Introd. a f. di Green

Motivazione. Formula per la soluzione esplicita del prob. di Dirichlet in un dominio (limitato)

$$\begin{cases} -\Delta u = f & x \in U \\ u = g & x \in \partial U \end{cases}$$

assegnata

contrapposto al problema Poisson Neumann

$$\begin{cases} -\Delta u = f & x \in U \\ \dfrac{\partial u}{\partial \nu} = g & x \in \partial U \end{cases}$$

non sempre risolubile

Funzione di Green

Teor. della div. : $\displaystyle\int_U \nabla \vec{\Phi} = \int_{\partial U} \vec{\Phi}\cdot\vec{\nu}$

con $\vec{\Phi} = u\vec{\nabla} v$

$$\int_U \vec{\nabla} u\cdot\vec{\nabla} v + u\Delta v = \int_{\partial U} u(\vec{\nu}\cdot\vec{\nabla} v)$$

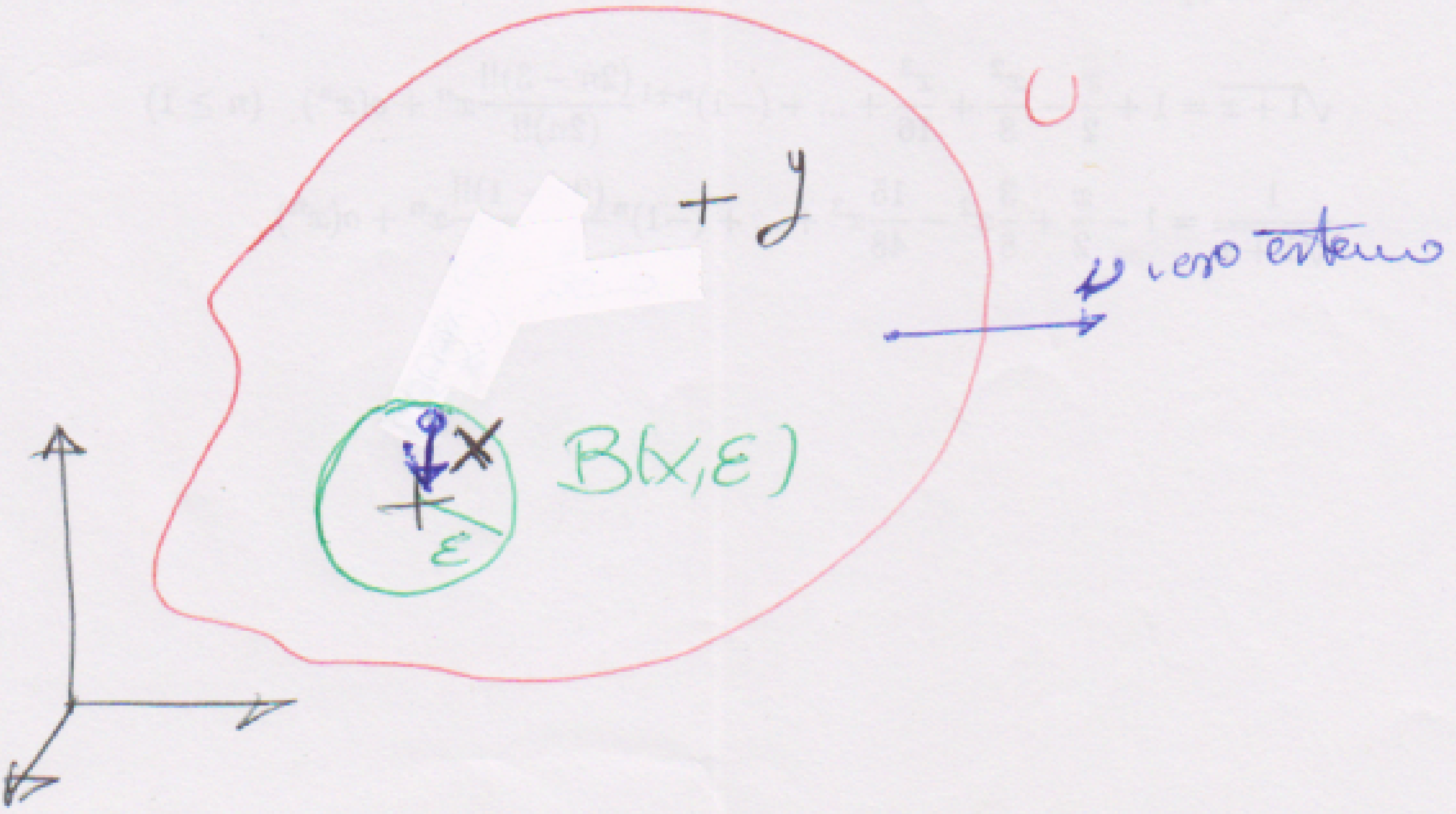
$$\vec{\nu}\cdot\vec{\nabla} v = \frac{\partial v}{\partial \vec{\nu}}$$

Scambiando $u \leftrightarrow v$ e sottraendo

$$\int_U (u\Delta v - v\Delta u)\, dV(y) = \int_{\partial U}\left(u\frac{\partial v}{\partial \vec{\nu}} - v\frac{\partial u}{\partial \vec{\nu}}\right) dS(y)$$

$u, v \in C^2(\bar{U})$

Considero la seguente situazione



Definisco $U^\varepsilon = U - B(x,\varepsilon)$

ed applico la relaz. precedente ad U^ε

con $v(y) = \phi(\bar{y}-\bar{x})$

↳ qui $\bar{x}$ è un vettore fissato

Ottengo

$$\int_{U^\varepsilon} \left(u(y)\, \Delta_y \phi(\bar{y}-\bar{x}) - \phi(\bar{y}-\bar{x})\, \Delta_y u(y) \right) dV(y)$$

$$= \int_{\underbrace{\partial U^\varepsilon}_{\partial U - \underbrace{\partial B(x,\varepsilon)}_{\overset{|||}{\partial B(x,\varepsilon)}}}} \left(u(y)\, \underbrace{\frac{\partial \phi}{\partial \nu}(\bar{y}-\bar{x})}_{\underset{(\bar{y}-\bar{x})}{\vec{\nabla}_y \phi \cdot \vec{\nu}}} - \phi(\bar{y}-\bar{x}) \frac{\partial u}{\partial \nu} \right) dS(y)$$

Utilizzo $\Delta_y \phi(\bar{y}-\bar{x}) = 0$ (ho escluso la singolarità)

$$-\int_{U-B(x,\varepsilon)} \phi(\bar{y}-\bar{x})\, \Delta_y u(y)\, dV(y) = \int_{\partial U} \left(u(y) \frac{\partial \phi}{\partial \nu}(x-y) \right) dS(y)$$

$$-\int_{\partial U} \phi(\bar{y}-\bar{x}) \frac{\partial u}{\partial \nu}\, dS(y) + \underbrace{\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \phi}{\partial \nu}(x-y)\, dS(y)}_{A} \underbrace{- \int_{\partial B(x,\varepsilon)} \phi(\bar{y}-x) \frac{\partial u}{\partial \nu}\, dS}_{B}$$

Stimo i termini A e B ed il loro comportamento quando $\varepsilon \to 0$

$$B \Rightarrow |B| \le \int_{\partial B(x,\varepsilon)} |\phi(\bar{y}-\bar{x})|\,|\vec{\nu}\cdot\nabla_y u(y)|\,dS(y)$$

considero $|\vec{\nu}\cdot\nabla_y u| \le \underset{\overset{\parallel}{1}}{|\nu|}\,|\nabla u|$

e $\|\nabla u\|_{y\in\partial B(x,\varepsilon)} < C$ in quanto ∂B è compatto e $u\in C^2 \Rightarrow \partial_{x_i} u$ hanno massimo in ogni compatto

$$|B| \le c \int_{\partial B(x,\varepsilon)} |\phi(y-x)|\,dS(y) =$$

$$= c \int_{\partial B(0,\varepsilon)} |\phi(y)|\,dS(y) = c\,|\partial B(0,\varepsilon)|\,\phi(\varepsilon)$$

$$|B| \le c\, n\,\alpha(n)\,\varepsilon^{n-1}\phi(\varepsilon) \begin{cases} \xrightarrow{n=2} c\,\varepsilon\,.\ln(\varepsilon) \xrightarrow{\varepsilon\to 0} 0 \\ \xrightarrow[n\ge 3]{} c\,\varepsilon^{n-1}\varepsilon^{2-n} \sim \varepsilon \end{cases}$$

Quindi $|B| \sim O(\varepsilon)$

Utilizzando $\underbrace{-\vec{\nu}\cdot\vec{\nabla}\phi\Big|_{\partial B(0,\varepsilon)}}_{\vec{\nu}^{\,i}\,\vec{\nabla}\phi\big|_{\partial B(0,\varepsilon)} \to \text{normale interna}} = \dfrac{1}{|\partial B(0,\varepsilon)|}$

valuto

$$A=\int_{\partial B(x,\varepsilon)} u(y)\,\frac{\partial\phi(x-y)}{\partial\vec{\nu}}\,dS = \frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)$$

$$= \fint_{\partial B(x,\varepsilon)} u(y)\,dS(y) \xrightarrow{\varepsilon\to 0} u(x)$$

Th Lebesgue media int.

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} u(y)\,\frac{\partial\phi(x-y)}{\partial\vec{\nu}}\,dS(y) = u(x)$$

Nel limite $\varepsilon\to 0$ ottengo la seguente espressione

$u\in C^2(U)\cap C(\bar{U}) \Rightarrow$

$$u(x) = -\int_U \phi(y-x)\,\Delta u(y)\,dV(y) + \int_{\partial U}\left(\phi(y-x)\frac{\partial u}{\partial\vec{\nu}}(y) - u(y)\frac{\partial\phi(y-x)}{\partial\nu}\right)dS(y)$$