# Differenziale di una applicazione.

Sia $F : N^n \longrightarrow M^m$ $C^\infty$

$\forall\, p \in N$ l'applicazione $F$ induce una applicazione **LINEARE**

$$F_* : T_pN \longrightarrow T_{F(p)}M$$

$$X_p \longmapsto F_*(X_p)_{F(p)}$$

differenziale di $F$ così definita:

sia $f \in C^\infty_{F(p)}(M)$

$$\boxed{F_*(X_p)(f) \overset{def}{=} X_p(f \circ F)}$$

esempio: consideriamo

$\mathbb{R}^n \quad (x^1, \ , x^n) \quad \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$

$\mathbb{R}^m \quad (y^1, \ , y^m) \quad \left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m}\right)$

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\underline{x} \longmapsto (F^1(\underline{x}), \dots, F^m(\underline{x}))$$

la matrice di $F_*$ in queste basi:

$f \in C^\infty(\mathbb{R}^m)$

$$F_*\left(\frac{\partial}{\partial x^i}_p\right)(f) \overset{\text{def.}}{=} \frac{\partial}{\partial x^i}(f \circ F)_p =$$

$$\sum_{j=1}^m \frac{\partial f}{\partial y^j}_{F(p)} \cdot \frac{\partial (F^j)}{\partial x^i}_p = \left(\sum_{j=1}^m \frac{\partial F^j}{\partial x^i}_p \cdot \frac{\partial}{\partial y^j}\right)(f)$$

$$J(F)_p = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}$$

matrice Jacobiana di $F$ in $p$

$m \times n$

$$\begin{array}{ccccc} F_* & : & T_pN & \longrightarrow & T_pM \\ & & \wr\! S & & \wr\! S \\ & & \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

fissate le basi

$$\begin{pmatrix} a_1 \\ \\ a_n \end{pmatrix} \longmapsto J(F) \begin{pmatrix} a_1 \\ \\ a_n \end{pmatrix}$$

Siano

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & M \arrow[r, "G"] & P \end{tikzcd}$$

$$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}$$

**Corollario**: se $F$ è un diffeomorfismo allora $F_{*p}$ è un isomorfismo $\forall\, p \in N$.

$$\begin{array}{ccccc} N & \xrightarrow{F} & M & \xrightarrow{F^{-1}} & N \\ p & \longmapsto & F(p) & \longmapsto & p \end{array}$$

$$F^{-1} \circ F = id_N$$

$$F^{-1}_{*F(p)} \circ F_{*p} = Id_p$$

analog. $F_{*p} \circ F^{-1}_{*F(p)} = Id_{F(p)}$

$\Bigg\}$ $F_{*p} : T_pN \to T_{F(p)}M$ è un isomorfismo.

Cambiamento di carte:

in $M$ varietà diff. prendiamo due carte:

$$(U, (x^1, \ ,x^n)) \quad e \quad (V, (y^1, \dots, y^n))$$

con $U \cap V \neq \emptyset$

$$x^i \overset{def}{=} r^i \circ \phi$$

$$y^i \overset{def}{=} r^i \circ \psi$$

sia $p \in U \cap V$, abbiamo visto che

$$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$$

$$\left\{ \frac{\partial}{\partial y^1}\Big|_p, \dots, \frac{\partial}{\partial y^n}\Big|_p \right\}$$

sono basi di $T_pM$. Qual è la relazione tra le due basi?

abbiamo

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right) \begin{bmatrix} a_{11} & & a_{1n} \\ & ? & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_{i=1}^{n} \underbrace{a_{ij}}_{?} \left.\frac{\partial}{\partial y^i}\right|_p$$

consideriamo la funzione

$$Y^k = r^k \circ \psi$$

① $$\left.\frac{\partial}{\partial x^j}\right|_p (Y^k) \overset{def}{=} \left.\frac{\partial}{\partial r^j}\right|_{\phi(p)} (Y^k \circ \phi^{-1}) =$$

$$\left.\frac{\partial}{\partial r^j}\right|_{\phi(p)} \Big(\underbrace{r^k \circ \psi \circ \phi^{-1}}_{(\psi \circ \phi^{-1})^k}\Big)$$

(2) $\displaystyle\sum_{i=1}^{n} a_{ij} \frac{\partial}{\partial y^i}\Big|_p (y^k) \overset{\text{def}}{=}$

$$= \sum_{i=1}^{n} a_{ij} \frac{\partial}{\partial r^i}\Big|_{\psi(p)} (\underbrace{r^k \circ \psi}_{y^k} \circ \psi^{-1}) =$$

$$= a_{kj}$$

$$\Rightarrow \quad a_{kj} = \frac{\partial}{\partial r^j}\Big|_{\phi(p)} (r^k \circ \underbrace{\psi \circ \phi^{-1}}_{(\psi \circ \phi^{-1})^k})$$

$$\Rightarrow \quad a_{kj} = \frac{\partial}{\partial r^j} (\psi \circ \phi^{-1})^k$$

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right) \underbrace{J(\psi \circ \phi^{-1})}_{\text{Jacobiana}}$$

**Prop** $\quad F : N \longrightarrow M$

$$F_{*|_p} : T_pN \longrightarrow T_{F(p)}M$$

$$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\} \qquad\qquad \left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m}\right\}$$

denotiamo $F^i = y^i \circ F$

rispetto a queste basi la matrice di $F_{*p}$ è la matrice $m \times n$

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{bmatrix}$$

Jacobiana di $F$

matematicamente è la Jacobiana di $\psi \circ F \circ \phi^{-1}$

# Teorema della funzione inversa

stessa dim

Sia $F : N^n \longrightarrow M^n$ $C^\infty$

e sia $p \in N$.

ossia $F_*$ isomorfismo

$\Updownarrow$

Se $\boxed{\det\left(\mathrm{Jac}(F)_p\right) \neq 0}$

allora esiste $U$ intorno di $p$ t.c. $F(U)$ è aperto in $M$ e

$$F : U \longrightarrow F(U)$$

è un **diffeomorfismo**

Quindi Jacobiana invertibile IN UN PUNTO determina comportamento in un INTORNO.

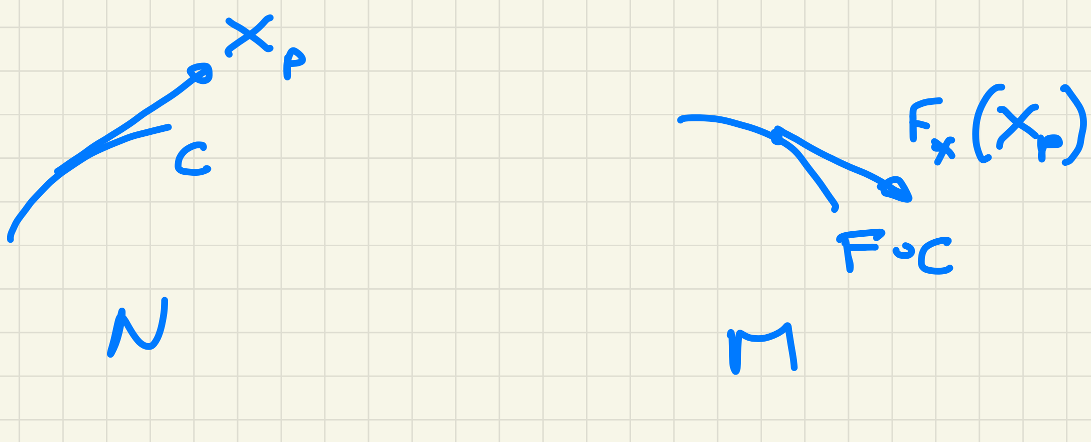
<u>OSS</u> :

abbiamo visto che, dato $p \in N$ $\forall$ $X_p \in T_pM$ $\exists$ curva $c : (-\varepsilon, \varepsilon) \to M$ t.c. $c(0) = p$ e $c'(0) = X_p$.

Sia $\gamma(t) = (F \circ c)(t)$.

Si osserva che

$$F_*(X_p) = \gamma'(0)$$



## Funzioni su $M$

**Def** Sia $f \in C^\infty(M)$

$\forall p \in M$ $f$ definisce una 1-forma, $df_p \in T_pM^*$

$$df_p(X_p) \overset{def}{=} X_p(f) \in \mathbb{R}$$

Notiamo che

$$f_{*|p} : T_pM \longrightarrow T_{f(p)}\mathbb{R} \simeq \mathbb{R}$$

fissata la base

$$\frac{\partial}{\partial t}\Big|_{f(p)}$$

Possiamo identificare

$$df \quad e \quad f_*$$

precisamente:

**Prop**

$$f_*(X_p) = df_p(X_p) \frac{\partial}{\partial t}_{f(p)}$$

**Dim**

?

$$f_*(X_p) = a \frac{\partial}{\partial t}\Big|_{f(p)}$$

applichiamo entrambi i membri alla funzione identità su $\mathbb{R}$

$$id(t) = t$$

$$f_*(X_p)(id) = X_p(id \circ f) = X_p(f)$$

$$a \frac{\partial}{\partial t}\Big|_{f(p)}(id) = a \frac{\partial}{\partial t}(t) = a$$

$$\Rightarrow \quad a = X_p(f)$$

## Espressione locale di df

Sia $(U, \phi) = (U, (x^1, \dots, x^n))$ una carta di $M$ in un intorno di $p$.

**OSS 1** $\{dx^1, \ , dx^n\}$ base

duale di $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$.

Infatti

$$dx^i|_p \left( \frac{\partial}{\partial x^j}\Big|_p \right) \overset{def}{=}$$

$$\frac{\partial}{\partial x^j}\Big|_p \left( x^i|_p \right) \overset{def}{=} \frac{\partial}{\partial r^j}\Big|_{\phi(p)} \left( \underbrace{r^i \circ \phi}_{x^i} \circ \phi^{-1} \right) =$$

$$= \delta_{ij}$$

Quindi $\{dx^1, \dots, dx^n\}$ base

di $T_p^* M$

## OSS 2

$f \in C^\infty(M)$

?

$$df = \sum_{i=1}^{n} a_i \, dx^i$$

$$df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j}(f)$$

$$\left(\sum_{i=1}^{n} a_i \, dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = a_j$$

$$\boxed{df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \, dx^i}$$