laplaciano $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

$\downarrow$

scalare

Integrali di Volume

$$\int_{\mathbb{R}^n} \omega(\bar{x}) \underbrace{dV^n(x)}$$

$\omega(x) : \mathbb{R}^n \to \mathbb{R}$

$\omega \in C^1(\mathbb{R}^n)$

ω sommabile

$\rightarrow$ misura di Lebesgue

$dV^n(\bar{x}) = dx_1 \, dx_2 \dots dx_n$

$\downarrow$

Tipicamente integriamo in $U \subset \mathbb{R}^n$, $B(x_0, R)$

Scaling dell'Int. di volume

$$\int_{B^n(0,r)} f(x)\, dV^n(x) \xrightarrow{\bar{x} = r\bar{y}} = \int_{B^n(0,1)} f(r\bar{y})\, r^n\, dV^n(y)$$

$$dV(x) = r^n\, dV(y)$$

Si verifica con la formula del cambio di variabile $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ $\quad \varphi(B(0,r)) = B(0,1)$

$\varphi \in C^1$ $\quad \varphi$ iniettiva

$$\int_{\varphi(E)} f(x)\, dV(x) = \int_E f(\varphi(y))\, \underbrace{|\det J_\varphi|}_{|J|}\, dV(y)$$

con $E \subset \mathbb{R}^n$

Matrice Jacobiana $\quad J: n \times n$

$$J = \begin{pmatrix} \frac{\partial \varphi_1}{\partial y_1} & \frac{\partial \varphi_1}{\partial y_2} & \frac{\partial \varphi_1}{\partial y_3} & \cdots & \frac{\partial \varphi_1}{\partial y_n} \\ \frac{\partial \varphi_2}{\partial y_1} & & & & \\ \vdots & & & & \\ \frac{\partial \varphi_n}{\partial y_1} & & \cdots & & \frac{\partial \varphi_n}{\partial y_n} \end{pmatrix}$$

Nel caso dell'integrale precedente

$$\varphi(\vec{y}) = r\,\vec{y} \qquad \frac{\partial \varphi_i}{\partial y_j} = r\,\frac{\partial y_i}{\partial y_j} = r\,\delta_{ij}$$

$$J = \left.\overbrace{\begin{pmatrix} r & 0 & 0 & \cdots \\ 0 & r & 0 & \cdots \\ & & \ddots & \\ & & & r \end{pmatrix}}^{n}\right\}n \;\rightarrow\; |\det J| = r^n$$

che verifica la formula

Notazioni

$$J = D\psi$$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \rho\cos\theta\cos\varphi & \rho\cos\theta\sin\varphi & -\rho\sin\theta \\ -\rho\sin\theta\sin\varphi & \rho\sin\theta\cos\varphi & 0 \end{pmatrix}$$

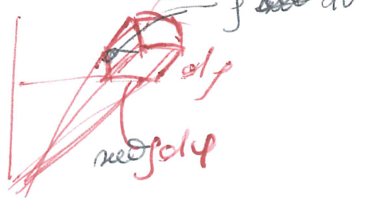
$$|J| = \det J = \rho^2\cos\theta \begin{vmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi \end{vmatrix} +$$

$$+ \rho^2\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi \end{vmatrix}$$

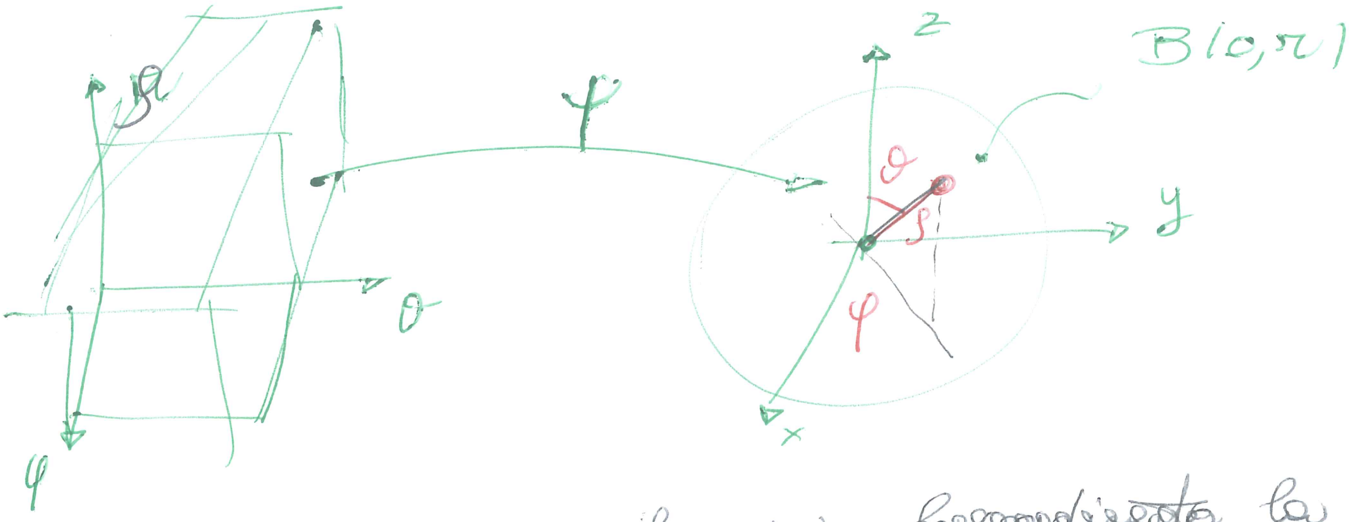
$$= \rho^2\cos\theta\,(\cos\theta\sin\theta\cos^2\varphi + \cos\theta\sin\theta\sin^2\varphi)$$

$$+ \rho^2\sin\theta\,(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi)$$

$$= \rho^2\cos^2\theta\sin\theta + \rho^2\sin^2\theta\sin\theta = \underbrace{\rho^2\sin\theta}_{\text{misura del volumetto } d\rho\, d\theta\, d\varphi}$$



Calcoliamo il volume di una sfera in $\mathbb{R}^3$ utilizzando le coordinate polari



ψ mappa $\mathbb{R}^3 \to \mathbb{R}^3$ che esprime ~~le coordinate~~ la posizione di un punto all'interno della sfera in coord. sferiche

$$\begin{cases} x = \rho \,\mathrm{sen}\,\theta \cos\varphi \\ y = \rho \,\mathrm{sen}\,\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \quad : \quad \psi\overbrace{(\rho, \theta, \varphi)}^{\vec{q}} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$|B(0,r)| \doteq \int_{B(0,r)} dV(x) \overset{\text{formula cambio variabili}}{=} \int_0^r d\rho \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \; |J|$$

$|J|$: det Jacob. della trasformaz.

$|B(0,r)|$ → misura di Lebesgue della sfera

$$|J| = \det(J)$$

$J \doteq$ mat. $n \times n$

$$|B(0,r)| = \int_0^r \int_0^\pi \int_0^{2\pi} \rho^2 \sin\theta \, d\varphi \, d\theta \, d\rho$$

$$= 2\pi \int_0^r \int_0^\pi \rho^2 \sin\theta \, d\theta \, d\rho =$$

$$2\pi \underbrace{\int_0^r \rho^2 d\rho}_{\frac{r^3}{3}} \underbrace{\int_0^\pi \sin\theta \, d\theta}_{2} = \frac{4}{3}\pi r^3$$