

Volume palla unitaria in \mathbb{R}^n

$$B^n(0,1) = \left\{ \vec{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq 1 \right\}$$

per comodità di notazione indico la compon. $x_n = z$

$$\vec{x} = (x_1, x_2, \dots, x_{n-1}, z) \equiv (\tilde{x}, z)$$

$\tilde{x} \in \mathbb{R}^{n-1}$

Fisso la coordinata $x_n = z$, ^{Indico} ~~trovo~~ la sezione

della palla col piano x_n costante con E_z

$$B^n(0,1) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n-1} x_i^2 \leq 1 - z^2 \right\} \equiv$$

$$\left\{ \begin{array}{l} (\tilde{x}, z) \in \mathbb{R}^n \\ \tilde{x} \in \mathbb{R}^{n-1} \mid \|\tilde{x}\|_{\mathbb{R}^{n-1}}^2 \leq 1 - z^2 \\ z \in \mathbb{R} \quad -1 < z < 1 \end{array} \right\}$$

$$E_z = \left\{ \tilde{x} \in \mathbb{R}^{n-1} \mid \|\tilde{x}\|^2 \leq 1 - z^2 \right\}$$

$$\equiv \begin{cases} B^{n-1}(0, \sqrt{1-z^2}) & -1 < z < 1 \\ \emptyset & |z| > 1 \end{cases}$$

La sezione di una palla con un piano

è una palla di dimensione minore

$$|B^m(0,1)| = \int_{B^m(0,1)} dx_1 dx_2 \dots dx_m =$$

$$\int_{B^m(0,1)} (dx_1 \dots dx_m) dt = \int_{-1}^1 dt \left(\int_{\mathbb{F}_t} dx_1 dx_2 \dots dx_{m-1} \right)$$

\parallel
 sezione della palla
 col piano

$$= \int_{-1}^1 dt \int_{B^{m-1}(0, \sqrt{1-t^2})} dV(\tilde{x})$$

$$\int_{B^{m-1}(0, \sqrt{1-t^2})} dV(\tilde{x}) \xrightarrow{\tilde{y} = \tilde{x} / \sqrt{1-t^2}} = \int_{B^{m-1}(0,1)} (1-t^2)^{\frac{m-1}{2}} dV(y)$$

$$dV(\tilde{x}) = (1-t^2)^{\frac{m-1}{2}} dV(y)$$

$$= (1-t^2)^{\frac{m-1}{2}} \int_{B^{m-1}(0,1)} dV(y) = (1-t^2)^{\frac{m-1}{2}} |B^{m-1}(0,1)|$$

$$|B^m(0,1)| = \int_{-1}^1 |B^{m-1}(0,1)| (1-t^2)^{\frac{m-1}{2}} dt$$

$$= 2 |B^{m-1}(0,1)| \int_0^1 (1-t^2)^{\frac{m-1}{2}} dt$$

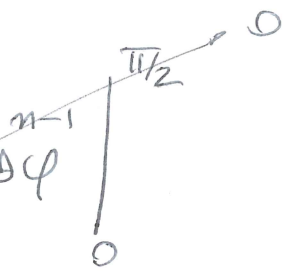
calcolano $\int_0^1 (1-t^2)^{\frac{n-1}{2}} dt$

cambio variable $z = \sin \varphi$
 $dt = \cos \varphi d\varphi$

$$= \int_0^{\pi/2} (\cos \varphi)^{\frac{n-1}{2}} \cos \varphi d\varphi$$

$$= \int_0^{\pi/2} (\cos \varphi)^{n-1} d\varphi$$

Integ. x parte

$$\int_0^{\pi/2} \cos^{(n-1)} \varphi \cos \varphi d\varphi = \cancel{\sin \varphi \cos \varphi}$$


D(xn phi)

$$+ \int_0^{\pi/2} (n-1) \cos^{n-2} \varphi \sin \varphi d\varphi$$

\parallel
 $1 - \cos^2 \varphi$

$$(n-1) \int_0^{\pi/2} \cos^n \varphi + \int_0^{\pi/2} \cos^n \varphi = (n-1) \int_0^{\pi/2} \cos^{n-2} \varphi d\varphi$$

$$\int_0^{\pi/2} \cos^n \varphi = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} \varphi d\varphi$$

Caso n pari

$$n = 2k$$

$$\begin{aligned} \int_0^{\pi/2} \cos^{2k} \varphi &= \frac{2k-1}{2k} \int_0^{\pi/2} \cos^{2k-2} \varphi d\varphi = \\ & \frac{2k-1}{2k} \frac{(2k-2)-1}{2k-2} \int_0^{\pi/2} \cos^{2k-4} \varphi d\varphi = \dots \\ & \frac{2k-3}{2k-2} \\ & = \frac{(2k-1)!!}{(2k)!!} \int_0^{\pi/2} \cos^0 \varphi d\varphi \\ & \quad \underbrace{\hspace{10em}}_{\pi/2} \end{aligned}$$

Caso n dispari

$$n = 2k+1$$

$$\int_0^{\pi/2} \cos^{2k+1} \varphi d\varphi = \frac{(2k)!!}{(2k+1)!!} \underbrace{\int_0^{\pi/2} \cos \varphi d\varphi}_1$$

Si è ottenuto

$$n = 2k$$

$$|B^{2k}(0,1)| = 2 |B^{2k-1}(0,1)| \frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2}$$

$$n = 2k+1$$

$$|B^{2k+1}(0,1)| = 2 |B^{2k}(0,1)| \frac{(2k)!!}{(2k+1)!!}$$

Caso dispari

$$|B^{2k+1}(0,1)| = 2^2 |B^{2k-1}(0,1)| \frac{(2k)!!}{(2k+1)!!} \frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2}$$

$$= 2^2 |B^{2k-1}(0,1)| \frac{\pi}{2k+1}$$

$$= 2 |B^{2k-3}(0,1)| \frac{\pi}{2k+1} \frac{\pi}{2k-1} = \dots$$

$$= \frac{2^k \pi^k}{(2k+1)!!} |B^1(0,1)| = \frac{2^{k+1} \pi^k}{(2k+1)!!}$$

Note

$$B^1(0,1) = [-1, 1]$$

analogamente per n pari $|B^{2k}(0,1)| = \frac{\pi^k}{k!}$

Integrali di superficie in \mathbb{R}^m

Superf. : definizione implicita

$$V = \{x \in \mathbb{R}^m : g(x) = 0\}$$

$\downarrow c^1$

$$g(x) : \mathbb{R}^m \rightarrow \mathbb{R}$$

Esempio tipico $g(x) = \sum x_i^2 - r^2$

$$g(x) = 0 \Rightarrow \underbrace{\sum x_i^2 = r^2}_{\text{surf. sferica}} = \partial B(0, r)$$

$$\int f(x) dS(x)$$

\downarrow

$$V \subset \mathbb{R}^{m-1}$$

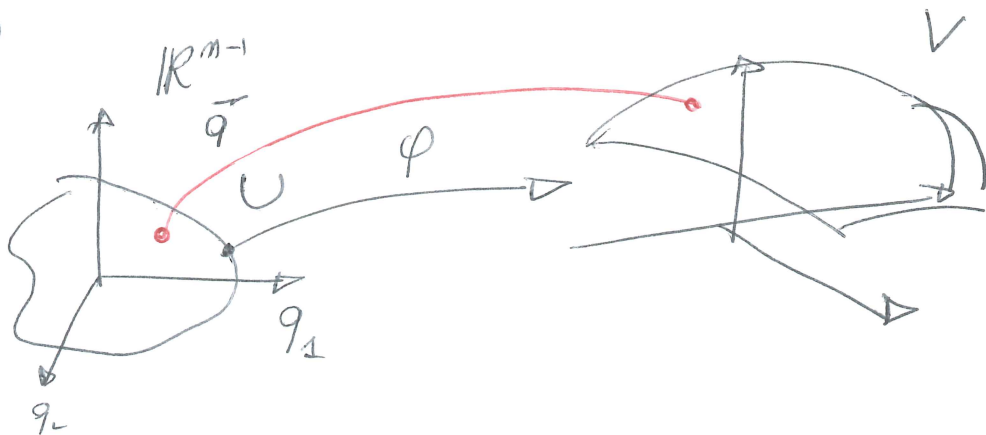
Superficie

misura di Hausdorff

$$f \in C^1(\mathbb{R}^m)$$

$$f : \mathbb{R}^m \rightarrow \mathbb{R}$$

Per valutare l'integrale immag. di disporre di una mappa $\varphi : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^m$ $\varphi(U) = V$ iniettiva



Treatto V come una varietà ed intollerando
 l'2 parametro, notata da φ si ottiene che
 l'integrale di superficie si scrive

$$\int_V f(x) dS^m(x) = \int_U f(\varphi(\bar{q})) \underbrace{|J|}_{\text{Lebesgue}} dV(\bar{q})^{m-1}$$

$$|J| = \sqrt{\det(JJ^T)}$$

J : matrice Jacobiana

$$J = \frac{\partial \vec{\varphi}}{\partial \bar{q}}$$

$$J = \begin{matrix} & \overbrace{\hspace{10em}}^m \\ \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{pmatrix} \frac{\partial \varphi_1}{\partial q_1} & \frac{\partial \varphi_2}{\partial q_1} & \frac{\partial \varphi_3}{\partial q_1} & \dots & \frac{\partial \varphi_m}{\partial q_1} \\ \frac{\partial \varphi_1}{\partial q_2} & \frac{\partial \varphi_2}{\partial q_2} & \frac{\partial \varphi_3}{\partial q_2} & \dots & \frac{\partial \varphi_m}{\partial q_2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial \varphi_1}{\partial q_{m-1}} & \frac{\partial \varphi_2}{\partial q_{m-1}} & \frac{\partial \varphi_3}{\partial q_{m-1}} & \dots & \frac{\partial \varphi_m}{\partial q_{m-1}} \end{pmatrix} \end{matrix}$$

$$J_{ij} = \frac{\partial \varphi_j}{\partial q_i}$$

$$J^T_{ij} = \frac{\partial \varphi_i}{\partial q_j}$$

J : $(m-1) \times m$
 righe col.

$J^T = m \times (m-1)$

$J \cdot J^T$: $(n-1) \times (n-1)$ quadrata

$$(JJ^T)_{ij} = \sum_{r=1}^m \underbrace{\frac{\partial \varphi_r}{\partial q_i}}_{J_{ir}} \underbrace{\frac{\partial \varphi_r}{\partial q_j}}_{J_{jr}^T}$$

Scaling dell'Integ. di superficie

$$\int_{\partial B(0, r)} f(x) dS^m(x) = \int_{\partial B(0, 1)} f(r\bar{y}) \boxed{r^{m-1}} dS^m(\bar{y})$$

$\bar{x} = r\bar{y}$

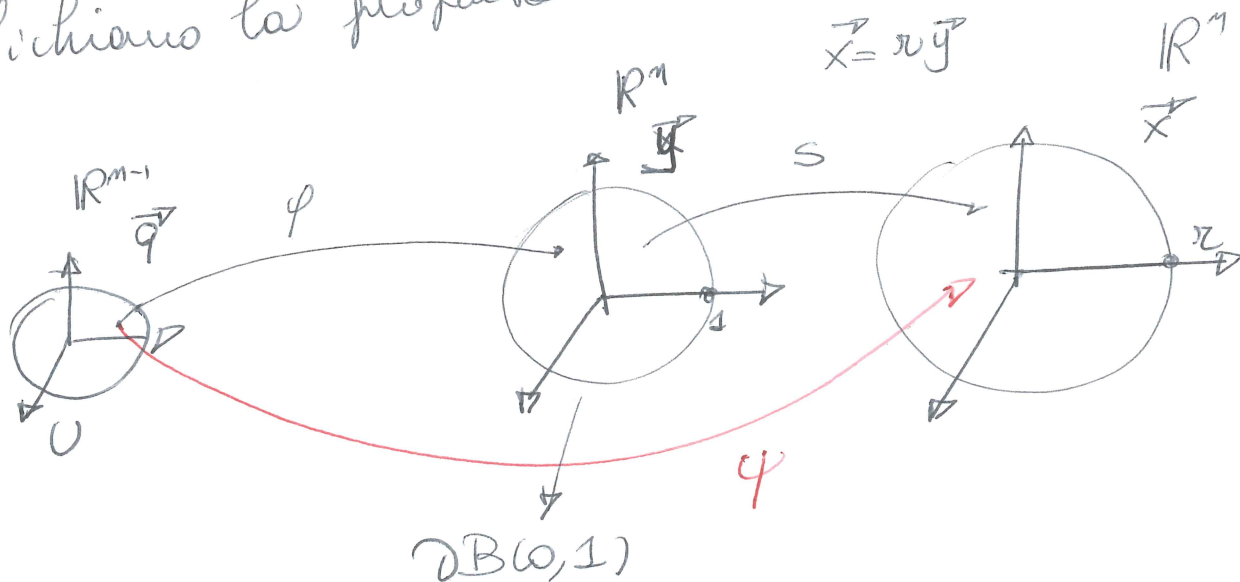
$$\bar{x} = r\bar{y} \Rightarrow dS^m(x) = r^{m-1} dS^m(\bar{y})$$

La relazione segue immutatamente dalla regola
 proprietà di scaling della misura di Hausdorff.

$$\mathcal{H}^{m-1}(\alpha \omega) = \alpha^{m-1} \mathcal{H}^{m-1}(\omega)$$

↓
scalare

Verifichiamo la proprietà direttamente



$$\psi = S \circ \phi$$

$$\psi(q) = r\phi(q)$$

Dalla definizione

$$\int_{\partial B(\omega, r)} f(\vec{x}) dS^m(x) = \int_U f(\varphi(q)) |J\varphi| dV(q)^{m-1}$$

$$|J\varphi| = \sqrt{\det(J\varphi J\varphi^t)}$$

$$(J\varphi J\varphi^t)_{ij} = \sum_n \frac{\partial \varphi_n}{\partial q_i} \frac{\partial \varphi_n}{\partial q_j} = \sum_n \frac{\partial \varphi_n}{\partial q_i} \frac{\partial \varphi_n}{\partial q_j}$$

$$= r^2 (J\varphi J\varphi^t)_{ij}$$

Ricordando che $\det(\lambda A) = \lambda^m \det(A)$
 \hookrightarrow mat $m \times m$

$$\det(J\varphi J\varphi^t) = (r^{m-1})^2 \det(J\varphi J\varphi^t)$$

$$\Rightarrow |J\varphi| = |J\varphi| r^{m-1}$$

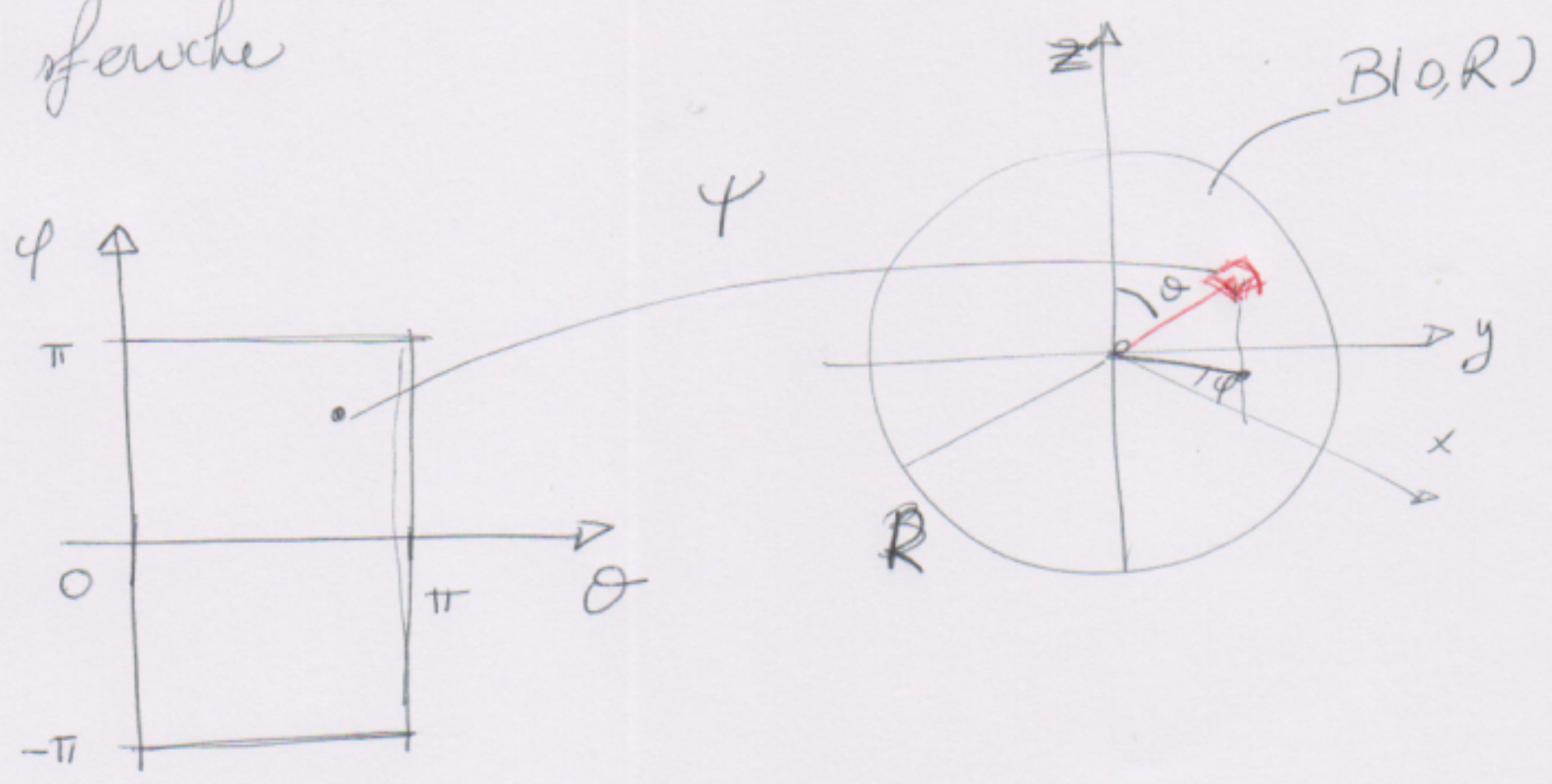
$$\int_{\partial B(\omega, r)} f(\vec{x}) dS^m(x) = \int_U \underbrace{f(r\varphi(q))}_{h(\varphi(q))} r^{m-1} |J\varphi| dV(q)^{m-1}$$

$$= \int_{\partial B(\omega, 1)} h(\varphi(y)) dS^m(y) = \int_{\partial B(\omega, 1)} f(r\vec{y}) r^{m-1} dS^m(y)$$

dalla def. di integ. di sup

Calcolo area superficie sfera

In analogia a quanto fatto x il calcolo del volume, descriviamo la sfera in coordinate sferiche



$$\psi \doteq \begin{cases} x = R \sin \theta \cos \phi \\ y = R \sin \theta \sin \phi \\ z = R \cos \theta \end{cases}$$

$$\psi(\phi, \theta) \rightarrow \mathbb{R}^3$$

$$|\partial B(0, R)| = \int_{\partial B(0, R)} dS(\vec{x}) = \int_{-\pi}^{\pi} \int_0^{\pi} |\det J J^T| d\phi d\theta$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix} = R \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \theta \cos \phi & \sin \theta \sin \phi & 0 \end{pmatrix}$$

$$JJ^T = R^2 \begin{pmatrix} \cos\theta \cos\varphi & \cos\theta \sin\varphi & -\sin\theta \\ -\sin\theta \sin\varphi & \sin\theta \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \cos\varphi & -\sin\theta \\ \cos\theta \sin\varphi & \sin\theta \\ -\sin\theta & 0 \end{pmatrix}$$

$$= R^2 \begin{pmatrix} \cos^2\theta \cos^2\varphi + \cos^2\theta \sin^2\varphi + \sin^2\theta & -\cos\theta \sin\theta \cos\varphi \sin\varphi + \cos\theta \sin\theta \cos\varphi \sin\varphi \\ -\sin\theta \cos\theta \sin\varphi \cos\varphi + \sin\theta \cos\theta \sin\varphi \cos\varphi & \sin^2\theta \sin^2\varphi + \sin^2\theta \cos^2\varphi \end{pmatrix}$$

$$= R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

$$\sqrt{\det JJ^T} = R^2 \sin\theta$$

$$|DB(\omega, R)| = \int_{-\pi}^{\pi} d\varphi \int_0^{\pi} d\theta \sin\theta R^2 = 4\pi R^2$$