# Petzval Theorem

The Petzval Field Curvature Theorem (after Joseph Petzval, 1807-1891) demonstrates that even thin lenses will have an aberration that causes the image plane to curve. The sign of the curvature is opposite for surfaces of positive and negative power, so that the only way to correct for it is to have both positive and negative surfaces in an optical system.
*(Not quite the **only** way to correct field curvature: The eye has only positive surfaces, so has significant field curvature – which is compensated for by curving the retina.)*

For a flat detector surface, however, the Petzval curvature must be corrected by balancing the positive and negative power surfaces – <u>nothing</u> else will correct it.

The only departure from Gaussian optics necessary to derive Petzval's theorem is to assume that the optical surfaces are true spheres, not paraxial approximations.

Let us consider an optical system of one spherical surface, with a spherical object surface co-centered with it:
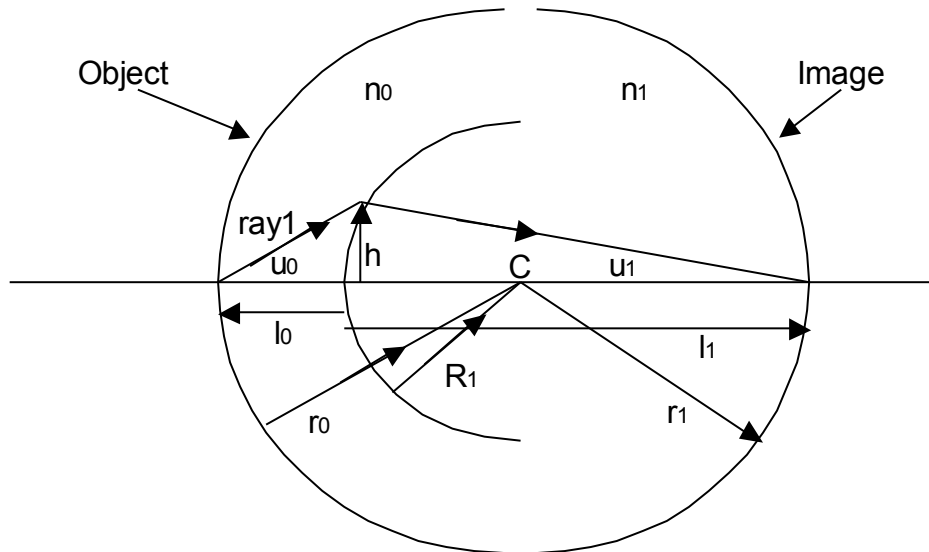
It is evident that any axis drawn through the center C is equivalent. Therefore any 'paraxial' rays traced along these alternate axes will produce an image at the same distance from C -- hence the image surface is also a sphere with center at C.
(Note that the usual notation has been modified, with the thought of continuing through many surfaces.)
Tracing ray1 through the surface, we get:

$$n_1 u_1 = n_0 u_0 - h \frac{n_1 - n_0}{R_1}.$$ Substituting $u_0 = -\frac{h}{l_0}, \quad u_1 = -\frac{h}{l_1}$,

gives: $\frac{n_1}{l_1} = \frac{n_0}{l_0} + \frac{(n_1 - n_0)}{R_1}.$ (Just the usual surface imaging formula.)

The overall goal here is to develop a formula relating the curvature of the object surface, the curvature of the image surface, and the power of the lens surface – hence we want to eliminate the usual lens and ray values such as object and image distances, ray angles and heights, etc. The following manipulations are not particularly obvious (except in retrospect), but accomplish that purpose:

Noting from the figure that $l_0 = R_1 - r_0$, $l_1 = R_1 - r_1$ (using the sign convention of measuring from the vertex), we can substitute into the imaging formula:

$$\frac{n_1}{R_1 - r_1} - \frac{n_0}{R_1 - r_0} = \frac{n_1 - n_0}{R_1}$$

Multiplying through by the denominators,

$$R_1 n_1 (R_1 - r_0) - R_1 n_0 (R_1 - r_1) = (R_1 - r_1)(R_1 - r_0)(n_1 - n_0),$$

and expanding terms we get,

$$\underline{R_1^2 n_1} - \underline{\underline{R_1 n_1 r_0}} - \underline{\underline{\underline{R_1^2 n_0}}} + \underline{\underline{\underline{\underline{R_1 n_0 r_1}}}} = \underline{R_1^2 n_1} - R_1 n_1 r_1 - \underline{\underline{R_1 n_1 r_0}} + n_1 r_1 r_0 - \underline{\underline{\underline{R_1^2 n_0}}} + R_1 n_0 r_1 + \underline{\underline{\underline{\underline{R_1 n_0 r_0}}}} - n_0 r_1 r_0$$

(where equivalent terms on each side have been underlined). Canceling those terms and re-arranging gives us:

$$R_1 n_0 r_0 - R_1 n_1 r_1 + (n_1 - n_0) r_1 r_0 = 0$$

Multiplying through by the carefully chosen term $\dfrac{1}{n_1 n_0 R_1 r_1 r_0}$ gives:

$$\frac{1}{n_1 r_1} - \frac{1}{n_0 r_0} + \frac{n_1 - n_0}{n_1 n_0 R_1} = 0$$

Finally, noting that $\dfrac{n_1 - n_0}{R_1} \equiv K_1$, the power of the lens surface, we arrive at our goal:

$$\boxed{\frac{1}{n_0 r_0} - \frac{1}{n_1 r_1} = \frac{K_1}{n_0 n_1}}$$

A formula that allows us to calculate the change in curvature from the image surface to the object surface as a function of the power of the lens surface.

Since the image surface for lens 1 is the object surface for lens 2 (not shown in the figure), we can chain this calculation through several surfaces (just like the imaging equation). Applying this equation to the next surface, we get $\dfrac{1}{n_1 r_1} - \dfrac{1}{n_2 r_2} = \dfrac{K_2}{n_1 n_2}$.

Substituting for $\frac{1}{n_1 r_1}$ in the first equation, we get: $\dfrac{1}{n_0 r_0} - \dfrac{1}{n_2 r_2} = \dfrac{K_1}{n_0 n_1} + \dfrac{K_2}{n_1 n_2}$, and for K

surfaces: $\dfrac{1}{n_0 r_0} - \dfrac{1}{n_k r_k} = \sum_{j=1}^{k} \dfrac{K_j}{n_{j-1} n_j}$. If the initial object plane is flat ($r_0 = \infty$), then we have the equation for the *Petzval Curvature of the image plane:*

$$\boxed{\frac{1}{r_k} = -n_k \sum_{j=1}^{k} \frac{K_j}{n_{j-1} n_j}}$$

Petzval Curvature: $$\frac{1}{r_k} = -n_k \sum_{j=1}^{k} \frac{K_j}{n_{j-1} n_j}$$

This equation clearly shows that positive power surfaces cause the image plane to curve negatively (that is, with the center of curvature to the left of the surface), and negative power surfaces cause it to curve positively. Also, the **only way** that the image plane can be flat is if the surface powers (weighted by the indices on either side of the surface) sum to zero.

It is equally clear (from the derivation) that this image plane curvature exists even for paraxial rays. We have chosen to ignore that in our development of 'paraxial' ray tracing, since our larger goal was to develop a simple mathematical model of a perfect imaging system. (It was in pursuit of this goal that we made the otherwise unjustified substitution of ray *tangents* for ray *angles* -- so that the perfect model would also cover finite height rays.)

Implications of Petzval Curvature for lens design:

I)   In multiple lens systems that must have net positive power, the negative lenses can be situated so that they contribute relatively less (given their power) to the total power than the positive lenses. If we use the usual way of generating multiple-lens power formulas (tracing a ray parallel to the axis through the system), but leave the formula in terms of ray heights, instead of lens spacings, we get:

$$K = K_1 + \frac{h_2}{h_1} K_2 + \frac{h_3}{h_1} K_3 + \cdots$$

By arranging the negative lenses where the parallel ray has a low height from the axis, we can minimize the negative contribution to the total power while balancing the Petzval sum. This is what is done in the Cooke Triplet style of camera lens:
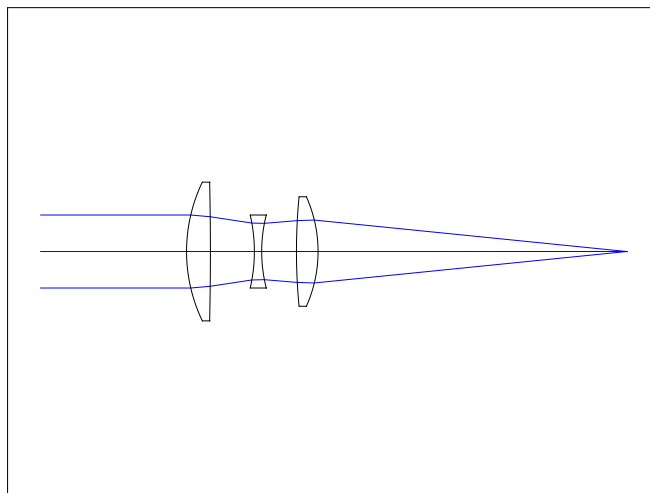
II)   The image plane doesn't have to be *exactly* flat – only flat enough that the Depth of Focus (DOF) of the optics includes the image plane:
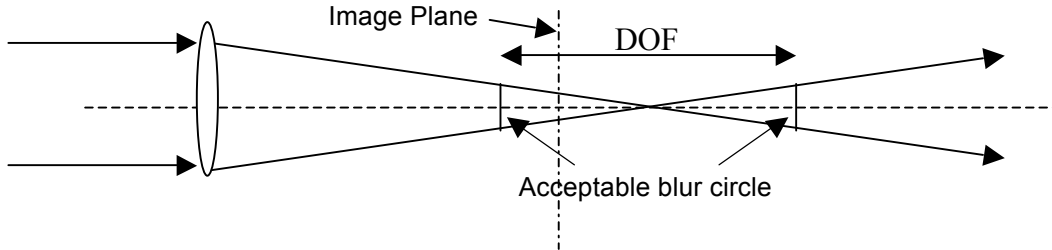

What constitutes an "acceptable blur circle"?
- Pixel size, for a digital detector. (Or even, 'Bayer-pattern' size, if the detector uses groups of red, green, and blue sensitive pixels).
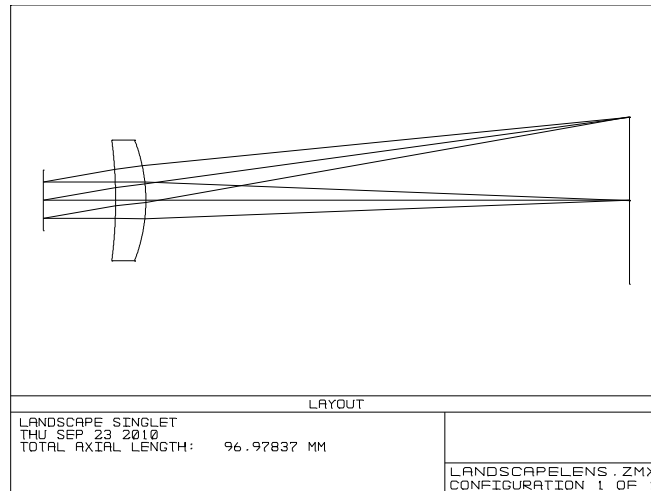
R	G
G	B

(What happens to the color information of an object point, if its PSF falls entirely on a single pixel of a color sensor using the above Bayer Pattern?)

- The size of the Airy Pattern, if the optics are 'diffraction limited'.
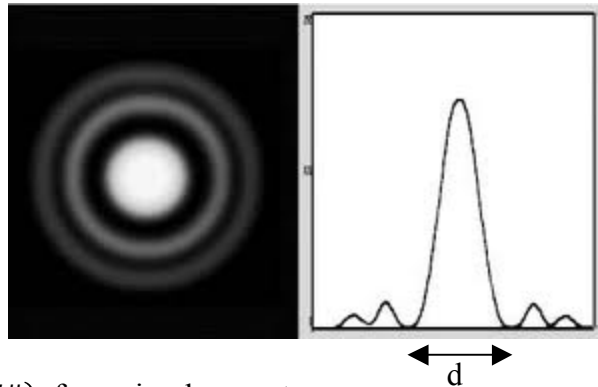
Where, $d = 1.22\lambda(F\#)$, for a circular aperture.

Example: The back landscape (Brownie box camera) lens.

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                              LAYOUT
LANDSCAPE SINGLET
THU SEP 23 2010
TOTAL AXIAL LENGTH:    96.97837 MM
                                    LANDSCAPELENS.ZMX
                                    CONFIGURATION 1 OF 1
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For this lens, n = 1.517, R1 = -80, R2 = -27.24, and f = 80. Using the formula for surface power, we get:

$$K_1 = \frac{n-1}{R_1} = -\frac{1}{154.7}$$

$$K_2 = \frac{1-n}{R_2} = \frac{1}{52.7}$$

and

$$P_{sum} = \frac{K_1}{n} + \frac{K_2}{n} = -4.26 \cdot 10^{-3} + 12.5 \cdot 10^{-3} = 8.24 \cdot 10^{-3}$$

So the Petzval sum is positive, indicating a negative image surface curvature (curved towards the front of the system). The lens works, because the aperture is kept small enough (high enough F/#) such that the pictures are acceptable for the film commonly used in these cameras.

Can we affect the Petzval Sum by redistributing the surface powers while maintaining the total focal length?

Note that, for a thin lens; $P_{sum} = \frac{K_1 + K_2}{n} = \frac{K}{n}$, so we can't affect the sum by 'bending' the lens. ('Bending is a term for re-distributing the surface powers of a thin lens while keeping the total power constant.)

For a thick lens, $K = K_1 + K_2 - K_1 K_2 d$ and $P_{sum} = \frac{K_1 + K_2}{n} = \frac{K + K_1 K_2 d}{n}$, so the Petzval sum can be reduced if $d > 0$ and $K_1 K_2 < 0$.

If Zemax is tasked with optimizing this lens, and no minimum blur circle is specified, it will try to make the lens extremely thick. This is an indication that the optical system doesn't have the ability to meet the merit function – one of them must be modified.