In the Catalogue of Clusters on the Franklin-Adams photographs of the sky (see *Memoirs R.A.S.*, **60**, part v.) this is described as a large open Cluster, 30′ diameter, but is erroneously designated M 14, following the N.G.C. (the N.G.C. error is corrected in *Memoirs R.A.S.*, **51**, 225).

Nizamiah Observatory, Hyderabad:
1918 *Aug.* 25.

The Five Aberrations of Lens-Systems.

By A. E. Conrady.

In 1856 Seidel published a classical paper (*Ast. Nach.*, **43**, Nos. 1027–1029) in which he extended the Gaussian theory of centred systems of spherical surfaces so as to include all aberrations of the third order, which had been treated as negligible by Gauss. As the result of this work Seidel proved that the images produced by such systems were subject to five, and only five, distinct forms of aberration, viz. ordinary spherical aberration, coma, astigmatism, curvature of the field, and distortion, of which the last four are peculiar to extra-axial points of the image.

Seidel's treatment of the problem was thorough and exhaustive, but decidedly lengthy and complicated. As a consequence the value of his work has only in recent times been widely recognised; but even now one hesitates to include it in the curriculum of ordinary students of practical optics.

Recently I suddenly realised that all the essential features of Seidel's theory, and in fact some additional deductions of considerable value which escaped him, can be derived in an extremely simple way from the well-known law of spherical aberration for points on the optical axis: that the lateral deviation of a marginal ray at the paraxial focus is proportional to the third power of the aperture.

In fig. 1 let AP represent the axial section of the first refracting surface of a centred system, C the centre of curvature of the surface, O the axial point of an object, and F its image, by paraxial rays; then the law of spherical aberration states that a ray starting from O at a finite but small angle with the optical axis will cut the paraxial focal plane at a distance $FG = k.y^3$, where k is a constant depending only on the radius of curvature and on the two conjugate distances, and that FG will lie in the plane containing the optical axis and the point of incidence of the ray. Now let us consider an extra-axial point O_1 of the object, and let us assume for the present that the latter lies on a sphere having its centre at C. If we draw the "auxiliary optical axis" O_1C which penetrates the refracting surface at A_1, we shall have precisely the same relations between O_1 and this auxiliary axis as between O and the principal optical axis; hence the focus of a thin "radial"

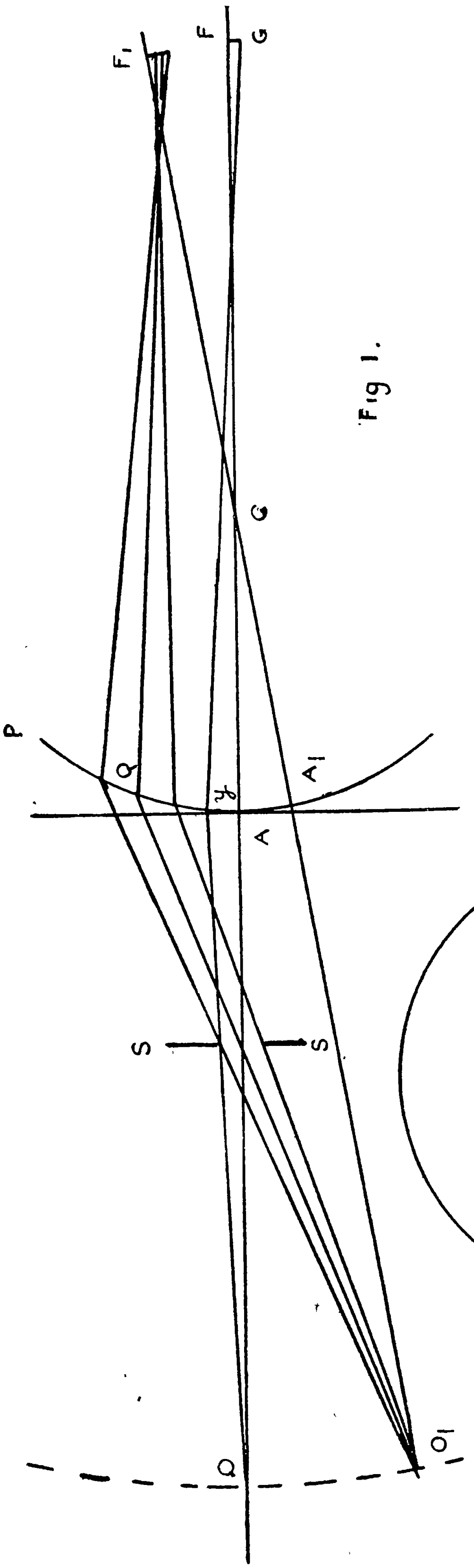

Fig 1.

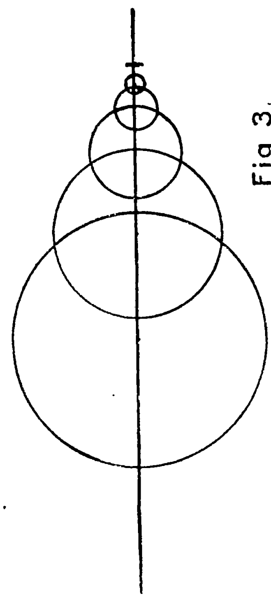

Fig 3.

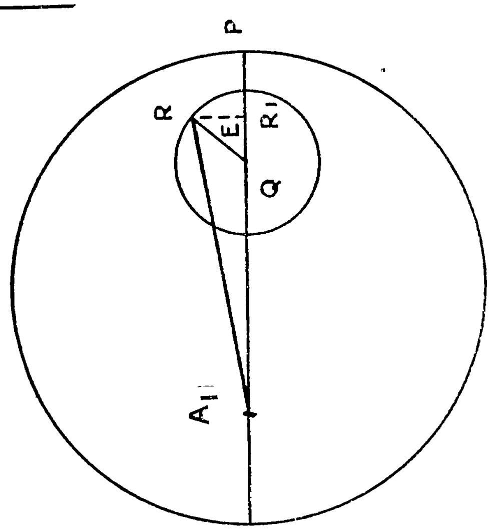

Fig 2.

pencil surrounding O_1C will be formed at a point F_1 of O_1C, produced so that $CF_1 = CF$, and we shall have the same aberration-constant k for rays at finite angles with O_1C as for those referred to OC.

Now let SS represent a stop or diaphragm concentric with the optical axis of the system which limits and regulates the pencils admitted to the system. This diaphragm will in general cause a more or less eccentric pencil to pass from O_1 to the refracting surface and thence towards the focus F_1. This oblique pencil will cut a plane touching the refracting surface at its axial pole A in a precise circle. Strictly speaking, its intersection with the refracting surface itself will be distorted, but under the limitations to small angles and curvatures, the amount of the distortion will be limited to small quantities of the third order and could only affect the aberrations of fifth and higher orders which we are neglecting; and the same consideration will apply to the projection of the intersection of our oblique pencil with the refracting surface upon a plane touching the latter at A_1. We now consider this latter projection of fig. 1, *i.e.* a view of it in the direction F_1C of our auxiliary optical axis (fig. 2). We then have A_1P as the trace of the plane containing the principal optical axis and the extra-axial object point, Q, as the point where the central or "principal" ray of our oblique pencil cuts the refracting surface, and a circle of radius y as the outline of the pencil. If we take any point R on the circumference of this circle, the ray passing through it will have a lateral spherical aberration in the plane of F_1 equal to $k.\overline{A_1R}^3$, and this will be in the reverse direction of A_1R if we assume spherical under-correction, which is usually taken as the positive sense of spherical aberration. We now resolve this displacement of our ray from its ideal coincidence with F_1 into components respectively parallel and perpendicular to the plane of the optical axis and easily find

$$\text{Horizontal component} = k\overline{A_1R}^3 \cdot \frac{A_1R'}{A_1R} = k\overline{A_1R}^2 . A_1R'$$

$$\text{Vertical component} = k\overline{A_1R}^3 \cdot \frac{RR'}{A_1R} = k\overline{A_1R}^2 . RR'$$

Introducing the symbols Y for the distance A_1Q and E for the angle PQR, we next have by inspection of our diagram

$$A_1R' = Y + y \cos E \qquad RR' = y \sin E$$

$$\overline{A_1R}^2 = (Y + y \cos E)^2 + y^2 \sin^2 E = Y^2 + 2Yy \cos E + y^2$$

and introducing these we find the components of the lateral aberration

$$\text{Horiz. comp.} = (Y^3 + 3Y^2y \cos E + Yy^2(1 + 2 \cos^2 E) + y^3 \cos E)k$$

$$\text{Vert. comp.} = (Y^2y \sin E + 2Yy^2 \sin E \cos E + y^3 \sin E)k$$

These equations display Seidel's five aberrations practically in the form in which he obtained them; but the last but one term of each is capable of a modification which renders its significance clearer. By adding $1-\cos^2 E-\sin^2 E=0$ to the coefficient $(1+2\cos^2 E)$ in the horizontal component, it becomes $2+\cos^2 E-\sin^2 E=2+\cos 2E$, whilst in the vertical component we can introduce $2\sin E\cos E=\sin 2E$; the equations then take the form

$$\text{Horiz. comp.}=(Y^3+3Y^2y\cos E+Yy^2(2+\cos 2E)+y^3\cos E)k$$
$$\text{Vert. comp.}=(Y^2y\sin E+Yy^2\sin 2E+y^3\sin E)k$$

But before discussing their significance we must remove the restriction to only one refracting surface and to the special position of the extra-axial object point on a sphere concentric with that surface.

If the object lies on a surface of any other finite curvature, or on a plane, the distance O_1A_1 will be altered by a small quantity of the second order; the distance A_1F_1 of its paraxial image will also be altered by a corresponding small quantity of the second order; but this will not sensibly alter the aberration-constant k for the surface, as it would only be affected by finite changes of the conjugate distances. No difficulty arises therefore from this generalisation.

The angle E which measures the location of a ray on the circumference of the pencil remains unchanged (within small quantities of the second order) from surface to surface for the reasons which were given with reference to the sensibly unaltered circular outline of the oblique pencil in different projections at small angles with each other.

The radius y of the oblique pencil will vary from surface to surface according to the separation between them and the convergence or divergence of the rays; but at each surface it will be in some fixed proportion to the radius S of the original stop SS (fig. 1); by putting $y_\gamma=c_\gamma\,.\,S$ we can therefore express all the y in terms of S.

For the second and subsequent surfaces the image produced by the first or preceding surface will take the place of the object, and the auxiliary optical axis defined by this object and the new centre of curvature may take all kinds of small inclinations to the principal optical axis, and the values of the Y will vary accordingly. But it is easily seen that so long as all the angles are small, the Y at any one surface will grow in exact proportion (within small quantities of the third order) with the angle V which the original extra-axial object-point subtends at the centre of the stop, *i.e.* with the semi-angle of the field of view. Hence we may put $Y_\gamma=d_\gamma\,.\,V$, where d_γ is another factor which is constant for any one surface, but varies widely for different surfaces.

We thus arrive at the general equations for the two components of the lateral aberration arising at the γ^{th} surface

$$\text{Horiz. comp.}=(d_\gamma{}^3V^3+3d_\gamma{}^2c_\gamma V^2S\cos E+d_\gamma c_\gamma{}^2VS^2(2+\cos 2E)$$
$$+c_\gamma{}^3S^3\cos E)k_\gamma$$

$$\text{Vert. comp.}=(d_\gamma{}^2c_\gamma V^2S\sin E+d_\gamma c_\gamma{}^2VS^2\sin 2E+c_\gamma{}^3S^3\sin E)k_\gamma.$$

It now only remains to determine how the total aberration existing at one surface combines with that arising at the next. Each surface produces a magnified or diminished image of the image presented to it by the previous surface, and this image is only subject to a distortion (represented, as we shall see, by the term in V^3), which is a small quantity of the third order. As the whole aberration-displacement of any ray is itself only a small quantity of the third order, the small distortion will be insensible in the image of so small an object and the displacement will therefore reappear in the new image simply magnified or diminished, but unchanged in its direction. Hence both components will retain their analytical form, however many surfaces there may be, and the final aberration may be written in the form

$$\text{Horiz. comp.} = m_1V^3 + 3m_2V^2S\cos E + m_3VS^2(2+\cos 2E) + m_4S^3\cos E$$
$$\text{Vert. comp.} = m_2V^2S\sin E + m_3VS^2\sin 2E + m_4S^3\sin E$$

in which m_1 m_2 m_3 and m_4 are constants which may have any positive or negative value, and which will be more and more independent of each other the greater the number of surfaces, primarily on account of the great variability of the value of the coefficient d_γ at successive surfaces.

The terms of our two final equations taken from right to left represent the five aberrations in the order adopted by Seidel. We will discuss them separately, each one as if the others did not exist or had been reduced to zero.

The two terms $m_4S^3\cos E$ and $m_4S^3\sin E$ when combined and applied to all values of E from 0 to 360°, represent a circle of radius m_4S^3 around the final image of the original object-point O, which would be produced by successive radial thin pencils. The marginal rays admitted by our stop SS would therefore be equally distributed around this circle, and indicate that they have a focus either shorter (if m_4 is positive) or longer (m_4 negative) than that produced by the radial pencils. This is ordinary spherical aberration, and the important conclusion (not usually emphasised in discussions of Seidel's analysis) is that these terms do not depend in any way on V, and therefore that the spherical aberration of any lens-system is, in first approximation, constant over the whole field. If it is corrected on the optical axis, it is also absent in the oblique pencils. This is of immense value in shortening the exact computation of lens-systems.

Coming to the terms in VS^2, we find the angle 2E in both components; therefore these components have the same value for E and for 180° + E, which means that a ray from any point on the circumference of the stop meets the one from the opposite point in our adopted focal surface.—For E = 0 and E = 180° the horizontal component (which will lie in the plane of the optical axis, because the vertical component is zero for these values of E) reaches its maximum value $= 3m_3VS^2$, and for the principal ray through the centre of the stop it reaches its minimum = zero, because S is then zero. This justifies the practical computer's practice of tracing these three rays in the plane of the optical axis by his simple formulæ and estimating the *Coma* by the distance of the intersection of the two marginal rays from the principal ray, for this is the maximum dimension of the Coma-figure. The latter is at once revealed when we discuss the two components for all possible values of E and S, for this shows that the rays for any particular diameter of the stop are distributed in pairs, round a circle of radius $m_3 VS^2$ with its centre at a distance $2m_3 VS^2$ from our reference-point. The totality of these circles corresponding to successive concentric zones of the diaphragm-opening produces the well-known comet-like shape (fig. 3).

In the discussion of the terms in V^2S we reap special advantages from our otherwise artificial reference of the phenomena to the final image which would result from successive images produced by thin radial pencils. It is well known (Seidel mentions it in a footnote) that this procedure leads to a final image which has the curvature defined by the Petzval-formula; hence the two astigmatic foci which are expressed in the terms in V^2S are referred to the Petzval-curvature as a reference-surface. Now the discussion of these terms in the manner already applied to the spherical aberration terms readily shows that in our reference-surface the rays from the circumference of the stop will be distributed over the circumference of an ellipse with a horizontal axis equal to three times its vertical axis, that the term in the horizontal component corresponds to the focal line perpendicular to the plane of the optical axis, and the term in the vertical component to the focal line in the plane of the optical axis, and that *the former always lies three times as far from the Petzval-curve as the latter*, but on the same side of it. This invariable relation between the Petzval-curvature and that of the two astigmatic focal surfaces seems to have escaped the attention of Seidel as well as of his successors in this branch of analytical optics, probably because all referred the phenomena to the paraxial focal *plane* and so obtained more complicated expressions for the astigmatic terms. This constant ratio is another fact of great value to practical computers, because the Petzval-curvature can be most easily computed by the well-known simple formula; and next in ease of computation comes that of a fan of rays in the plane of the optical axis which reveals the location of the vertical focal line or the "tangential focus." By the law of the three-to-one relation we can then, without any separate calculation, locate the horizontal focal line or "sagittal focus" as lying between the two others, at one-third of their separation from the Petzval-curve. As special applications of this law we can predict that if the field of a system is made flat for tangential rays (a favourite state of correction) the curvature of the field of the sagittal rays will be two-thirds of the Petzval-curvature and that the other special correction frequently found in photographic objectives, when the two astigmatic surfaces have equal but opposite curvature, must make the curvature of both

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numerically equal to half the Petzval-curvature. It also follows that if the astigmatism is corrected, the stigmatic image must necessarily have the Petzval-curvature. In the light of these results the Petzval theorem acquires a much wider significance than has ever been attributed to it.

The only remaining term of our equations is that in V^3, which occurs only in the horizontal component. It depends on the angle of the field only, and is thus present in undiminished magnitude, however small the diaphragm opening may be. It represents distortion, but it must be stated that with regard to this term the equation is not exhaustive, because it is based on the *curved* images produced by the radial pencils. This curvature of the intermediate images can easily be shown to lead to additional terms of the V^3 order, which would have to be included to secure a numerically correct value of the distortion.

It is, however, hardly worth while to develop these terms, for I am strongly of opinion that the proper use of approximate analytical solutions of optical problems consists in deriving general conclusions from them, and not in working out numerical results, for the latter rarely are sufficiently accurate to be of practical utility.

I should like to add that the most characteristic of the fifth order aberrations are displayed with the same clearness and facility as the third order ones if the lateral aberration is put $= k_1y^3 + k_2y^5$ and the resulting expression resolved into components exactly as for the third order aberration only. A complete discussion of the fifth order would, however, lead to great complications, because the various simplifying assumptions used in the paper are no longer admissible when the fifth order terms are included.

Imperial College,
1918 *Nov.* 7.

Differential Transit Observations. By W. E. Cooke, M.A.

At the Sydney Observatory we are just commencing transit observations of the "Intermediate Stars" in the zones 52°–65° South Declination; and in aiming at the greatest possible efficiency I have introduced the following features:—

Mechanical.

The scheme is arranged for two observers, one (A) recording transits, and the other (B) reading the circle. An effort has been made to afford all possible comfort, and to avoid the necessity for either observer to move from his position during the evening; also to eliminate all unnecessary observations. In other words, to secure the maximum output and accuracy with the minimum effort.

The Observing Couch.—A lies down on an upholstered couch, with or without a small pillow under his head, according to taste,