Physics of the spin-glass state

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The modern point of view on the physics of the spin-glass state is considered. The physical meaning of the phenomenon known as the "replica symmetry breaking" is discussed.

1. INTRODUCTION

The main idea of this review is to describe in simple terms that new area of physics which is known by the title "Spin-Glasses". After almoust twenty years of hard work this area of human knowledge has been developed to contain a lot of information about all kinds of quite remarkable phenomena, both experimental and theoretical. At present, a general qualitative picture of what is going on there, seems to be getting more or less clear. Unfortunately, this general understanding belongs only to a relatively small circle of specialists. Even for many physicists who were working on spin-glasses for years, the state of affairs in this field remains to be just a mass of thousands of contradictory experiments and dozens of doubtful theories none of which has anything to do with experimental realities. Even for many of those who know the main points of the modern understanding of the physics of spin-glasses, it remains to be just the result of some mysterious "magic" which is known as "replica symmetry breaking scheme" and which has no reliable justification in the "real" world of theoretical physics.

In this review I am not going to describe the history of the subject. I am not going to tell about all those brilliant papers which were filling physics journals during years and years and which eventually resulted in the modern understanding of the problem. For that reason the number of "historical" references will be minimal. Such kind of a topic is already described in numerous reviews and books, and those who are interested in that side of the story may refer to the book of Ref. 1 or to the review of Ref. 2. However, the history and the result of the history are not the same things, and it is mainly the resulting state of affairs which I propose to describe in the present review. On the other hand, it should be noted, that the physics of the spin-glass state which will be considered here is not the result of a theoretical derivation. It is rather the result of a "logical jump" from more or less strict theoretical and experimental facts to the thing which same to be called the "real world."

The point is that now the concept of the physics of the spin-glass state can be formulated, in a sense, apart from those artificial models for which it has been originally derived. It appears to look so natural and aesthetically attractive that it would be quite odd if there would be no such thing in Nature. Moreover, now it is obvious that the problem of spin-glass state has appeared to be much more general than that of the original studies of the low temperature properties of random magnets. It includes now the whole spectrum of problems from that of the optimizations in economy to biological systems. Therefore, it would be more probable that it is the absence of the spin-glass phenomena which might be an exception.

The review is structured in such a way that its different parts are directed to different readers.

The First part, consisting of three Chapters, is a sort of a detailed introduction combined with the conclusions. Almost without any sort of calculations, here it is explained what the problem is, and the modern understanding of the physics of the spin-glass state is formulated in general terms. Besides, in Chapter 4, recent experiments on real spin glasses are briefly described to demonstrate how this bit of abstract physics can be really *measured*. The reader who is mainly interested in the qualitative results only, and not in the process of their derivation, may restrict himself to this First part only.

In the Second part of the review the "magic" of the replica symmetry breaking scheme is demonstrated and the physics behind it is discussed. This part contains the detailed derivation of the physical picture of the spin-glass state for the model of a spin-glass with long-range spin-spin interactions. At the end of this part the physics behind the replica symmetry breaking is discussed again, but here it is done on a bit higher level than in the First part of the review. The Second part is directed to readers who are interested not only in the results, but also in the process of their derivation.

The Concluding remarks are written mainly not to let the reader, who has managed to read through the entire review, get the feeling as if everything is perfectly clear. At the present level of understanding, the problem of spin glasses is, of course, not solved. Presumably, the most adequate way to formulate the present state of affairs would be to say that now the problem is (hopefully) *correctly formulated.*

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PART 1

2. WHAT IS THE PROBLEM?

2.1. The model

There are many different statistical models of spin glasses. One of the simplest and, on the other hand, a rather general model is the one in terms of the classical Ising spins, described by the Hamiltonian:

$$H=-\frac{1}{2}\sum_{i\neq j}^{N} J_{ij}\sigma_i\sigma_j. \qquad (2.1)$$

This system consists of N Ising spins $\{\sigma_i\}$ $(i=1,2,...,N)$, taking values ± 1 which are placed at the vertices of some lattice, numbered by the index i. The spin-spin interactions J_{ij} *are random in their values and signs*. The model itself is defined by the choice of the distribution function $P(J_{ij})$ for these spin-spin interactions. The simplest and, on the other hand, a sufficiently realistic from the experimental point of view distribution function is the one with nonzero interactions for the nearest neighbors only and their probability distribution being Gaussian and independent for all pairs of the interacting spins:

$$P(J_{ij})=\frac{1}{\sqrt{2\pi J^2}}\exp\left\{-\frac{J_{ij}^2}{2J^2}\right\}. \qquad (2.2)$$

The parameter J is the characteristic value of the spin-spin interactions.

The motivation for the Hamiltonian (2.1) from the point of view of the description of the realistic spin-glass systems is well described in the review of Ref. 2. For the moment, however, the concrete choice of the model is not so important, since for now we are just going to realize on a qualitative level what the problem is in general.

2.2. Frustrations

The main problem is due to the fact that the random interactions J_{ij} are *quenched* i.e. for any concrete sample they are fixed. This results in the following phenomenon. Consider three arbitrary interacting spins (Fig. 1). Let us assume for simplicity, that all the interactions are equal in their values and different only in signs.

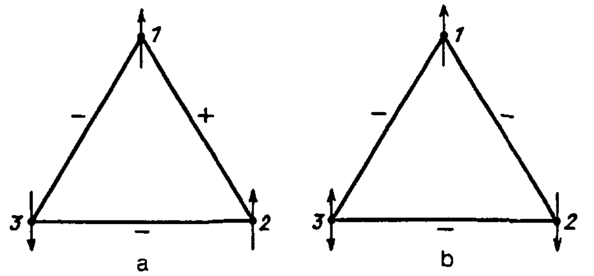

FIG. 1. The frustrations in a system of three spins. (a) The product of the interactions along the triangle is positive. In this case there is no frustration in the system, and the ground state is not degenerate. (b) The frustrated triangle: the product of the interactions along the triangle is negative, and the ground state of the system is degenerate.

Now, if the three interactions J_{12}, J_{23} and J_{13} have happened all to be positive, or two of them have happened to be negative while the third one is positive (these are the cases when the product of the interactions along the triangle is positive), then the ground state of such three spin system will be unique (except for the global change of signs of all the spins) (Fig. 1a).

However, if the product of the interactions along the triangle has happened to be negative (one of the interactions is negative, or all three interactions are negative), then the ground state of such a system will be degenerate. One may fix, e.g., the first spin to be "up," then if one goes along the triangle clockwise (in the case when all three interactions are negative, Fig. 1b) the second spin must be oriented "down," while the orientation of the third spin will happen to be undefined: according to $J_{23}<0$ it should be oriented "up," while according to $J_{13}<0$ it should be oriented "down."

One can easily check that a similar phenomenon takes place in any closed spin chain, consisting of an arbitrary number of spins, in which the product of the interactions along the chain is negative. This phenomemon is called *frustration*[1]

An important point is that not any disorder appears to be relevant for the thermodynamic properties of the system. It is the frustrations, which are that relevant part of the disorder, which essentially changes the behavior of the system in comparison with the corresponding one without disorder. In other words, if the disorder does not produce frustrations, then it is not relevant for the basic properties of the ground state of the system.

It might also happen that the disorder is just "fictitious," in a sense that it can be removed by a proper redefinition of the variables of the system. A trivial example is the Mattice magnet, which is also described by the Hamiltonian (2.1), where the spin-spin interactions are defined as follows: $J_{ij}=\xi_i\xi_j$, and the quenched ξ_i's are taking values ± 1 with equal probability. In such system the interactions J_{ij} are also random in signs, but after a simple redefinition of the spin variables $\sigma_i\to\sigma_i\xi_i$, an ordinary ferromagnetic Ising model will be recovered. One can easily check, that with such definition of the random interactions, there are no frustrations in the system. The frustrations are that part of the disorder, which is not removable by any transformation of the variables.

Now, in a multispin system one may draw a large number of all possible closed spin chains. In general, if the spin-spin interactions are random in sign, then one could expect that there could exist a lot of frustrations in the system. This, in turn, would result in a tremendous degeneracy of the ground state, or, at least, it would produce a large number of low-lying states with energies very close to the ground state. (In the Ising spin glass described by the Hamiltonian (2.1) with long-range interactions the total *number of such states seems to be of the order of* $\exp(\lambda N)$, where λ is some number smaller than log 2, while the total number of states in the system is equal to $2^N=\exp[(\log 2)N]$.)

2.3. Selfaveraging

Note again that all the thermodynamics we are talking about takes place for *quenched* random spin-spin interactions. Therefore, strictly speaking, all the results we may hope to get for the observable quantities for the given concrete system, must be expected to depend on the concrete interaction matrix J_{ij}, i.e. the result could be expected to contain some $N(N-1)/2$ random parameters. Apparently, results of such kind are more or less impossible to calculate, and moreover, they are useless. Intuitively it is clear, however, that the quantities which are usually called observables, should depend on some general averaged characteristics of the random interactions. This brings us to the concept of *selfaveraging*.

The traditional way of speculating why the selfaveraging phenomenon should be expected to take place, is the following. The free energy of the system is known to be proportional to the volume of the system, which in our case is N. Therefore, in the thermodynamic limit $N\to\infty$ the main contribution to the free energy in such a *macroscopic* system must come from the volume, and not from the boundary, which usually produces effects of the next order in the small parameter $1/N$.

Any macroscopic system could be divided again, into, a macroscopic number of macroscopic subsystems. Then the total free energy of the system would consist of the sum of the free energies of the subsystems, plus the contribution which comes from the interactions of the subsystems on their boundaries. If all the interactions in the system are short range (which occurs in any normal system), then the contributions from the mutual interactions of the subsystems are just the boundary effects which vanish in the thermodynamic limit. Therefore, the total free energy could be represented as the sum of the macroscopic number of terms. Each of these terms would be a random quenched quantity since it contains as parameters the elements of the random spin-spin interaction matrix. Next, in accordance with the law of large numbers, the sum of many random quantities can be represented as their average value, obtained from their statistical distribution, times their number (all this is true, of course, only under certain requirements on the characteristics of the statistical distribution in the limit $N\to\infty$). Therefore, the conclusion which comes out from such speculations is that the free energy of a macroscopic system must be selfaveraging over the realizations of the random interactions in accordance with their statistical distribution.

The free energy is known to be given by the logarithm of the partition function. So, it looks as if the only thing which should be performed for the calculation of the observable thermodynamics is to produce the averaging of the logarithm of the partition function over the given distribution of random J_{ij}'s simultaneously with the calculation of the partition function itself. It is quite clear that such a program is not easy, but nevertheless, on the level of this sort of speculation it looks as if this is just a technical problem (well, presumably a very difficult one), but no more than that.

2.4. The ergodicity breaking

The above arguments are highly instructive for two reasons. First, they show on a quite simple and qualitative level what an ordinary physics of disordered systems might be expected to look like. And second, because in general for spin-glasses they are not applicable. What makes spin-glasses to be such a special situation, will be discussed in the next Chapter, but now let us consider a few general points about statistical mechanics.

Everything would be quite simple if the free energy were an analytic function of the temperature (and the other parameters), or, in other words, if there would be no phase transitions due to spontaneous breaking of some kind of symmetry which exists in the system under consideration.

Here is a simple example of how clear and formally absolutely correct arguments lead to incorrect conclusions. Consider an ordinary ferromagnetic Ising model. It is described by the Hamiltonian (2.1) in which all the spin-spin couplings are equal to some positive constant. Since the Hamiltonian is invariant with respect to a global change of the signs of all the spins of the system, any thermodynamic quantity which is odd in spins must be identically equal to zero. In particular, this must also be true for the quantity which describes the general magnetization of the system. For any finite N these arguments are perfectly correct. Formally they remain correct also in the limit $N\to\infty$. However, what happens in this limit, is that the free energy becomes a non-analytic function of the temperature at some critical point T_c. As a result, below T_c spontaneous symmetry breaking is known to occur, and a nonzero average magnetization appears. The formal argument, showing why the above symmetry speculations appear to be not correct, is to say that in the limit $N\to\infty$ the partition function is formally divergent, and divergent quantities very often might produce quite unexpected things. In terms of physics, the phenomenon is qualitatively very simple: below T_c the space of all the states of the system becomes divided into two equal parts separated by an infinite (in the limit $N\to\infty$) barrier. For that reason in the observable thermodynamics only half of all the states contribute, and these are the states which are on one side from the barrier. And that is why in the observable thermodynamics the average magnetization is nonzero.

In the actual calculations of the thermodynamics for such systems, all the results are coming out quite automatically both above and below T_c, and usually it is done withought any reference to the semiphilosophical background described above. However, even in this simple situation it is useful to remember, that below T_c not the entire partition function is used in the calculations but only one half of it, and it is this one half, which gives the observable physics. This phenomenon is called the breakdown of ergodicity and it is the property of any phase transition.

The example of the ferromagnetic system is very simple because we are able to guess right away what kind of symmetry might be expected to be broken at low temperatures. We can easily guess that it is the symmetry with respect to a general change of the magnetization, and knowing that we are able to conclude which part of all the states of the system should be taken into account.

In spin glasses spontaneous symmetry breaking also takes place. But it is much more difficult to tell right away, which one. The main problem is that the symmetry which might be broken is directly connected to the quenched disorder in the system. Moreover, what actually happens in spin glasses is that spontaneous symmetry breaking takes place not just at a certain T_c, but it occurs at any temperature below T_c. In other words, below T_c a continuous sequence of phase transitions of symmetry breaking takes place, and correspondingly the free energy is nonanalytic at any temperature below T_c. And it is this phenomenon, which makes a spin-glass to be such a special thing.

Now, even at a qualitative level the situation in spin-glasses looks highly non-trivial. Let us imagine, that somehow we would be able to compute the total free energy, averaged over the disorder. Then, what we would get, most probably, would make no sense for observable physics. The problem is that, to obtain observable physics we have to calculate the partition function only within a certain part of the space of states in accordance with what kind of symmetry breaking takes place, while this concrete symmetry breaking, most probably, is directly connected with the concrete realization of the quenched disorder. On the other hand, the observable physics (at least to a certain extent) must be selfaveraging.

2.5. The possible scenarios

The only fact which we have learned up to now about spin glasses is that there could exist a lot of metastable states (local minima of energy), and that the ground state could be strongly degenerate.

Now, based on simple physical arguments let us try to anticipate what types of low temperature behavior could be expected to take place in such systems.

First. If the barriers separating local minima of the free energy remain finite at low temperatures, then the thermodynamic state of the system would correspond to a paramagnetic state, although the time relaxations could appear to be anomalously slow. In other words, the thermal averages of the magnetizations at each site could be expected to be zero: $\langle\sigma_i\rangle=0$ (the notation $\langle ... \rangle$ means the thermal average), and all the time correlation functions, such as $\langle\sigma_i(0)\sigma_i(t)\rangle$ could be expected to tend to zero (possibly, more slowly than exponentially) at $t\to\infty$ as well.

Second. It could also happen that there exists a certain spin state which would have its free energy much lower than all the other local minima, so that at low enough temperatures the system would "freeze" in this state. In more concrete terms it would mean that the system undergoes a phase transition, such that in the low temperature phase the thermal averages of spin magnetizations at each site $\langle\sigma_i\rangle$ would not be zero any more. In this "frozen" state the values of the magnetizations $\langle\sigma_i\rangle$ will fluctuate in their values and signs from site to site (since the state is disordered), so that the parameter, which describes the magnetization of the system: $m=1/N\Sigma_i\langle\sigma_i\rangle$ would be zero as in a paramagnetic state. However, the other order parameter (usually called the Edwards–Anderson order parameter):[4]

$$q=\frac{1}{N}\sum_i \langle\sigma_i\rangle^2 \tag{2.3}$$

would be nonzero in this case.

In this situation the thermodynamics of the system would essentially differ from that of both the paramagnetic and the ferromagnetic state. The time relaxations could also be anomalously slow, since in that region of the phase space where the system is frozen, numerous local minima of the free energy could exist, and correspondingly there could exist a whole spectrum of the energy barriers separating them.

Third. Something much more sophisticated. This is the case, when there could exist a *large number* of states, in which the system could get "frozen" at low temperatures.

The first case could take place when there are strong fluctuations in the system. This usually happens in low-dimensional systems (such as the two dimensional Ising model), or if the spin variables are continuous (as in the Heisenberg model), which results in the existence of the soft Goldstone modes, which could easily "melt" any "freezing."

It is more difficult to say when the second scenario might take place. However, it is also quite reasonable to expect that it is realizable in certain disordered statistical systems. All the thermodynamics which would be observed in this case is described in all details in the papers by Fisher and Huse.[5]

The above first two cases will not be considered here. The reason is not that it is something too simple, or that there is something basically wrong in them, or that they are hardly realizable in Nature. The reason is that, in a sense, *it is not interesting*: these two scenarios do not imply the existence of a new physics. They are not simple, but they are routine.

The second scenario, in principle, corresponds to a new type of phase transitions in magnetic systems. However, in most general terms it is just a highly complicated version of the ferromagnetic phase transition. In an ordinary ferromagnet the system is "freezing" in the ordered state with all the spins pointing, say, "up," while here it is "freezing" in some other state, which is defined by the quenched spin-spin interaction matrix. One could imagine that there exists some sort of a tricky transformation (depending on J_{ij}'s) of the spin variables, such that in terms of the new variables the "frozen" state becomes ordered. The other metastable states which could exist near this ground state could make the whole physics much more complicated in comparison with both the paramagnetic state and the ferromagnetic one. However, on a qualitative level it is not anything new. The point is that in all these situations the region of the space of states where the system is being localized is *unique*.

In what follows we are going to consider the third scenario only, and it is this case in which qualitatively new physics comes into play. The question whether it takes place in Nature or not, remains open, although recent experiments (which will be discussed in Chapter 4) indicate that presumably it is just this scenario, which Nature prefers to realize in "everyday" life.

3. PHYSICS OF THE SPIN-GLASS STATE

3.1. The continuous sequence of phase transitions

In an ordinary phase transition from the paramagnetic to the ferromagnetic state spontaneous symmetry breaking takes place, so that below T_c the system could be in one of the two states characterized by the order parameters: $\langle\sigma_i\rangle = +m$, or $\langle\sigma_i\rangle = -m$. The order parameter m goes to zero as $T\to T_c$. These two "valleys" in the space of states are separated by an infinite barrier, so that once trapped in one of the valleys below T_c the system could never go over to the other one. This phenomenon is also called ergodicity breaking. If the temperature is further decreased, no other symmetry breaking takes place in an ordinary ferromagnetic system.

In a spin-glass system there is also a certain critical temperature T_c, and above T_c ergodicity is not broken, so that the system is in the paramagnetic state. At T_c a phase transition of ergodicity breaking takes place, so that just below T_c the space of states is divided into many valleys (their number goes to infinity in the thermodynamic limit), separated by infinite barriers of free energy.

At some temperature $T=T_c-\delta T$ each valley is characterized by non-zero values of average spin magnetization at each site $\langle\sigma_i\rangle_{(\alpha)}$ (which, of course, fluctuate in sign and magnitude from site to site). Here $\langle...\rangle_\alpha$ denotes the thermal average inside the valley number α. Note that, as we have seen in the example of the ferromagnetic system, only such "restricted" thermal averages make physical sense if ergodicity is broken. The order parameter, which could describe the degree of "freezing" of the system inside the valleys could be defined as follows:

$$q=\frac{1}{N}\sum_i [\langle\sigma_i\rangle_{(\alpha)}]^2. \qquad (3.1)$$

What usually happens, is that the value of q turns out to be the same for all the valleys. As $T\to T_c$, $q\to 0$.

The most important point is that at any further decrease of temperature in all the valleys new phase transitions of ergodicity breaking occur, so that each valley is divided into many new smaller ones separated by infinite barriers of free energy (Fig. 2). The state of the system in any of these new valleys is again characterized by the order parameter (3.1), and its value grows as the temperature decreases.

This process of fragmentation of the space of states into smaller and smaller valleys goes on *continuously* with the temperature decreasing down to zero temperature. It means that at any temperature below T_c the system is in the state of a phase transition of ergodicity breaking.

However, this is not all. The other point is that at any temperature below T_c in each of the valleys there are also many metastable states separated by *finite* free energy barriers. There are barriers of any height, so that the spectrum of the values of the barriers goes continuously up to infinity. This results in the phenomenon that during any arbitrarily large but finite time real thermodynamic equilibrium inside the valleys is never achieved. On the other hand, experiment shows that there exist certain thermodynamic observables, such as specific heat, which behave as if thermodynamic equilibrium is achieved: they do not depend on time, and they are reproducible (which means that they are the same in all the valleys). On the other hand, other observables depend on time explicitly on any time scale that is accessible experimentally at present.

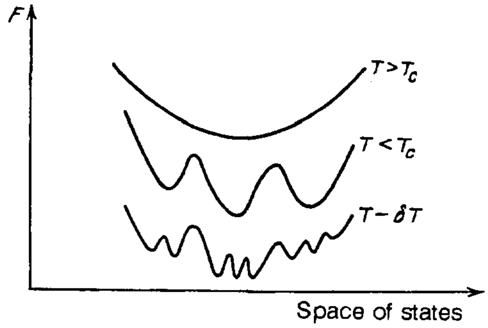

FIG. 2. The qualitative structure of the free energy landscape at different temperatures.

For that reason the possibility to use the traditional approaches of statistical mechanics for spin-glasses looks rather puzzling: as if it could be used in a sense, but only up to a certain extent, although it is not quite clear up to what extent.

3.2. The order parameter

Anyway, keeping in mind all the above reservations about the possibility to use traditional statistical mechanics for spin glasses, let us try to define the physical order parameter, which would reflect all that complicated structure of the space of states as fully as possible.

It is more or less clear that the order parameter (3.1) defined inside one valley only, does not contain any information about the other valleys, as well as it does not tell us anything, about what is, so to say, the topology of these valleys inside the phase space.

Consider the following series of imaginary experiments. At some given temperature below T_c starting from an arbitrarily disordered spin state, we let the system relax to thermal equilibrium. In each new experiment we start from a new initial random spin state. In the result of each such experiment we will obtain some "equilibrium" values for the average spin magnetization at each site $\langle\sigma_i\rangle_{(\alpha)}$, where α denotes the number of the experiment. Since it is expected that there are many valleys in phase space in which the system could get trapped, these site magnetizations could be found to be different in different experiments. "Equilibrium" is undersood here rather conditionally: it is simply assumed that site magnetizations are quantities that attain their equilibrated values relatively quickly.

Let us assume that we have performed an infinite number of such experiments. Then, we can introduce the quantity, which would describe to what extent the states which have been obtained in different experiments are close to each other:

$$q_{\alpha\beta}=\frac{1}{N}\sum_{i}^{N}\langle\sigma_i\rangle_{(\alpha)}\langle\sigma_i\rangle_{(\beta)}. \qquad (3.2)$$

Obviously $|q_{\alpha\beta}|\leqslant 1$, and the maximum value of $q_{\alpha\beta}$ is achieved when the states in experiments α and β are the same (in this case the overlap (3.2) coincides with that of (3.1), which has been introduced for one valley only). One can easily see, that the less correlated the two states are, the smaller is their overlap value (3.2). If the two states are not correlated at all, then their overlap (in the thermodynamic limit) is equal to zero. In this sense the overlaps $q_{\alpha\beta}$ define a sort of a metric in the space of states (the quantity which is inverse to $q_{\alpha\beta}$ could be called the "distance" in the space of states).

Now, to describe the statistics of all possible overlaps in the space of states one could introduce the probability distribution function:

$$P(q)=\sum_{\alpha\beta}\delta(q_{\alpha\beta}-q). \qquad (3.3)$$

It is this function $P(q)$ that could be considered as the physical order parameter, and it is in terms of the function $P(q)$ that the spin-glass phase looks essentially different from any other phase. Although in the procedure described above this function has been introduced in a rather speculative way, it will be shown in the second part of the review, that it could be defined as the normal thermodynamical quantity, and for a model with long range interactions it can be calculated explicitly.

Possible types of the functions $P(q)$ are shown in Fig. 3. In a paramagnet there is only a single thermodynamic state which is characterized by zero site magnetizations, and therefore the function $P(q)$ is the δ-function at $q=0$ (Fig. 3a). In the ferromagnetic state bellow T_c there are two states characterized by the site magnetizations $\pm m$, and therefore the function $P(q)$ is the two δ-functions at $q=+m^2$, and at $q=-m^2$ (Fig. 3b). Obviously in the case of, so to say, "fake" spin glass (scenario 2 of section 2.5) the function $P(q)$ looks the same as in the ferromagnetic state.

In the true spin-glass state the function $P(q)$ looks essentially different (Fig. 3c). Here, between the two δ-functions at $q=\pm q_{\max}(T)$ there is a continuous curve. The $q_{\max}(T)$ is the maximum possible overlap which is the "selfoverlap" (3.1). Since the number of the valleys in the system is macroscopic and (for some reasons) their selfoverlaps are all equal, the function $P(q)$ has two δ-functions at $q=\pm q_{\max}(T)$ (the symmetry of $P(q)$ is due to the fact that the Hamiltonian of the system is symmetric with respect to the global change of the signs of all the spins). The existence of the continuous curve on the interval $(0,\pm q_{\max}(T))$ is the result of the continuous process of fragmentation of the valleys into smaller and smaller ones. The *hierarchy* of the states which appears in this way just can not be non-correlated.

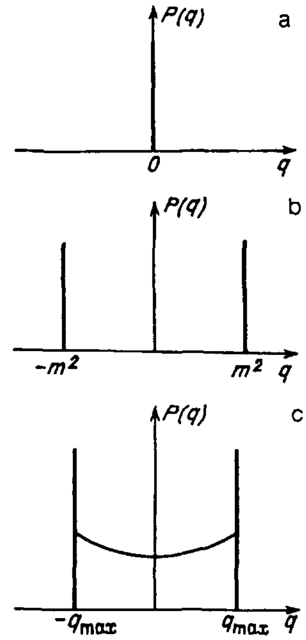

FIG. 3. The probability distribution function $P(q)$: (a) in the paramagnetic phase; (b) in the ferromagnetic phase; (c) in the spin-glass phase.

3.3. Ultrametricity

It is more or less clear that according to the qualitative picture of the spin-glass state described in section 3.1, a sort of a hierarchical structure of the spin-glass states could be expected to appear (Fig. 2): inside each valley there exist many smaller ones, inside the smaller valleys there exist many still smaller ones, and so on. It can be proved that all this rather sophisticated stuff could be described in terms of well-defined thermodynamical quantities.

In the previous section we have introduced the function $P(q)$, which gives the probability to find two spin-glass states which would have an overlap equal to q. Let us introduce now a somewhat more complicated probability distribution function $P(q_1,q_2,q_3)$, which would give the probability that three arbitrary spin-glass states would have their paired overlaps simultaneously equal to q_1, q_2 and q_3:

$$P(q_1,q_2,q_3)=\sum_{\alpha\beta\gamma}\delta(q_{\alpha\beta}-q_1)\delta(q_{\alpha\gamma}-q_2)\delta(q_{\beta\gamma}-q_3). \qquad (3.4)$$

For the model of a spin glass with long range interactions this function can also be calculated explicitly (this will be done in Part 2 of this review), and the result may look surprising at first. It can be shown that the function $P(q_1,q_2,q_3)$ is not equal to zero only if at least two of the three overlaps are equal and their value is not bigger than the third one. In other words, the function $P(q_1,q_2,q_3)$ is non-zero only in one of the three cases: $q_1=q_2<q_3$, or

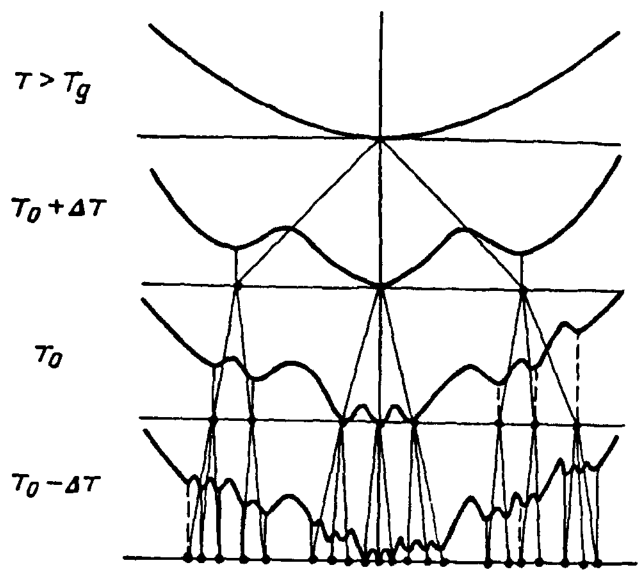

FIG. 4. The hierarchical tree of the spin-glass states.

$q_1 = q_3 \leqslant q_2$, or $q_3 = q_2 \leqslant q_1$. In all other cases $P(q_1, q_2, q_3)$ is identically equal to zero. It means that in the space of spin-glass states there exist no triangles with all three sides different. The spaces having such a metric property are called ultrametric. In mathematics ultrametric structures have been already known since the end of the last century, and they came into physics only recently due to spin glasses. Ultrametricity from the point of view of physics is described in all detail in the review of Ref. 6.

The most simple illustration of ultrametric structures can be made in terms of a hierarchical tree (Fig. 4). The space of the spin-glass states is identified with the set of endpoints of the tree. The metric in this space is defined in such a way, that the overlap (the distance) between any two states depends only on the number of generations to their closest "ancestor" on the tree (as the number of generations increases, the value of the overlap decreases). One can easily check (just playing with any choice of arbitrary three points in the set), that the space defined in such a way is ultrametric.

Actually, such a tree of states is not just an abstract auxiliary illustration. It really corresponds to the hierarchical fragmentation of the space of states into valleys, as has been described in section 2.5 (Fig. 2). For the vertical axis in Fig. 4 the value of q should be assigned, and then for any given temperature $T < T_c$ the set of all the spin-glass states which exist at this temperature would be obtained at the crossection of the tree at the level $q = q_{max}(T)$. The horizontal direction in this picture is the infinite-dimensional space of states.

As the temperature is decreased to a new $T' < T$, each of the states, which live on the level $q_{max}(T)$ gives birth to numerous "descendants," which are the endpoints of the tree at a new level $q_{max}(T') > q_{max}(T)$. As $T \to 0$, $q_{max}(T) \to 1$, and this is the "lowest" (most detailed) level of the tree at which one finds all possible states which could exist in the spin-glass.

Correspondingly, as the temperature increases to some value $T'' > T$, all the states having common ancestors at the level $q_{max}(T'') < q_{max}(T)$ merge together into these ancestors. As $T \to T_c$, $q_{max}(T) \to 0$, and this is the level of the (paramagnetic) "grand-ancestor" of all the spin-glass states.

Since the function $q_{max}(T)$ is determined by the temperature, it means that it is the temperature which uniquely determines that level of the tree at which the "horizontal" crossection should be made, and which, in turn, defines all the spin-glass states existing at this temperature. Everything which is below this level is "invisible," and everything which is above this level is the "evolution history" of the spin-glass states. Therefore it is the temperature that determines the scale in the space of the spin-glass states, in the sense that all the states which have overlaps bigger than $q_{max}(T)$ are nondistinguishable. In this sense one could also say that there exists a special sort of scaling in the low-temperature spin-glass phase: changing the temperature one just changes the scale in the space of states.

Although one could assume that such kind of structure of the space of states is just a very special property of the very artificial model of the spin glass (which is really very far from the experimental realities), in fact rather general arguments could be given which indicate that it is the absence of such kind of ultrametric structures that might be something special in disordered frustrated systems.

Consider a disordered system which just by its construction due to numerous frustrations contains a macroscopic number of local minima of the free energy. Let us assume then (and this is the crucial assumption) that the local minima states are sufficiently "strong" to collapse into closed valleys which would factorize the space of states into many pieces at low temperatures. Then, the most natural way of such fragmentation of the phase space would be that which has been described in section 3.1: first the space is separated into most uncorrelated (distant) parts, then, as the temperature decreases, these valleys are separated into smaller and slightly correlated ones, and so on. In fact this is nothing else, but the random branching process which goes on in infinite-dimensional space (the space of the states in the thermodynamic limit $N \to \infty$ becomes infinite-dimensional). The ultrametricity in this situation appears automatically whatever the concrete properties of the random branching process are. The point is that in any random branching process in the infinite-dimensional space, any two branches once separated never come close again. Therefore, the ultrametric hierarchy of states described above could be a rather general property of random systems (note that even disordered social systems are well known always to form strict hierarchies).

The results of recent experiments that have been made on real spin-glass materials also indicate in favor of this assumption.

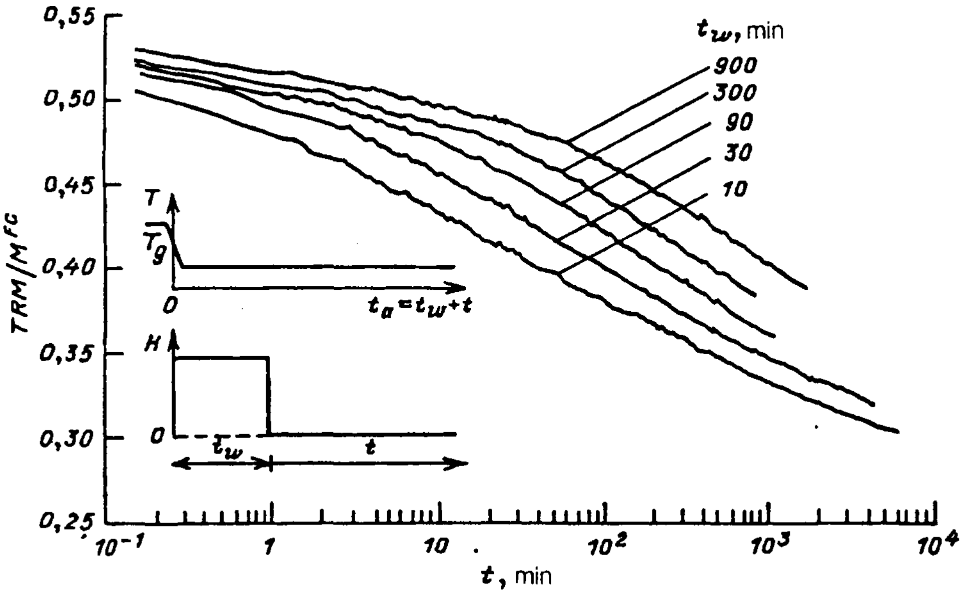

FIG. 5. The relaxation behavior of the magnetization after the external magnetic field is switched off in the aging experiments in nonzero magnetic field.

4. EXPERIMENTS

In this Chapter we will consider recent experiments which have been done on *real* spin-glasses. The idea of these experiments was to check to what extent the qualitative picture of the spin-glass state described above, is valid in the real world. The main problem, as usual, is that the concepts and quantities which are very convenient in the theoretical speculations are very far from the experimental realities, and it is a matter of true experimental art to construct convincing experimental procedures which would be able to confirm (or reject) a theoretical scheme.

A series of such brilliant experiments has been performed by M. Ocio, J. Hammann, F. Lefloch and E. Vincent (Saclay), and M. Lederman and R. Orbach (UCLA).[7]

Most of these experiments have been done on crystals of $CdCr_{1.7}In_{0.3}S_4$. The magnetic disorder there is due to the competition of the ferromagnetic nearest neighbor interactions and the antiferromagnetic higher order neighbor interactions. This spin-glass has already been systematically studied some time ago,[8] and its spin-glass phase transition point $T = 16.7$ K is well established. Some of the measurements have also been performed on the metallic spin glasses of the type[9] AgMn and the results obtained were qualitatively the same. It indicates that presumably the qualitative physical phenomena observed, do not depend very much on the concrete realization of the spin-glass system.

4.1. Aging

The phenomenon of *aging* in spin glasses is already known since some years ago.[10] It is not directly connected with the hierarchy of the spin-glass states, but it explicitly demonstrates the absence of true thermodynamic equilibrium in spin glasses.

The procedure of the experiments is the following. The sample is cooled down into the spin-glass state in the presence of a weak uniform magnetic field h. Then at a constant temperature $T < T_c$ it is kept in the same magnetic field during some time t_w. Then the magnetic field is switched off, and measurements of the relaxation of the thermoremanent magnetization (TRM) is performed. The results of these measurements for different values of t_w is shown in Fig. 5 (note, that the values of t_w are quite macroscopic: they are minutes, hours, days).

The first thing these plots show is that the observed relaxation is slow and non-exponential (that is why the results are shown on a logarithmic scale). More important, however, is that the relaxation appears to be a non-steady-state one: the processes which take place in the system after switching off the field depend on the time t_w when this occurred. The spin-glass becomes stiffer with time: the bigger is t_w, the slower is the relaxation. Therefore, any experiment of this kind depends on two time scales: the observation time t, and the time which has passed after the system came into the spin-glass state, the "aging" time t_w. It is also important to note that at all experimentally accessible time scales there are no signs of reaching thermal equilibrium, which would take place if the relaxation curves would approach a certain limiting curve corresponding to $t_w = \infty$.

Note also that it is not the magnetic field itself, which might be thought to be responsible for the observed phenomnenon. The magnetic field here is just the instrument which makes it possible to observe the phenomenon. One could also perform the "mirror" experiment: the system is cooled down into the spin-glass state in zero magnetic field, then it is kept at a constant temperature $T < T_c$ during some time t_w and after that the magnetic field is switched on and the relaxation of the magnetization is measured. Again, the results of the measurements essentially depend on t_w. Moreover, for any value of t_w the curves obtained in these two kinds of experiments turn out to be symmetric: if one plots the value of the sum of the magnetic moments

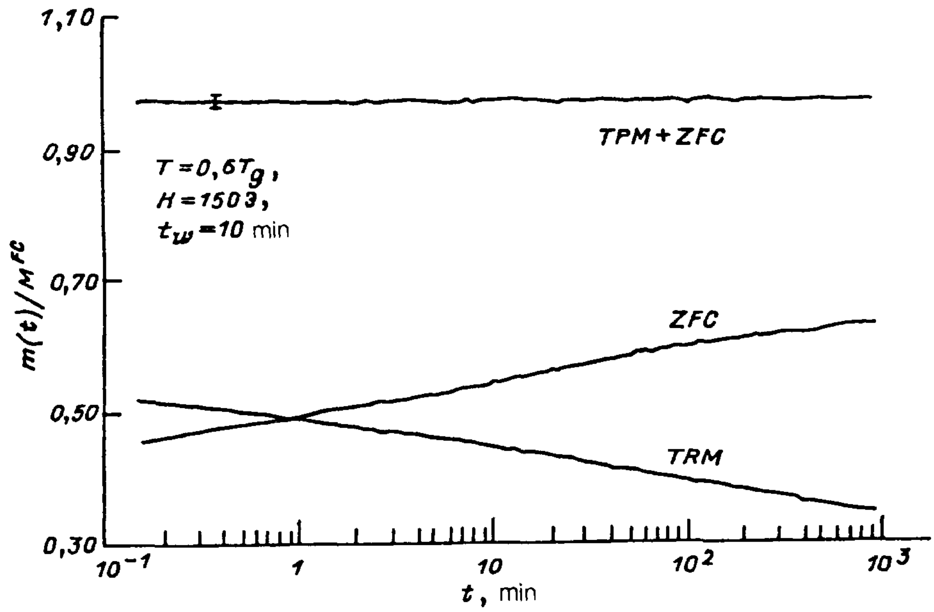

FIG. 6. The relaxation behavior of the magnetization after the external magnetic field is switched on in the aging experiments in zero magnetic field.

obtained in these "mirror" experiments as a function of time, one finds that it is a time-independent constant (Fig. 6).

4.2. Temperature cycles and the hierarchy of states

Now we consider two types of experiments which were specially designed to observe the effects which might appear due to the existence of a hierarchical tree of the spin-glass states and a continuous hierarchy of phase transitions in the low temperature phase.

In the experiments of the first type, the sample is cooled down in a weak magnetic field into the spin-glass phase, and then it is kept at constant temperature $T<T_c$ during some time t_{w_1}. After that the temperature is changed down to $T-\Delta T$ (where the value of ΔT is small), and the sample is kept at this temperature during some time t_{w_3}. Then the temperature is changed up to the original value T again, and the sample is kept at this constant temperature during some time t_{w_2}. After that the magnetic field is switched off and the relaxation of the magnetization is measured. The results for different values of ΔT are shown in Fig. 7.

The main result of these measurements is the following. It is clear from the plots of Fig. 7 that if the value of the temperature step ΔT is not too small, then all the relaxation curves obtained appear to be identical to those in the ordinary aging experiments (Sec. 4.1) with the waiting time $t_w = t_{w_1} + t_{w_2}$. This means that as regards the process of equilibration at temperature T, the system has effectively appeared to be completely frozen during the

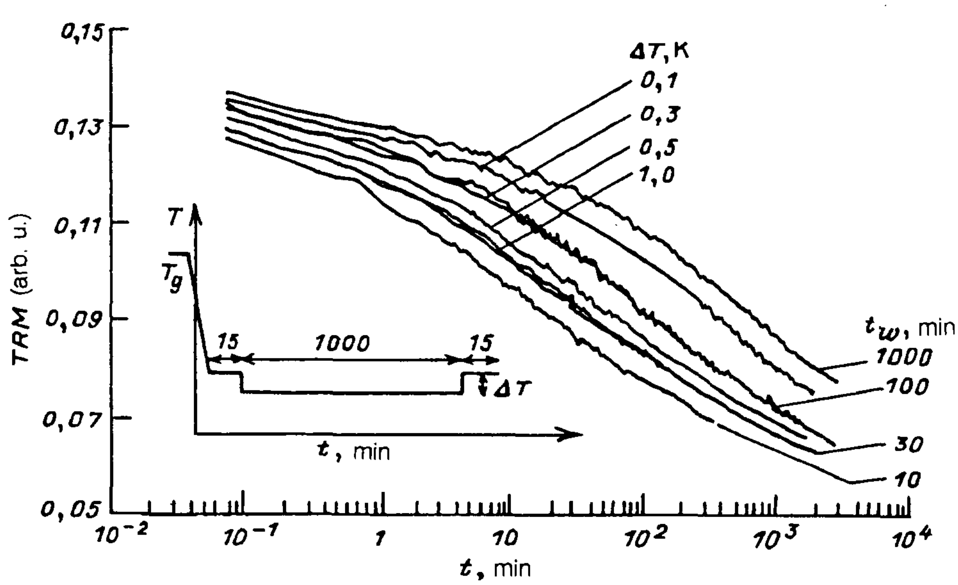

FIG. 7. The relaxation behavior of the magnetization in the aging experiments with the cooling temperature cycles.

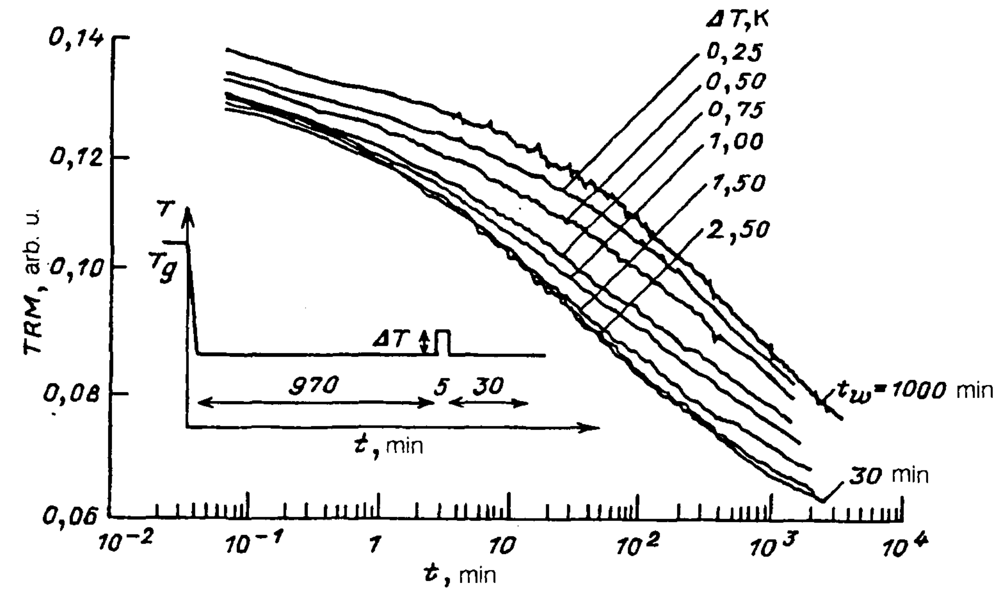

FIG. 8. The relaxation behavior of the magnetization in the aging experiments with the heating temperature cycles.

entire time t_{w_3} when it was kept at the temperature $T-\Delta T$.

In the experiments of the second type, again, the sample is cooled down into the spin-glass phase in the presence of a weak magnetic field, and then it is kept at a constant temperature $T<T_c$ during some time t_{w_1}. After that, for a relatively short time interval the sample is heated up to $T+\Delta T$ (where the value of ΔT is small), then it is cooled down again to the temperature T and it is kept at this constant temperature during some time t_{w_2}. After that, the magnetic field is switched off and the relaxation of the magnetization is measured. The results for different values of ΔT are shown in Fig. 8.

What happens in this case is that if the value of the temperature step ΔT is not too small, then all the relaxation curves obtained turn out to be *identical* to those in the ordinary aging experiments (Sec. 4.1) with the waiting time $t_w = t_{w_3}$. This means that even slight heating is enough to wipe out all the aging which has been "achieved" at the temperature T during all the preceding time, and to start the aging process all over again. (Note that the temperature $T+\Delta T$ is still essentially below T_c.)

Such a quite asymmetric response of the system with respect to the considered temperature cycles of cooling and heating fits well into the qualitative physical picture of a continuous hierarchy of phase transitions and the tree of the spin-glass states.

The qualitative interpretation of the results described above is the following. The process of aging which is assumed to be a very slow drift down to the true thermal equilibrium, is imagined as a process of jumping over higher and higher energy barriers as time goes on. After the waiting time t_w the system covers a certain part of phase space, which could be characterized by the maximum energy barriers of the order of $\Delta_{max} \simeq T\log(t_w/\tau)$ (here τ is some characteristic microscopic time). It is assumed that to any scale in the space of states there corresponds a certain characteristic scale of the energy barriers.

The results of the experiments with the temperature cycles of cooling could be interpreted as follows. During the time period t_{w_1} when the system has been kept at the temperature T, it covers a certain restricted part of the phase space inside one of the valleys existing at this temperature. After cooling down to the temperature $T-\Delta T$ this part of the phase space is divided into several smaller valleys separated by infinite barriers. Correspondingly, each of the metastable states inside the valley, which were separated by finite barriers, are divided into many new ones. The barriers separating the states become higher, and some of them become infinite (that is why the valley is divided into many smaller valleys). Then, during the time t_{w_3} the system begins to occupy these new states remaining locked in by infinite barriers in some part of the phase space which is smaller than it was before, at the temperature T. Therefore, whatever time has passed at the temperature $T-\Delta T$ the system could occupy only those states, which are descendants of the states already occupied at the temperature T, and not more. Note that all these effects are just a direct consequence of the *phase transition* which occurred in the system due to cooling from the temperature T to the temperature $T-\Delta T$. Then, after heating back to the temperature T all these descendant states merge together into their ancestors, and all the aging which has been achieved at the temperature $T-\Delta T$ is wiped out. After that, the process of aging at the temperature T would continue again, as if there were no time interval which the system spent at the temperature $T-\Delta T$.

In experiments with the temperature cycles of heating the effects to be expected are different. The states occupied by the system during the time t_{w_1} at the temperature T, after heating to the temperature $T+\Delta T$ would merge together into a much smaller number of their ancestor states. If ΔT is taken to be such that $q(T+\Delta T)<q'$, where $q(T)$

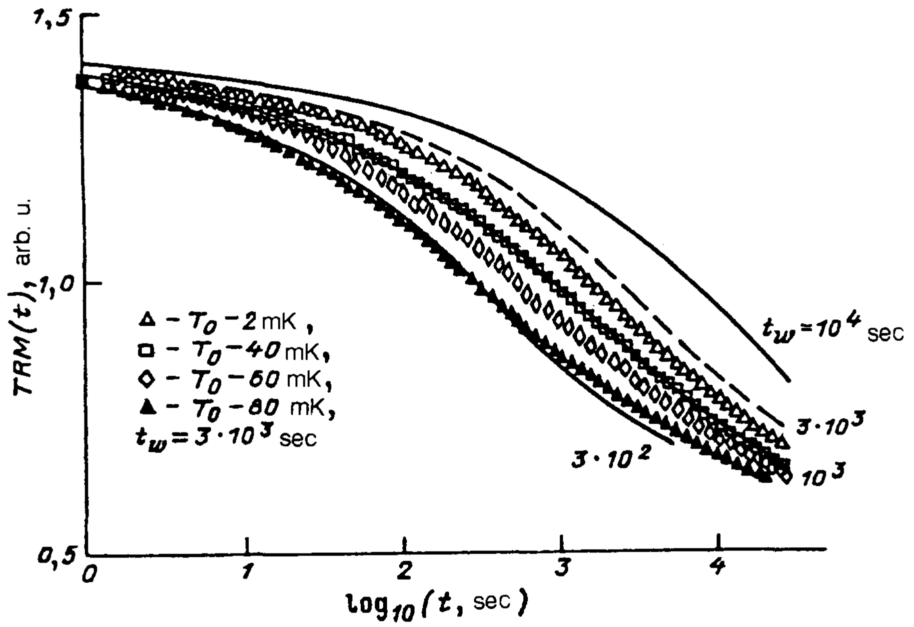

FIG. 9. The relaxation behavior of the magnetization at temperature T after aging at temperature $T-\Delta T$.

is the selfoverlap of the states at the temperature T and q' is the selfoverlap of the common ancestor of the states occupied during time interval t_{w_1}, then after the heating all the occupied states would merge into their common ancestor state. Within this limited part of the phase space it would effectively look as if a paramagnetic phase transition occurred. Therefore, all the aging which has been achieved at the temperature T would be wiped out, and after cooling back to the temperature T the process of aging would start again from the very beginning.

In simple terms the results of the considered experiments could be summarized as follows. If the spin-glass system is aging at some tempetature $T < T_c$, then any temporary heating would eliminate all the aging achieved, while any temporary cooling for any time period, just postpones the aging processes at this temperature.

4.3. The temperature dependence of the energy barriers

The scheme of the above experiments could be slightly changed so that it would make it possible to estimate the temperature dependence of the (finite) free energy barrier heights.

The experiments have been done on the metallic spin glasses AgMn ($T_c = 10.4$ K). The scheme of the experiments is the following. First, the spin glass ages in a weak magnetic field during the waiting time t_w at some temperature $T-\Delta T$. Then the sample is quickly heated to the temperature T, and simultaneously the magnetic field is switched off. After that, measurements of the relaxation of the magnetization are done.

The results are shown in Fig. 9. These plots clearly show, that if the value of ΔT is not too small, the relaxation curves obtained are practically identical with those in the ordinary aging experiments (Sec. 4.1) at the same temperature T but with some other waiting time $t_w^{\text{eff}} < t_w$.

This phenomenon is also well explained in terms of the hierarchical structure of the spin-glass states if we accept the idea that the barrier heights themselves essentially depend on the temperature. The free energy barriers at the temperature $T-\Delta T$ must be higher than those at the temperature T. In other words, the region of phase space occupied by the system at the temperature $T-\Delta T$ is bounded by barriers which are lower at the temperature T. Correspondingly, the time needed to occupy this part of the phase space at the temperature T is smaller than that at the temperature $T-\Delta T$.

What is most important, is the fact that the relaxation curves obtained during long observation times become practically identical. Note, that at the moment when the measurements are starting, the value of the temperature and the magnetic field in both cases are the same. If the value of t_w^{eff} is chosen correctly, then the relaxation curves also become the same. It means that the region of phase space occupied by the system by the time of the beginning of the measurements in both cases must be the same.

This region is characterized by the maximum barriers overcome by the system during aging at the temperature T:

$$\Delta(T, t_w^{\text{eff}}) = T \log\left(\frac{t_w^{\text{eff}}}{\tau}\right) \quad (4.1)$$

and correspondingly, during aging at the temperature $T-\Delta T$:

$$\Delta(T-\Delta t, t_w) = (T-\Delta T) \log\left(\frac{t_w}{\tau}\right). \quad (4.2)$$

Since the relaxation processes both after aging at T during the time interval t_w and after aging at $T-\Delta T$ during the time interval t_w^{eff} are the same, the initial state of the system must also be the same. Therefore, one can conclude that $\Delta(T-\Delta T)$ and $\Delta(T)$ are the heights of *the same* barrier at different temperatures.

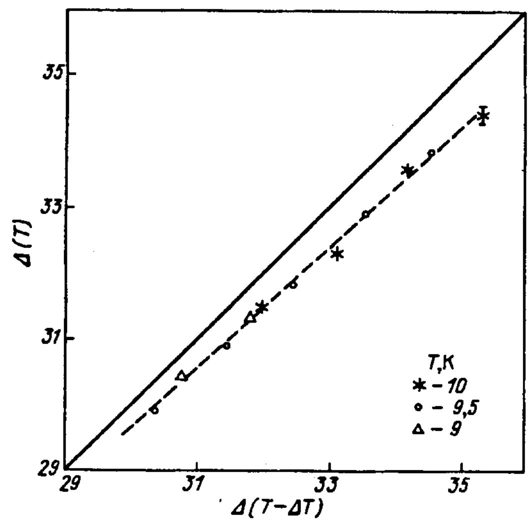

FIG. 10. The dependence (obtained from the data of Fig. 9) of the values of the free energy barriers at temperature T on their values at temperature $T-\Delta T$.

On the basis of this conclusion and using the experimental plots of Fig. 9, one can get the dependence of the value of $\partial\Delta/\partial T$ on Δ at the given temperature. In Fig. 10 the dependence of $\Delta(T-\Delta T)$ on $\Delta(T)$ is shown for $T=9$ K, 9.5 K and 10 K and a fixed value $\Delta T=20$ mK. These plots demonstrate that within experimental error the dependences obtained at different T coincide. In Fig. 11 the corresponding dependence of the value of $\partial\Delta/\partial T$ on Δ is shown. Within experimental error $\partial\Delta/\partial T$ depends only on the value of Δ and does not depend directly on the temperature. The dashed line in Fig. 11 is the power law approximation to the experimental data:

$$\frac{d\Delta}{dT}\simeq a\Delta^6; \quad a=2.9\times 10^{-7}. \tag{4.3}$$

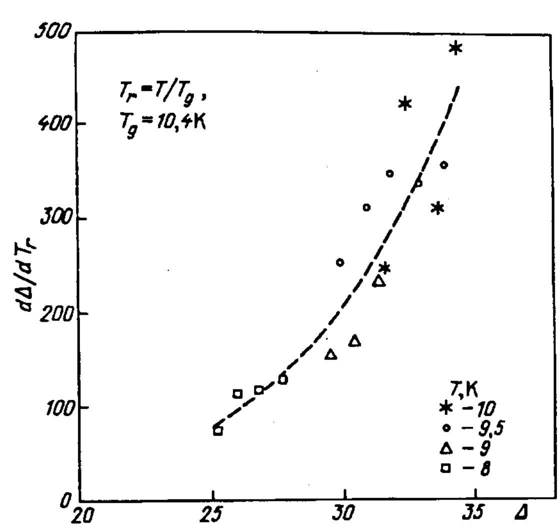

FIG. 11. The dependence of the growth rate of the barriers, $d\Delta/dT$, on the values of these barriers Δ.

Integrating this equation, one gets:

$$\Delta(T)\simeq\left[\frac{T-T^*}{T_c}\right]^{-1/5}; \quad T>T^*. \tag{4.4}$$

The temperature T^* is the integration constant, which, in fact, labels the barrier under consideration. Each barrier can be characterized by the limiting temperature T^* at which this particular barrier becomes infinite.

In conclusion, the experiments considered in this Chapter clearly demonstrate the existence of the aging phenomena in the spin-glass phase over the whole range of temperatures below T_c and at all observable time scales. The experiments demonstrate also the existence of the whole spectrum of the free energy barrier heights up to infinity, at any temperature below T_c. What is more important, however, is that the measurements show that the barrier heights strongly depend on the temperature and at any $T<T_c$ there are certain barriers which become infinite. This is a clear indication of the existence of the ergodicity breaking phase transition at any $T<T_c$. It is this phenomenon, that results in the continuous process of fragmentation of phase space into ever smaller valleys as the temperature decreases. In these terms the critical temperature T_c is just the maximum possible value of T^*.

PART 2

In the subsequent three Chapters the formal calculations for the special model of spin-glass with long range interactions will be performed. Parisi replica symmetry breaking scheme will be considered in detail. As a result, the physical picture of the spin-glass phase discussed in the first part of the review will be obtained.

5. THE REPLICA METHOD

5.1. The model

The Sherrington and Kirkpatric (SK) model of a spin-glass[11] is described by the Ising Hamiltonian:

$$H=-\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j. \tag{5.1}$$

The spin-spin interactions J_{ij} in this system are the random quenched variables which are *independent* for any pair of sites (i,j), and which are described by the Gaussian distribution:

$$P[J_{ij}]=\prod_{i<j}\left[\sqrt{\frac{N}{2\pi}}\exp\left(-\frac{J_{ij}^2N}{2}\right)\right]. \tag{5.2}$$

According to the above definitions, each spin can interact with any other spin of the system. For that reason no concept of space structure (dimensionality, type of the lattice, nearest neighbors etc.) exists in this model. The space here is just the set of N sites in which the Ising spins are placed, and all these spins, in a sense, could be considered as nearest neighbors. One could also interpret the SK model as that of the lattice spin system in an infinite-(in the limit $N\to\infty$) dimensional space.

According to the probability distribution (5.2) one gets:

$$\langle\langle J_{ij}\rangle\rangle=0;\quad \langle\langle J_{ij}^2\rangle\rangle\frac{1}{N} \tag{5.3}$$

where $\langle\langle ...\rangle\rangle$ means averaging over the realizations of the random J_{ij}'s:

$$\langle\langle(...)\rangle\rangle\equiv\int \mathbf{D}\mathbf{J}P[\mathbf{J}](...)$$
$$=\prod_{i<j}\left[\int_{-\infty}^{+\infty}\frac{dJ_{ij}}{\sqrt{2\pi N}}\exp\{-\tfrac{1}{2}J_{ij}^2N\}\right](...). \tag{5.4}$$

One could easily check that due to the normalization of the order of $1/N$ for the average square value of the J_{ij}'s (5.3), the average energy of the system estimated using the Hamiltonian (5.1) appears to be of the order of N, as it should be in any adequately defined physical system.

It is clear, however, that microscopic structure of the model defined above is highly unphysical. Nevertheless, this model has two big advantages: first, it is exactly solvable, and second, the results obtained from that solution appear to be quite non-trivial and on a qualitative level they could be easily generalized for a "normal" random physical system. Therefore, if it would be discovered (e.g. in experiments) that real spin-glasses demonstrate that special sort of the physical properties described in the first part of the review, then it is not so important, in a sense, what was that original artificial model, which has initiated the correct result. One could argue then, that such physics of the spin-glass state is a sort of a big universality class, which covers a wide spectrum of the disordered statistical systems, including even such an exotic one as the SK model.

As for the SK model itself, the reason why it is exactly solvable is very simple: in the infinite-dimensional space the mean field theory is exact.

5.2. The replicas

Formally, the replicas are introduced as follows. The physical (selfaveraging) free energy of the quenched random system is known to be obtained by averaging over the randomness of the logarithm of the partition function:

$$F\equiv\langle\langle F_J\rangle\rangle=-\frac{1}{\beta}\langle\langle\ln Z_J\rangle\rangle \tag{5.5}$$

where the partition function

$$Z_J=\sum_{\sigma}\exp\{-\beta H[\sigma]\} \tag{5.6}$$

should be calculated for fixed random J_{ij}'s. To perform this procedure of averaging the following trick is invented. Let us consider the *integer* power n of the partition function (5.6). That would be the partition function of the n non-interacting *identical* replicas of the original system (i.e. having identical fixed spin-spin couplings J_{ij}):

$$Z_J^n=\left(\prod_{a=1}^{n}\sum_{\sigma^a}\right)\exp\left[\beta\sum_{a=1}^{n}\sum_{i<j}^{N}J_{ij}\sigma_i^a\sigma_j^a\right]. \tag{5.7}$$

Here the subscript a labels the replicas. Let us introduce the quantity:

$$F_n=-\frac{1}{\beta n}\ln Z_n \tag{5.8}$$

where

$$Z_n\equiv\langle\langle Z_J^n\rangle\rangle. \tag{5.9}$$

Now, if a *formal* limit $n\to 0$ would be taken in the expression (5.8), then the original expression for the physical free energy (5.5) would be recovered:

$$\lim_{n\to 0}F_n=-\frac{1}{\beta}\langle\langle\ln Z_J\rangle\rangle=F. \tag{5.10}$$

So that, the scheme of the replica method is the following. First one calculates the quantity F_n for integer n. Then, the analytic continuation of the obtained function of the parameter n should be made for an arbitrary non-integer n. After that the limit $n\to 0$ has to be taken.

Although this procedure may look rather doubtful at first, actually it appears to be not that crazy. Of course, in general no one was able to prove yet that the replica method must give the correct results. And nevertheless, there are several arguments in favor of the replica method:

First, if the free energy appears to be an analytic function of the temperature and the other parameters (so that it can be represented as a series in powers of β), which usually takes place in the high temperature phase, then the replica method can be easily proved to be correct in a strict sense. Well, of course, in the spin-glass phase the free energy function is far from being an analytic one.

Second, in all cases, when the calculations could be performed by some other method, the results of the replica method are confirmed.

Third, (which is presumably most important) I would argue that the replica method is not just a formal trick, but actually it is a physically sensible procedure. In general terms I would say, that if in the disordered system there are numerous alternatives for the ground state, which are essentially defined by the quenched disorder and which produce the ergodicity breaking, then in the calculations of the thermodynamics while summing over all these alternatives one is inevitably forced, in a sense, to take the same system (with the same realization of the disorder) many times. As for the limit $n\to 0$, it will be demonstrated later on that, in terms of the Parisi replica symmetry breaking scheme, it corresponds, in a sense, to the limit $n\to\infty$, which is quite natural, since the number of the alternative states in the thermodynamic limit of the SK model is infinite.

Actually, the replica formalism could be introduced in a purely physical way.[12] Let us consider a general spin system described by some Hamiltonian $H[J;\sigma]$, which depends on the spin variables $\{\sigma_i\}$ and the spin-spin interactions J_{ij} (for the moment, the concrete form of the Hamiltonian is not important). If the interactions J_{ij} are quenched, the free energy of the system would depend on the concrete realization of the J_{ij}'s:

$$F[J]=-\frac{1}{\beta}\log Z[J] \tag{5.11}$$

where

$$Z[J]=\sum_{\sigma}\exp(-\beta H[J;\sigma]) \tag{5.12}$$

is the partition function.

Now, let us assume that the spin-spin interactions are not perfectly quenched, so that they can also change their values, but the characteristic time scale of their changes is much larger than the time scale on which the spin degrees of freedom reach thermal equilibrium. In this case the free energy (5.11) would still make sense, and it would become the energy function (the Hamiltonian) for the J_{ij}'s degrees of freedom.

Besides, the space in which the interactions J_{ij} take on their values should be specified separately. The J_{ij}'s could be variables, taking on values $\pm J_0$, or they could be the continuous variables taking on values in some restricted interval, or something else. In the quenched case this space of J_{ij}'s is defined by some statistical distribution function $P[J]$. In the case of partial annealing this function $P[J]$ has the meaning of an internal potential for the J_{ij}'s, which restricts the space of their values.

Let us assume now, that the spin and the interaction degrees of freedom *are not thermally equilibrated*, so that the interaction degrees of freedom have their own temperature T', which is different from that of the spin degrees of freedom T. In this case for the total partition function of the system one gets:

$$\begin{aligned}\mathscr{Z}&=\int DJP[J]\exp(-\beta' F[J])\\&=\int DJP[J]\exp\left(\frac{\beta'}{\beta}\log Z[J]\right)\\&=\int DJP[J](Z[J])^n\end{aligned} \tag{5.13}$$

where $n=T/T'$. Correspondingly, the total free energy of the system would be:

$$\mathscr{F}=-T\log\{\langle\langle(Z[J])^n\rangle\rangle\} \tag{5.14}$$

where

$$\langle\langle(Z[J])^n\rangle\rangle\equiv\int DJP[J](Z[J])^n. \tag{5.15}$$

This way we have arrived at the replica formalism, in which the "number of replicas" $n=T/T'$ is a finite parameter.

To obtain the physical (selfaveraging) free energy in the replica approach in the case of quenched random J_{ij}'s one takes the limit $n\to 0$. From the point of view of partial annealing considered above, this situation corresponds to the limit of the infinite temperature T' in the subsystem of J_{ij}'s. This is natural in a sense that in this case the thermodynamics of the spin degrees of freedom produces no effect on the distribution of the spin-spin interactions.

In the case that the spin and the interaction degrees of freedom are thermally equilibrated $T'=T$ ($n=1$), and we arrive at the trivial case of the totally annealed disorder whatever is the difference of the characteristic time scales of the J_{ij}'s and the spins. This is also natural because the thermodynamic description formally corresponds to infinite times, and the characteristic time scales of the dynamics of the internal degrees of freedom are becoming to be of no importance.

If $n\neq 0$ and $n\neq 1$, we arrive at the situation which could be called partial annealing, and which is the intermediate case between quenched disorder and annealed disorder.

5.3. Calculation of the free energy

To calculate the replica free energy F_n, according to Eqs. (5.8) and (5.9), one has to calculate the annealed average of the n-replica partition function:

$$Z_n=\sum_{\sigma_i^a}\int DJ_{ij}\exp\left\{\beta\sum_{a=1}^{n}\sum_{i<j}^{N}J_{ij}\sigma_i^a\sigma_j^a-\frac{1}{2}\sum_{i<j}J_{ij}^2N\right\} \tag{5.16}$$

(here and everywhere in what follows all kinds of pre-exponential factors are omitted). Integration over the J_{ij}'s gives:

$$Z_n=\sum_{\sigma_i^a}\exp\left\{\frac{\beta^2}{2N}\sum_{i<j}^{N}\left(\sum_{a=1}^{n}\sigma_i^a\sigma_j^a\right)^2\right\} \tag{5.17}$$

or

$$Z_n=\sum_{\sigma_i^a}\exp\left\{\frac{1}{4}\beta^2Nn+\frac{\beta^2N}{2}\sum_{a<b}^{n}\left(\frac{1}{N}\sum_{i}^{N}\sigma_i^a\sigma_i^b\right)^2\right\}. \tag{5.18}$$

The summation over the sites in the above equation can be linearized by introducing the replica matrix Q_{ab}:

$$Z_n=\prod_{a<b}^{n}\left(\int dQ_{ab}\right)\sum_{\sigma_i^a}\exp\left\{\tfrac{1}{4}\beta^2Nn-\frac{\beta^2N}{2}\sum_{a<b}^{n}Q_{ab}^2+\beta^2\sum_{a<b}^{n}\sum_{i}^{N}Q_{ab}\sigma_i^a\sigma_i^b\right\}. \tag{5.19}$$

The parameters Q_{ab} have in fact a rather clear physical interpretation. According to the above equation, the equilibrium values of the matrix elements Q_{ab} are defined by the equations $\delta Z_n/\delta Q_{ab}=0$, which give:

$$Q_{ab}=\frac{1}{N}\sum_{i}^{N}\langle\sigma_i^a\sigma_i^b\rangle. \tag{5.20}$$

Since the expression in the exponent of Eq. (5.19) is linear in the spatial summation, the total partition function factorizes into independent site partition functions:

$$Z_n = \prod_{a<b}^{n} \left(\int dQ_{ab} \right) \exp\left\{ \frac{1}{4}\beta^2 Nn - \frac{\beta^2 N}{2} \sum_{a<b}^{n} Q_{ab}^2 \right.$$
$$\times \prod_{i}^{N} \left[\sum_{\sigma_i^a} \exp\left\{ \beta^2 \sum_{a<b}^{n} Q_{ab}\sigma_i^a\sigma_i^b \right\} \right] \quad (5.21)$$

or

$$Z_n = \prod_{a<b}^{n} \left(\int dQ_{ab} \right) \exp\left\{ \frac{1}{4}\beta^2 Nn - \frac{\beta^2 N}{2} \sum_{a<b}^{n} Q_{ab}^2 \right.$$
$$\left. + N \log\left[\sum_{\sigma_a} \exp\left(\beta^2 \sum_{a<b}^{n} Q_{ab}\sigma_a g j_b \right) \right] \right\}. \quad (5.22)$$

This equation can be represented as folows:

$$Z_n = \int D\hat{Q} \exp(-\beta n N f[\hat{Q}]) \quad (5.23)$$

where

$$f[\hat{Q}] = -\tfrac{1}{4}\beta + \frac{\beta}{2n} \sum_{a<b}^{n} Q_{ab}^2$$
$$- \frac{1}{\beta n} \log\left[\sum_{\sigma_a} \exp\left(\beta^2 \sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b \right) \right]. \quad (5.24)$$

So that now all the problems are transfered into the replica space.

In the thermodynamic limit in the leading order in N the integral for the partition function (5.23) is determined by the saddle point of the function $f[\hat{Q}]$:

$$Z_n \simeq \left[\det \frac{\delta^2 f}{\delta \hat{Q}^2} \right]^{(-1/2)} \exp(-\beta n N f[\hat{Q}^*]) \quad (5.25)$$

where $\hat{Q}^*$ is the matrix which corresponds to the minimum of the function f, and which is defined by the saddle-point equation:

$$\frac{\delta f}{\delta Q_{ab}} = 0. \quad (5.26)$$

According to the general scheme of the replica method, the quantity $f[\hat{Q}^*]$ would be the density of the free energy of the system. In the case that the ground state solution would turn out to be not unique, then the free energy would be given by the sum over all the solutions.

At this stage it seems as if to solve the problem, one only has to calculate the expression for the replica free energy (5.24) as a function of $\hat{Q}$, then one has to solve the saddle-point equations (5.26) to obtain $\hat{Q}^*$ and the corresponding value for $f[\hat{Q}^*]$, and finally to take the limit $n \to 0$. Unfortunately, the problem is that in general this procedure can not be carried out, since for an arbitrary matrix $\hat{Q}$ the expression (5.24) can not be calculated. Note also, that the limit $n \to 0$ is a somewhat special point, and one has to be careful about what sort of solutions of Eqs. (5.26) should be taken into account. The point is that for $n<1$ the contributions to the physical free energy come from the maxima and not from the minima of the replica free energy (5.24). The reson is that at $n<1$ the number of independent parameters in the matrix $\hat{Q}$ is becoming negative, and turns everything upside down (we will see this a bit later).

Therefore, the following procedure of solving the problem is proposed. First, one has simply to guess the correct form of the matrix $\hat{Q}^*$, which would depend on some finite number of parameters, and after that these parameters should be obtained from the saddle-point equations (5.26). This way one would be able to find the extremum inside some subspace of the space of all the matrices $\hat{Q}$. If it would be possible to prove then, that the Hessian $\delta^2 f/\delta\hat{Q}^2$ in this extremum point is positively defined, then it would mean that the true extremum is found.

5.4. The replica-symmetric solution

Since all the replicas in our system are equivalent, one could naively guess that the adequate form of the matrix $\hat{Q}^*$ is such that all its elements are equal:

$$Q_{ab} = q; \quad \text{for all } a \neq b. \quad (5.27)$$

Such an ansatz is called the replica symmetric (RS) approximation. Actually, this is just the hypothesis that there exists only one ground state in the system.

All the calculations in the RS approximation are simple. For the replica free energy (5.24) one gets:

$$f(q) = -\tfrac{1}{4}\beta + \frac{\beta}{2n}\frac{n(n-1)}{2} q^2$$
$$- \frac{1}{\beta n} \log\left[\sum_{\sigma_a} \exp\left\{ \frac{1}{2}\beta^2 \left(\sum_{a}^{n} \sigma_a \right)^2 q - \frac{1}{2}\beta^2 n q \right\} \right] \quad (5.28)$$

or

$$f(q) = -\tfrac{1}{4}\beta + \tfrac{1}{2}\beta q + \tfrac{1}{4}(n-1)\beta q^2 - \frac{1}{\beta n} \log\left[\int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} \right.$$
$$\times \exp$$
$$\left. (-\tfrac{1}{2}z^2) \prod_{a=1}^{n} \left(\sum_{\sigma_a = \pm 1} \exp\{\beta\sigma_a \sqrt{q} z\} \right) \right]. \quad (5.29)$$

In the limit $n \to 0$ one obtains:

$$f(q) = -\tfrac{1}{4}\beta(1-q)^2 - \frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}}$$
$$\times \exp(-\tfrac{1}{2}z^2) \ln(2\cosh(\beta\sqrt{q}z)). \quad (5.30)$$

The saddle-point equation for the function $f(q)$ with respect to q gives:

$$q = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2 \right) \tanh^2(\beta\sqrt{q}z). \quad (5.31)$$

One can easily see that for $T>1$ the only solution of this equation is $q=0$. For $T<1$ there is a nontrivial solution $q \neq 0$:

if $(1-T) \equiv \tau \ll 1$, then $q(\tau) \simeq \tau$;

if $T \to 0$, then $q \to 1$.

According to Eq. (5.20), the obtained solution $q(T)$ is the physical order parameter:

$$q = \frac{1}{N} \sum_i^N \langle \sigma_i \rangle^2. \qquad (5.32)$$

The fact that q is not equal to zero in the low temperature phase means that the spins of the system are frozen in some random state. Since there is only one solution for $q(T)$, such a disordered ground state is unique. Producing some more calculations one could easily obtain the results for all the observable thermodynamical quantities, such as the specific heat, the susceptibility, the entropy etc. Therefore, in terms of the considered replica symmetric ansatz a complete solution of the problem could be easily obtained.

All that would be very fine, if it were correct. Actually it is not. One of the simplest ways to see that there is something wrong with the obtained solution is to calculate the entropy. One could easily discover then, that at low enough temperatures the entropy becomes negative! (For $T=0$ the entropy $S = -1/2\pi \simeq -0.17$).

The detailed calculations of the Hessian $\delta^2 f/\delta \hat{Q}^2$ for the obtained RS solution demonstrate the reason for that paradox: this solution turns out to be unstable ($\det(\delta^2 f/\delta \hat{Q}^2) < 0$) in the entire low temperature region $T < 1$ (Ref. 13). It means that the true solution is somewhere beyond the replica-symmetric subspace of the matrices $\hat{Q}$.

6. THE REPLICA SYMMETRY BREAKING

The strategy of finding the true solution for the replica matrix $\hat{Q}$ in the limit $n \to 0$ is called the Parisi replica symmetry breaking (RSB) scheme.

First, let us guess some other trial structure (not replica symmetric) for the matrix $\hat{Q}$, and within this new subspace let us calculate the extremum for the replica free energy f. After that, one should calculate the Hessian $\delta^2 f/\delta \hat{Q}^2$ and check the stability of the obtained solution. Since the RS solution has turned out to be not satisfactory, we should try with some other structure which would contain more parameters.

Actually, the situation appears to be much more sophisticated since (as we will see later) no ansatz which contains a finite number of parameters could provide a stable solution. Nevertheless, trying different structures for $\hat{Q}$, and calculating the eigenvalues of the Hessian, one at least is able to judge which ansatz could be better (so to say, which is less unstable). Such a procedure could point the correct "direction" in the space of the matrices $\hat{Q}$ toward the true solution.

The Parisi RSB scheme is an infinite sequence of ansatzes which step by step approximate the true solution better and better. Then this true solution can be formulated and adequately described in terms of continuous functions as the limit of a certain sequence. Moreover, in this limit one is able to prove the stability of the obtained solution (actually the stability appears to be marginal: the most negative eigenvalue of the Hessian turns to zero), and it is

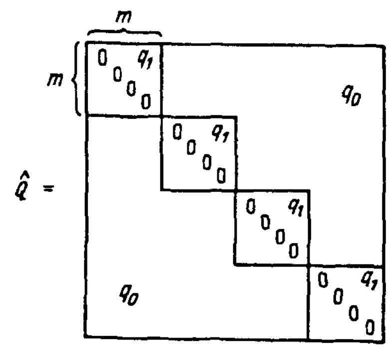

FIG. 12. The structure of the matrix Q_{ab} at the one-step replica symmetry breaking.

also possible to define simple physical quantities which make it possible to demonstrate what is the physics behind the obtained solution.

Consider now, how step by step, the solution is approximated. Note, however, that at the present stage trying to understand right away "why is it so?," might not be the best idea. A bit later, after passing through all these formal constructions, the feeling that this is just the most natural construction, comes automatically.

6.1. The one-step replica symmetry breaking

At the first step it is "natural" to divide all n replicas into n/m groups with m replicas in each one (until now it is assumed, of course, that both m and n/m are integers). Then, the trial matrix $\hat{Q}$ is defined as follows: $Q_{ab} = q_1$, if the replicas a and b belong to the same group, and $Q_{ab} = q_0$, if the replicas a and b belong to different groups. The diagonal elements are, of course, zeros. In a compact form such a structure could be written as follows:

$$Q_{ab} = \begin{cases} q_1 & \text{if } I\left(\frac{a}{m}\right) = I\left(\frac{b}{m}\right) \\ q_0 & \text{if } I\left(\frac{a}{m}\right) \neq I\left(\frac{b}{m}\right) \end{cases} \qquad (6.1)$$

where $I(x)$ is the integer valued function, which is equal to the smallest integer bigger or equal to x. The qualitative structure of this matrix is shown in Fig. 12.

In this ansatz we have three parameters: q_1, q_0 and m, and these parameters have to be defined from the corresponding saddle-point equations.

Now, using the explicit form of the matrix $\hat{Q}$, for the replica free energy (5.24) one gets:

$$f[\hat{Q}] = -\tfrac{1}{4}\beta + \frac{\beta}{2n} \sum_{a<b}^n Q_{ab}^2 - \frac{1}{\beta n} \log Z([\hat{Q}]) \qquad (6.2)$$

where

$$Z([\hat{Q}]) = \sum_{\sigma_a} \exp\left(\beta^2 \sum_{a<b}^n Q_{ab}\sigma_a\sigma_b\right). \qquad (6.3)$$

Simple calculations give:

$$\sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b=\frac{1}{2}\left[q_0\left(\sum_{a=1}^{n}\sigma_a\right)^2+(q_1-q_1)\right.$$
$$\left.\times\sum_{k=1}^{n/m}\left(\sum_{c_k=1}^{m}\sigma_{c_k}\right)^2-nq_1\right]. \quad (6.4)$$

Here k numbers the replica blocks and c_k numbers the replicas inside the blocks. After the Gaussian transformation in $Z[\hat{Q}]$ for each of the squares in the above equation, one gets:

$$Z[q_1,q_0,m]$$
$$=\int\frac{dz}{\sqrt{2\pi q_0}}\exp\left(-\frac{z^2}{2q_0}\right)\prod_{k=0}^{n/m}\left(\int\frac{dy_k}{\sqrt{2\pi(q_1-q_0)}}\right.$$
$$\times\exp\left(-\frac{y_k^2}{2(q_1-q_0)}\right)\sum_{\sigma_a}\exp\left\{\beta\left[z\sum_a^n\sigma_a\right.\right.$$
$$\left.\left.+\sum_{k=0}^{n/m}y_k\left(\sum_{c_k=1}^{m}\sigma_{c_k}\right)\right]-\frac{1}{2}\beta^2 nq_1\right\}. \quad (6.5)$$

The summation over the spins gives:

$$Z[q_1,q_0,m]=\int\frac{dz}{\sqrt{2\pi q_0}}\exp\left(-\frac{z^2}{2q_0}\right)$$
$$\times\left[\int\frac{dy}{\sqrt{2\pi(q_1-q_0)}}\exp\left[-\frac{y^2}{2(q_1-q_0)}\right]\right.$$
$$\left.\times\left\{2\cosh\beta(z+y)\right\}^m\right]^{n/m}\exp\left(\frac{1}{2}\beta^2 nq_1\right). \quad (6.6)$$

For the second term in Eq. (6.2) one obtains:

$$\frac{\beta}{2n}\sum_{a<b}^{n}Q_{ab}^2=\frac{\beta}{4n}\left[q_1^2 m(m-1)\frac{n}{m}+q_0^2\left(n^2-m^2\frac{n}{m}\right)\right]$$
$$=\frac{\beta}{4}\left[q_1^2(m-1)+q_0^2(n-m)\right]. \quad (6.7)$$

Now the limit $n\to 0$ has to be taken. Originally m has been defined as an integer in the interval $1\leqslant m\leqslant n$. The formal analytic continuation turns this interval into $0\leqslant m\leqslant 1$, where m in no longer an integer. It should be noted here that since there is no reliable mathematical background for the replica method anyway, I am not going to waste time and words to produce some sort of respected formal justification for it here. My purpose now, is just to *tell* what this method is, and how this method works. So I'm telling: in the RSB scheme the parameter m originally introduced as an integer in the interval $1\leqslant m\leqslant n$, turns into a non-integer in the interval $0\leqslant m\leqslant 1$ in the limit $n\to 0$.

Nothing in this world is perfect and the replica formalism is not an exception. One should just rely, as much as possible, on common sense, while the main support for all that formal scheme is that after all the rules of the game have once been fixed, whenever the method is applied to any other concrete system, it produces reasonable results (which quite often could be verified in some other ways). An ideal absence of any logical jumps can lead only to triviality. This is not the case in the RSB scheme.

Thus, taking the limit $n\to 0$ in Eqs. (6.6) and (6.7) for the free energy of Eq. (6.2) one gets:

$$f(q_1,q_0,m)$$
$$=-\tfrac{1}{4}\beta(1+mq_0^2+(1-m)q_1^2-2q_1)-\frac{1}{m}\int\frac{dz}{\sqrt{2\pi q_0}}$$
$$\times\exp\left(-\frac{z^2}{2q_0}\right)\ln\left[\int\frac{dy}{\sqrt{2\pi(q_1-q_0)}}\right.$$
$$\left.\times\exp\left(-\frac{y^2}{2(q_1-q_0)}\right)(\cosh\beta(z+y))^m\right]-\ln 2.$$
$$(6.8)$$

One can easily check that in the cases $m=0$ and $m=1$ the replica symmetric solution is recovered (section 5.4) with $q=q_0$ and $q=q_1$ respectively.

Note now another essential point. Actually, in the replica formalism one is looking for the maxima and not for the minima of the free energy. The formal reason is that in the limit $n\to 0$ the number of components of the order parameter $\hat{Q}$ becomes negative. For example, in the case of the one-step RSB each line of the matrix $\hat{Q}$ contains $(m-1)<0$ components which are equal to q_1, and $(n-m)\to -m<0$ components which are equal to q_0. This phenomenon can also be easily demonstrated for the case when the free energy contains only the term $\beta/n\Sigma_{a<b}Q_{ab}^2$ and contains no interaction terms (which is the case in the high temperature limit):

$$\lim_{n\to 0}\left[\frac{\beta}{n}\sum_{a<b}Q_{ab}^2\right]=-\frac{\beta}{2}\left[(1-m)q_1^2+mq_0^2\right]. \quad (6.9)$$

It is obvious that for $0\leqslant m\leqslant 1$ the "correct extremum" in which the Hessian is positive, is the maximum and not the minimum with respect to q_0 and q_1.

To get the saddle point equations for the case under consideration one has simply to differentiate Eq. (6.8) with respect to q_0, q_1 amd m. We omit this simple exercise. The result of the numerical solution of these saddle point equations is the following:

1) In the low-temperature phase $T<1$ the function f has indeed a maximum at a certain point: $0\leqslant m(T)\leqslant 1$; $0\leqslant q_0(T)\leqslant 1$; $0\leqslant q_1(T)\leqslant 1$ (both for $T\to 1$ and $T\to 0$ one gets $m(T)\to 0$).

2) Although the entropy at low temperatures still becomes negative, this negative value appears to be much smaller than that of the replica-symmetric solution: $S(T=0)\simeq -0.01$ (while for the RS solution $S(T=0)\simeq -0.17$)

3) The most negative eigenvalue of the Hessian near T_c is equal to $-c(T-T_c)^2/9$ (c is some positive number), while for the RS solution it is equal to $-c(T-T_c)^2$. So that, in a sense, the instability is reduced by a factor 9.

The conclusion is that, although the considered one-step RSB solution has also turned out to be not perfect, it is a much better approximation to the true solution than

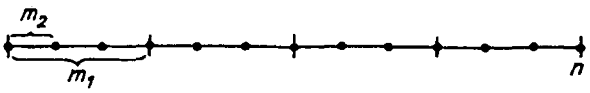

FIG. 13. Classification of the replicas into blocks at the two-step replica symmetry breaking.

the RS one. Therefore one could try to move further and further in the chosen "direction" in the replica space.

6.2. The full scale replica symmetry breaking scheme

Let us try to generalize the structure of the matrix $\hat{Q}$ considered in the previous section to more steps of the replica symmetry breaking. Introduce a series of integers m_i $(i=1,2,...,k+1)$ such that $m_0=n$, $m_{k+1}=1$ and all m_i/m_{i+1} would be integers. Next, let us divide n replicas into n/m_1 groups such that each group would consist of m_1 replicas; divide each group of m_1 replicas into m_1/m_2 subgroups so that each group would consist of m_2 replicas; and so on (Fig. 13). We define all non-diagonal elements of the matrix $\hat{Q}$ as follows:

$$Q_{ab}=q_i,I\left(\frac{a}{m_{i+1}}\right)=I\left(\frac{b}{m_{i+1}}\right) \tag{6.10}$$

where $\{q_i\}$ is a set of $(k+1)$ parameters ($k=1$ corresponds to the one-step RSB case considered in the previous section).

This definition of the matrix elements could also be represented using the hierarchical tree shown in Fig. 14: the element Q_{ab} is equal to that q_i corresponding to the level i of the tree in the "vertical" direction, at which the lines outgoing from the points a and b meet.

Explicitely the matrix $\hat{Q}$ for the case $k=2$ is shown in Fig. 15.

Now we have to calculate the free energy, Eqs. (6.2) and (6.3), which would depend on $(k+1)$ parameters q_i and k parameters m_i. After that, the limit $n\to 0$ should be taken. The parameters m_i have been defined such that $1\leqslant m_{i+1}\leqslant m_i\leqslant n$. Just as in the case of the one-step RSB these conditions turn into $0\leqslant m_i\leqslant m_{i+1}\leqslant 1$ in the limit $n\to 0$. Finally, the parameters q_i and m_i have to be defined from the saddle-point equations:

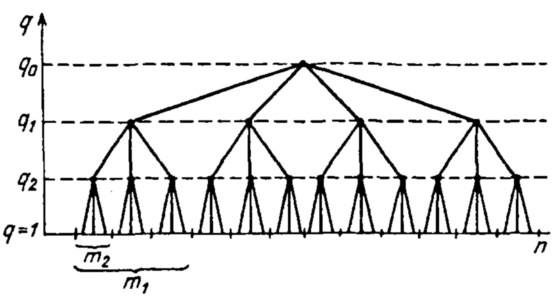

FIG. 14. The definition scheme for the matrix elements Q_{ab} at the two-step replica symmetry breaking.

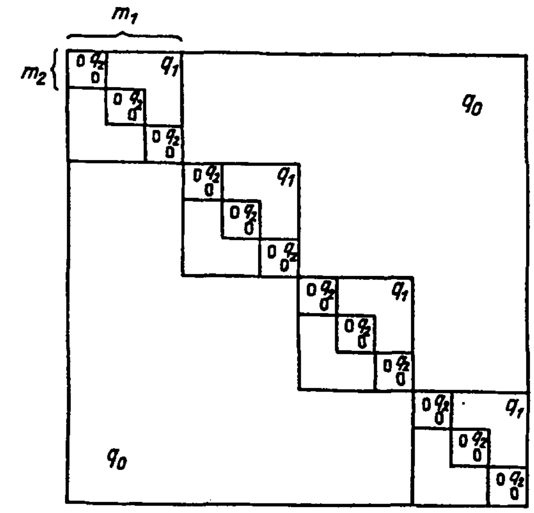

FIG. 15. The explicit form of the matrix Q_{ab} at the two-step replica symmetry breaking.

$$\frac{\partial f}{\partial q_i}=0;\quad \frac{\partial f}{\partial m_i}=0. \tag{6.11}$$

The explicit calculations of the free energy are similar to those of the one-step RSB. After somewhat painful algebra the result obtained is the following:

$$\begin{aligned}
&f[q_0,q_1,...,q_k;m_1,m_2,...,m_k]\\
&=-\frac{1}{4}\beta\left[1+\sum_{i=1}^{k}(m_{i+1}-m_i)q_i^2-2q_k\right]-\frac{1}{m_1}\int\frac{dz_0}{\sqrt{2\pi q_0}}\\
&\times\exp\left(-\frac{z_0^2}{2q_0}\right)\ln\left\{\int\frac{dz_1}{\sqrt{2\pi(q_1-q_0)}}\right.\\
&\times\exp\left(-\frac{z_1^2}{2(q_1-q_0)}\right)\left[\int\frac{dz_2}{\sqrt{2\pi(q_2-q_1)}}\right.\\
&\times\exp\left(-\frac{z_2^2}{2(q_2-q_1)}\right)\Bigg]\cdots\left[\int\frac{dz_k}{\sqrt{2\pi(q_k-q_{k-1})}}\right.\\
&\times\exp\left(-\frac{z_k^2}{2(q_k-q_{k-1})}\right)\\
&\times\left.\left.\left.\left(2\cosh\beta\left(\sum_{i=0}^{k}z_k\right)\right)^{m_k}\right]^{m_{k-1}/m_k}\cdots\right]^{m_2/m_3}\right]^{m_1/m_2}\Bigg\}.
\end{aligned} \tag{6.12}$$

Unfortunately, it is hardly possible to obtain solutions of the corresponding saddle-point equations for an arbitrary k (for $k=3$ the numerical solution for the zero temperature entropy gives the result $S(T=0)\simeq -0.003$).

Now to obtain the exact solution of the problem the limit $k\to\infty$ should be taken. Therefore, in this RSB scheme the limit $n\to 0$ is achieved, in a sense, via the limit $n\to\infty$. Actually, this trick is not something extraordinary: in terms of p-adic numbers (which might be the most suitable language for the structures considered here) this sort of transition is quite natural.

In the limit $k\to\infty$ instead of the infinite set of parameters q_i as an order parameter it is convenient to introduce the *function* $q(x)$, defined as follows:

$$q(x)=q_i, \quad \text{for } m_i\leqslant x\leqslant m_{i+1}. \tag{6.13}$$

In these terms the free energy becomes a functional of the function $q(x)$, and the problem is to find the maximum of this functional with respect to the function $q(x)$:

$$\frac{\delta f}{\delta q(x)}=0. \tag{6.14}$$

This is the saddle-point equation for the order parameter function $q(x)$. Unfortunately, the solution of this equation for an arbitrary temperature $T<1$ could be found only numerically. Nevertheless, near T_c all the calculations could be performed analytically which makes it possible to obtain the function $q(x)$ explicitly, and to get a qualitative understanding of what is going on as the temperature changes. We will consider this useful exercise in the next Chapter, but first I shall try to answer the inevitable question: "What does all this mean?"

However, before turning to this hard question, I would like to conclude this formal Chapter with the following interesting mathematical interpretation of the problem under consideration. It can be shown (see Refs. 14, 15) that the functional $f[q(x)]$ (in the presence of an external magnetic field h) could be represented in a compact form as follows:

$$f[q(x)]=-\tfrac{1}{4}\beta\left[1+\int_0^1 \mathrm{d}x q^2(x)-2q(1)\right]-\frac{1}{\beta}A[q(x)] \tag{6.15}$$

where

$$A[q(x)]=\int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{\sqrt{2\pi q(0)}}\exp\left(-\frac{z^2}{2q(0)}\right)g(0;\quad z+h) \tag{6.16}$$

and the function $g(x,y)$ is obtained from the following nonlinear differential equation:

$$\frac{\partial g(x;\ y)}{\partial x}=-\frac{1}{2}\frac{\mathrm{d}q(x)}{\mathrm{d}x}\left[\frac{\partial^2 g(x;\ y)}{\partial y^2}+x\left(\frac{\partial g(x;\ y)}{\partial y}\right)^2\right] \tag{6.17}$$

with the boundary condition:

$$g(1;\ y)=\ln[2\cosh(\beta y)]. \tag{6.18}$$

If the function $q(x)$ is a monotonic one (which can be proved to be the case), then the inverse function $x(q)$ can be defined, and the saddle-point equations $\delta f/\delta q(x)=0$ could be written in the compact form:

$$q=\int \mathrm{d}y m^2(q;\ y) \tag{6.19}$$

where the function $m(q;\ y)\equiv\partial g/\partial y$ is obtained from the equation

$$\frac{\partial m(q;\ y)}{\partial q}=-\frac{1}{2}\left[\frac{\partial^2 m(q;\ y)}{\partial y^2}+2x(q)m(q;\ y)\frac{\partial m(q;\ y)}{\partial y}\right]. \tag{6.20}$$

Thereby the problem can be formally reduced simply to that of solving a nonlinear differential equation. Of course, in practice it does not help much (due to the universal non-decreasing troubles law). Nevertheless, some people might happen to feel a bit more comfortable if they know about such a compact mathematical formulation of the problem.

7. PHYSICS OF REPLICA SYMMETRY BREAKING

Let us forget about replicas for a while, but let us keep in mind that according to the studies of the previous Chapter there are numerous ground state solutions in the low temperature RSB state. This fact is a direct consequence of the symmetry of the replica free energy with respect to permutations of replicas in the matrix $\hat{Q}$: if there is one RSB solution for the matrix $\hat{Q}$, then any other matrix obtained via permutations of the replica indices in this matrix will also be a solution.

In this Chapter physical quantities will be introduced to describe what is the physics behind that rather formal RSB structure of the spin-glass state considered in the previous Chapter.

7.1. Pure states

Consider first a simple ferromagnetic system. It is well known that if the temperature falls below a certain T_c spontaneous symmetry breaking takes place in the system, so that at each site nonzero spin magnetizations $\langle\sigma_i\rangle=\pm m$ appear. Nevertheless, in any *finite* system (before the thermodynamic limit $N\to\infty$ is taken) all the thermal averages $\langle\sigma_i\rangle$ are identically equal to zero, since due to the symmetry of the Hamiltonian with respect to the global change of the signs of all the spins the states with the magnetizations $+m$ and $-m$ give equal contribution to the partition function. In the thermodynamic limit, however, these two states become separated by an infinite barrier. Therefore, if the system happened to be in one of these states, it will never be able (during any finite time) to go over into the other one. In this sense, the observable state is not a Gibbs one (which is obtained by summing over all the states in the partition function), but one of two such states with nonzero site magnetizations. To distinguish them from the Gibbs state they could be called "pure" states.

The pure states could also be characterized by the property that all the connected correlation functions such as $\langle\sigma_i\sigma_j\rangle_c\equiv\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle$, become zero at large distances in these states. Note here that in the model with long-range interactions all the sites, in a sense, could be considered to be at large distances from each other.

The obtained solutions with replica symmetry breaking indicate that in the low temperature spin-glass state very many pure states could exist in the thermodynamic limit $N\to\infty$. These states are nothing else but the solutions of the saddle-point equations for the free energy, which are just a bit more sophisticated traditional mean field equations for a system with disorder.

Therefore, the Gibbs state of the spin glass could be considered as the result of the summation over all the pure states weighted according to their energies. For example, the thermodynamic average of the site magnetizations could be represented as follows:

$$\langle\sigma_i\rangle = m_i = \sum_\alpha w_\alpha m_i^\alpha . \qquad (7.1)$$

Here the α's label the pure states and the w_α are their statistical weights which formally could be written as

$$w_\alpha = \exp(-F_\alpha) \qquad (7.2)$$

where F_α is the free energy of the pure state number α.

The representation of the thermodynamic state as a linear combination of the pure states in which all the intensive quantities have vanishing long-distance fluctuations, is actually, a central point in the exact definition of the concept of spontaneous symmetry breaking in statistical mechanics.

In the same way the two-point correlation function can be represented as the linear combination

$$\langle\sigma_1\sigma_2\rangle = \sum_\alpha w_\alpha \langle\sigma_1\sigma_2\rangle_\alpha \qquad (7.3)$$

where $\langle\sigma_1\sigma_2\rangle_\alpha \simeq \langle\sigma_1\rangle_\alpha\langle\sigma_2\rangle_\alpha$is the two-point correlation function in the pure state number α. Similar expressions could be written for any many-point correlation functions.

7.2. The physical order parameter

One could ask now, how can the pure states be distinguished one from another. To answer this question it is natural to introduce the concept of "distance" in the space of states. The distance between the states α and β could be defined, for example, as follows:

$$d_{\alpha\beta} = \frac{1}{N} \sum_i^N (m_i^\alpha - m_i^\beta)^2 \qquad (7.4)$$

where $m_i^\alpha = \langle\sigma_i\rangle_\alpha$ and $m_i^\beta = \langle\sigma_i\rangle_\beta$. One could also define the overlap between the two states which, in a sense, is the quantity complementary to the distance:

$$q_{\alpha\beta} = \frac{1}{N} \sum_i^N m_i^\alpha m_i^\beta . \qquad (7.5)$$

It is obvious that $0 \leqslant |q_{\alpha\beta}| \leqslant 1$.

To describe the statistics of the overlaps between all the pairs of pure states it is natural to introduce the probability distribution function:

$$P_J(q) = \sum_{\alpha\beta} w_\alpha w_\beta \delta(q_{\alpha\beta} - q). \qquad (7.6)$$

Note, that the function $P_J(q)$ could depend on the concrete realization of the quenched interactions J_{ij}. The probability distribution function averaged over the disorder is defined as follows:

$$P(q) = \langle\langle P_J(q)\rangle\rangle . \qquad (7.7)$$

The function $P(q)$ gives the probability to find two pure states having the mutual overlap equal to q, provided that these states are taken with the probability of their appearence in the statistical ensemble.

It is easy to understand that in a ferromagnet the function $P(q)$ is just one δ-peak at $q=0$ at the temperatures $T > T_c$, and in the low-temperature phase it is the sum of the two δ-peaks at $q = \pm m^2$ (Fig. 3).

In a spin-glass which could have many pure states at low temperatures, the function $P(q)$ could turn out to be not that trivial. Moreover, since the statistical weights w_α strongly fluctuate depending on the realizations of the J_{ij}'s (which is due to the long-range structure of the spin-spin interactions) the function could also strongly fluctuate depending on the J_{ij}'s.

It is the function $P(q)$ which is the physical order parameter and it is in terms of this order parameter that the nontrivial nature of the spin-glass state could be understood. The non-trivial structure of this function (which as will be shown below, can be calculated in terms of the replica method) demonstrates that the properties of the spin-glass state are essentially different from those of the traditional magnets. If, for example, a disordered system exibits only two "frozen" low-temperature ground states (which differ by the global change of the signs of the spins), then the function $P(q)$ in this system will be the same as in a ferromagnet, i.e. it will be just the sum of two δ-functions. Therefore, although originally defined as a random one, this system, in a sense, belongs to the "class" of "normal" magnets and not to the spin glasses. One could say that in terms of the function $P(q)$ the phase transition into the low-temperature phase in the "normal" magnets is characterized by one (or, may be, several) bifurcation of one δ-function in $P(q)$ (at $T > T_c$) into two (or, may be, several) δ-functions (at $T < T_c$).

The probability distribution function $P(q)$ is a much more general concept than ordinary order parameters which usually describe the phase transitions. The fact that it is a *function* is just a realization of the phenomenon that for the description of the spin-glass phase one needs an *infinite* number of order parameters.

7.3. The order parameter *P(q)* and the replicas

Consider now how the order parameter function $P(q)$ can be calculated in terms of the replica method.

Consider the following series of correlation functions:

$$q_J^{(1)} = \frac{1}{N} \sum_i \langle\sigma_i\rangle^2$$

$$q_J^{(2)} = \frac{1}{N^2} \sum_{i_1 i_2} \langle\sigma_{i_1}\sigma_{i_2}\rangle^2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad (7.8)$$

$$q_J^{(k)} = \frac{1}{N^k} \sum_{i_1 \ldots i_k} \langle\sigma_{i_1} \ldots \sigma_{i_k}\rangle^2 .$$

Using the representation of the Gibbs averages in terms of the pure states for the correlation functions (7.8) one gets:

$$q_J^{(1)}=\frac{1}{N}\sum_i\Big(\sum_\alpha w_\alpha\langle\sigma_i\rangle_\alpha\Big)\Big(\sum_\beta w_\beta\langle\sigma_i\rangle_\beta\Big)$$

$$=\sum_{\alpha\beta}w_\alpha w_\beta q_{\alpha\beta}=\int dqP_J(q)q;$$

$$q_J^{(2)}=\frac{1}{N^2}\sum_{i_1i_2}\Big(\sum_\alpha w_\alpha\langle\sigma_{i_1}\sigma_{i_2}\rangle_\alpha\Big)\Big(\sum_\beta w_\beta\langle\sigma_{i_1}\sigma_{i_2}\rangle_\beta\Big)$$

$$=\sum_{\alpha\beta}w_\alpha w_\beta\Big(\frac{1}{N}\sum_{i_1}\langle\sigma_{i_1}\rangle_\alpha\langle\sigma_{i_1}\rangle_\beta\Big)\Big(\frac{1}{N}\sum_{i_2}\langle\sigma_{i_2}\rangle_\alpha\langle\sigma_{i_2}\rangle_\beta\Big)$$

$$=\sum_{\alpha\beta}w_\alpha w_\beta(q_{\alpha\beta})^2=\int dqP_J(q)q^2;$$

....

$$q_J^{(k)}=\int dqP_J(q)q^k. \qquad (7.9)$$

For the derivation of the result (7.9) one uses the fact that the connected correlation functions in the pure states vanish in the limit $N\to\infty$. For example, for the two-point correlation function the difference $\langle\sigma_i\sigma_j\rangle_\alpha-\langle\sigma_i\rangle_\alpha\langle\sigma_j\rangle_\alpha$ must be of the order of $N^{-\delta}$, where $\delta>0$, so that

$$\lim_{N\to\infty}\left[\frac{1}{N^2}\sum_{ij}(\langle\sigma_i\sigma_j\rangle_\alpha-\langle\sigma_i\rangle_\alpha\langle\sigma_j\rangle_\alpha)^2\right]=0. \qquad (7.10)$$

From Eqs. (7.8) and (7.9) one can easily get the results for the corresponding correlation functions averaged over the disorder:

$$q^{(1)}\equiv\langle\langle q_J^{(1)}\rangle\rangle=\langle\langle(\langle\sigma_i\rangle^2)\rangle\rangle=\int dqP(q)q$$

....

$$q^{(k)}\equiv\langle\langle q_J^{(k)}\rangle\rangle=\langle\langle(\langle\sigma_{i_1}...\sigma_{i_k}\rangle^2)\rangle\rangle=\int dqP(q)q^k \qquad (7.11)$$

where $i_1\neq i_2\neq...\neq i_k$.

The principal point in the above considerations is that the function $P(q)$ originally defined to describe the pure states, can be calculated (at least theoretically) from the multipoint correlation functions in the *Gibbs states*. It makes it possible to avoid that delicate point, that each particular pure state is not quite well defined.

Let us calculate now the multipoint correlation functions in terms of the replicas. Thereby the connection of the formal RSB scheme with the physical order parameter will be established.

According to the definition of the probability distribution function $P(q)$ (7.6), it can be represented in terms of the Gibbs average for the two identical systems as follows:

$$P_J(q)=\frac{1}{Z^2}\sum_\sigma\sum_s\exp(-\beta H[\sigma])$$
$$\times\exp(-\beta H[s])\delta\Big(\frac{1}{N}\sum_i\sigma_is_i-q\Big); \qquad (7.12)$$

$$P(q)=\langle\langle P_J(q)\rangle\rangle.$$

This expression gives the relative number of pairs of the thermodynamically relevant states having their overlap equal to q. Accordingly, one can get the representation for the spin correlation moments (7.8). For $q^{(1)}$ one gets:

$$q^{(1)}=\left\langle\left\langle\frac{1}{Z^2}\sum_\sigma\sum_s(\sigma_is_i)\exp(-\beta H[\sigma]-\beta H[s])\right\rangle\right\rangle$$

$$=\lim_{n\to0}\Big(\prod_{a=1}^n\sum_{\sigma^a}\Big)(\sigma_i^b\sigma_i^c)\exp\Big(-\beta\sum_{a=1}^nH[\sigma^a]\Big) \qquad (7.13)$$

$$\equiv\lim_{n\to0}\langle\langle(\langle\sigma_i^b\sigma_i^c\rangle)\rangle\rangle \quad (b\neq c).$$

Here a and b are two fixed different replicas (the summation over the rest $(n-2)$ replicas in Eq. (7.13) gives the factor Z^{n-2} which turns into Z^{-2} in the limit $n\to0$).

In a similar way one gets:

$$q^{(2)}=\lim_{n\to0}\langle\langle(\langle\sigma_{i_1}^a\sigma_{i_2}^a\sigma_{i_1}^b\sigma_{i_2}^b\rangle)\rangle\rangle;\quad(i_1\neq i_2;a\neq b)$$

....

$$q^{(k)}$$
$$=\lim_{n\to0}\langle\langle(\langle\sigma_{i_1}^a...\sigma_{i_k}^a\sigma_{i_1}^b...\sigma_{i_k}^b\rangle)\rangle\rangle;\quad(i_1\neq i_2\neq...\neq i_k;a\neq b). \qquad (7.14)$$

In calculations of the previous Chapter it has been demonstrated that the free energy of the model under consideration factorizes into the independent site replica free energies. Therefore, the result (7.14) for $q^{(k)}$ can be represented as follows:

$$q^{(k)}=\lim_{n\to0}[\langle\langle(\langle\sigma_i^a\sigma_i^b\rangle)\rangle\rangle]^k=\lim_{n\to0}[Q_{ab}]^k \qquad (7.15)$$

where (see Sec. 5.3)

$$Q_{ab}=\langle\langle(\langle\sigma_i^a\sigma_i^b\rangle)\rangle\rangle \qquad (7.16)$$

is the replica matrix introduced in the previous Chapter which defines the structure of the spin-glass state, and which is obtained from the saddle point equation for the replica free energy.

Since in the RSB solution the matrix elements of Q_{ab} are not all equal, in evaluating the thermodynamic average one has to sum over all the saddle point solutions for the matrix Q_{ab}. All such solutions are obtained by doing the permutations of rows and columns in one of the solutions for this matrix. The summation over all the permutatations (in this particular case is just a summation over the subscripts a and b. Therefore, the correct result for the correlator $q^{(k)}$ should be written as follows:

$$q^{(k)}=\lim_{n\to0}\frac{2}{n(n-1)}\sum_{a<b}[Q_{ab}]^k \qquad (7.17)$$

where $2/n(n-1)$ is the normalization factor which is equal to the inverse number of different pairs of replica indices.

The results (7.17) and (7.11) demonstrate that the RSB solution for the matrix Q_{ab} considered in the previous Chapter makes it possible to calculate (at least in principle) the order parameter distribution function $P(q)$ which has been introduced originally by purely physical qualitative considerations without any replicas. From these two equations one gets the following explicit expression for the function $P(q)$:

$$P(q)=\lim_{n\to 0}\frac{2}{n(n-1)}\sum_{a<b}\delta(Q_{ab}-q). \tag{7.18}$$

Let us calculate this function for the concrete RSB solution considered in the previous Chapter. According to the Parisi RSB scheme with k steps of replica symmetry breaking (Sec. 6.2) one gets:

$$\lim_{n\to 0}\frac{1}{n}\sum_{a,b}[Q_{ab}]^l=-\sum_{i=0}^{k}(m_{i+1}-m_i)q_i^l. \tag{7.19}$$

In the continuous limit at $k\to\infty$ one obtains:

$$\lim_{n\to 0}\frac{1}{n}\sum_{a,b}[Q_{ab}]^l=-\int_0^1 dxq^l(x). \tag{7.20}$$

So that for the correlator $q^{(k)}$, Eq. (7.17), one gets:

$$q^{(k)}=\int_0^1 dxq^k(x). \tag{7.21}$$

If the function $q(x)$ is a monotonic one, then one can introduce the inverse function $x(q)$, and the result (7.21) can be rewritten as follows:

$$q^{(k)}=\int_0^1 dq\frac{dx(q)}{dq}q^k. \tag{7.22}$$

On the other hand, according to Eq. (7.11):

$$q^{(k)}=\int_0^1 dqP(q)q^k. \tag{7.23}$$

So that, one gets:

$$P(q)=\frac{dx(q)}{dq}. \tag{7.24}$$

This is that key result, which defines the connection between the physical order parameter function $P(q)$ and the formal RSB solution, the function $q(x)$. The above result could also be represented in the integral form:

$$x(q)=\int_0^q dq'P(q') \tag{7.25}$$

which gives the answer to the question, what is the meaning of the function $q(x)$. The answer is simple: the function $x(q)$ inverse to $q(x)$ gives the probability to find a pair of pure states which would have an overlap not bigger than q.

Now, to understand at least at the qualitative level what is the structure of the function $P(q)$ in the spin-glass phase, it would be very useful to obtain the RSB solution for $q(x)$ explicitly near the critical temperature T_c.

7.4. Replica symmetry breaking solution near T_c

Near the critical temperature $T_c=1$ the solution for the function $q(x)$ can be obtained analytically. Near the point of the phase transition the order parameter $q(x)$ could be expected to be small in $\tau=(1-T_c)\ll 1$, and therefore one can produce an expansion in powers of the matrix Q_{ab} for the replica free energy (5.24):

$$f[\hat{Q}]=-\tfrac{1}{4}\beta+\frac{\beta}{2n}\sum_{a<b}^{n}Q_{ab}^2 - \frac{1}{\beta n}\log\left[\sum_{\sigma_a}\exp\left(\beta^2\sum_{a<b}^{n}Q_{ab}\sigma_a\sigma_b\right)\right]. \tag{7.26}$$

The expansion of the above expression in powers of Q_{ab} is straightforward. The result of the expansion up to the fourth order is:

$$f[\hat{Q}]=\lim_{n\to 0}\frac{1}{n}\left[-\frac{1}{2}\tau\,\mathrm{Tr}(\hat{Q})^2-\frac{1}{6}\mathrm{Tr}(\hat{Q})^3-\frac{1}{12}\sum_{a,b}Q_{ab}^4 +\frac{1}{4}\sum_{a,b,c}Q_{ab}^2Q_{ac}^2-\frac{1}{8}\mathrm{Tr}(\hat{Q})^4\right]. \tag{7.27}$$

Here in all the terms, but the first one, one takes $T=1$.

Detailed studies of the stability of the replica symmetric solution show that it is the term $\Sigma_{a,b}Q_{ab}^4$ which makes the RS solution unstable below T_c, and it is this term which is responsible for the replica symmetry breaking. This indicates that for the RSB solution near T_c, the last two terms of the fourth order in (7.27) should be expected to be of higher order in τ than all the previous terms. Therefore, to obtain the solution most easily we just neglect these last two terms, and then using the explicit form of the obtained solution for $q(x)$ we can easily check that these neglected terms are of a higher order in τ.

According to the general scheme of replica symmetry breaking (Sec. 6.2) one easily gets:

$$\lim_{n\to 0}\frac{1}{n}\sum_{a,b}[Q_{ab}]^l=\sum_{i=0}^{k}(m_i-m_{i+1})q_i^l\to-\int_0^1 dxq^l(x) \tag{7.28}$$

$$\lim_{n\to 0}\frac{1}{n}\mathrm{Tr}(\hat{Q})^3=\int_0^1 dx\left[xq^3(x)+3q(x)\int_0^x dyq^2(y)\right]. \tag{7.29}$$

For the free energy one obtains:

$$f[q(x)]=\frac{1}{2}\int_0^1 dx\left[\tau q^2(x)-\frac{1}{3}xq^3(x) -q(x)\int_0^x dyq^2(y)+\frac{1}{6}q^4(x)\right]. \tag{7.30}$$

Variation of this expression with respect to the function $q(x)$ gives the following saddle-point equation:

$$2\tau q(x)-xq^2(x)-2q(x)\int_x^1 dyq(y)-\int_0^x dyq^2(y) +\tfrac{2}{3}q^3(x)=0. \tag{7.31}$$

The solution of this equation is simple. Taking the derivative of Eq. (7.31) with respect to x one gets:

$$q'(x)\left[2\tau-2xq(x)-2\int_x^1 dyq(y)+2q^2(x)\right]=0. \tag{7.32}$$

This equation results in the following:

$$2\tau-2xq(x)-2\int_x^1 dyq(y)+2q^2(x)=0 \qquad (7.33)$$

or

$$q'(x)=0. \qquad (7.34)$$

The last equation means that $q(x)=$const which corresponds to the replica symmetric solution which should not be considered. Consider Eq. (7.33). Taking the derivative with respect to x once again, one gets:

$$q(x)=\tfrac{1}{2}x. \qquad (7.35)$$

Assuming that the function $q(x)$ is a continuous one, the following general solution for the saddle-point equation (7.31) is given by

$$q(x)=\begin{cases} q_0, & 0\leqslant x\leqslant x_0 \\ \frac{1}{2}x, & x_0\leqslant x\leqslant x_1 \\ q_1, & x_1\leqslant x\leqslant 1 \end{cases} \qquad (7.36)$$

where

$$x_1=2q_1;\quad x_0=2q_0. \qquad (7.37)$$

Substituting this solution into the original saddle-point equation at the points $x=x_0$ and $x=x_1$ one gets:

$$\begin{aligned} &q_0[2\tau-2q_1+2q_1^2]-\tfrac{4}{3}q_0^3=0, \\ &q_1[2\tau-2q_1+2q_1^2]-\tfrac{4}{3}q_0^3=0. \end{aligned} \qquad (7.38)$$

The solution of these equations is:

$$q_0=0,$$

$$q_1=\tau+O(\tau^2). \qquad (7.39)$$

Now one can easily check even without any calculations that the last two terms of the fourth order in Eq. (7.27) are of a higher order in τ. Since they contain additional summations over the replicas, in the limit $n\to 0$ this results in an additional integration over x (or to an additional power of x). According to the structure of the obtained RSB solution (7.36), (7.37) and (7.39) this results in an additional power of τ.

Note that the obtained RSB solution could be easily generalized for the case of a nonzero external magnetic field (see e.g., Ref. 1). If the value of the field h is small then in the leading order in τ and h the value of q_1 and the form of the function $q(x)$ in the interval $x_0<x<x_1$ do not change, while $q_0\simeq 3/4h^{2/3}$ and $x_0\simeq 3/2h^{2/3}$. Therefore, if the value of the field reaches

$$h_c(\tau)\simeq(\tfrac{4}{3}\tau)^{2/3} \qquad (7.40)$$

(when $x_0=x_1$ and $q_0=q_1$) then the solution for $q(x)$ becomes replica symmetric. The equation for the line $h_c(T)$ (which is usually called the de Almeida–Thouless (AT) line [13]) could be obtained for a whole range of temperatures and the magnetic fields. It can be shown that for $h>h_c(T)$ the replica symmetric solution is stable.

Using the obtained result for the function $q(x)$ and Eq. (7.24) for the distribution function $P(q)$, one gets:

$$P(q)=x_0\delta(q-q_0)+x_1\delta(q-q_1)+p(q) \qquad (7.41)$$

where $p(q)$ is the smooth function in the interval $q_0\leqslant q\leqslant q_1$. In the considered case $\tau\ll 1$, $p(q)=2$.

The result (7.41) shows that for the overlaps of pairs of the pure states taken at random in accordance with their thermodynamic weights, one finds:

1) there exists a finite probability x_1 that these states will turn out to be the same state, in which case their overlap will turn out to be the maximum possible one equal to q_1;

2) there exists a finite probability x_0 (in the case of a nonzero magnetic field) that these states will turn out to be the most "distant" from each other, in which case their overlap will be the minimum possible one equal to q_0;

3) there exists a finite probability $1-x_0-x_1$ of the intermediate situation. If one takes a small interval $(q, q+\delta q)$ near some number q such that $q_0\leqslant q\leqslant q_1$, then there exists a finite probability $p(q)\delta q$ that one finds a pair of pure states having an overlap in this interval.

Although it is very difficult to calculate the functions $q(x)$ and $P(q)$ analytically for arbitrary values of the temperature and the magnetic field, their qualitative behavior remains more or less the same as in the case considered above. The only difference is that the form of the functions $q(x)$ and $P(q)$ in the intervals $x_0\leqslant x\leqslant x_1$ and $q_0\leqslant q\leqslant q_1$ are not as trivial as those near T_c, and the dependences of x_0, x_1, q_0 and q_1 on the temperature and the magnetic field are much more complicated.

The qualitative behavior of the functions $q(x)$ and $P(q)$ for different values of the temperature and the magnetic field is shown in Fig. 16.

8. ULTRAMETRICITY

The obtained RSB solutions for the functions $q(x)$ and $P(q)$ show that the structure of the space of pure states in the spin-glass phase is highly non-trivial. Unfortunately the function $P(q)$ is not enough to understand what this structure is. To get an insight of the topology of the space of pure states one needs to know the higher order correlation properties of the overlaps of the states. Such calculations will be done in this Chapter.

8.1. The formal proof of ultrametricity of pure states

Let us consider the distribution function $P(q_1,q_2,q_3)$ which would describe the joint statistics of the overlaps of arbitrary three pure states. By definition this function gives the probability that arbitrary three pure states α, β and γ would have their mutual overlaps $q_{\alpha\beta}$, $q_{\alpha\gamma}$ and $q_{\beta\gamma}$ to be equal correspondingly to q_1, q_2 and q_3:

$$P(q_1,q_2,q_3)=\left\langle\left\langle \sum_{\alpha\beta\gamma} w_\alpha w_\beta w_\gamma \delta(q_1-q_{\alpha\beta}) \times\delta(q_2-q_{\alpha\gamma})\delta(q_3-q_{\beta\gamma})\right\rangle\right\rangle. \qquad (8.1)$$

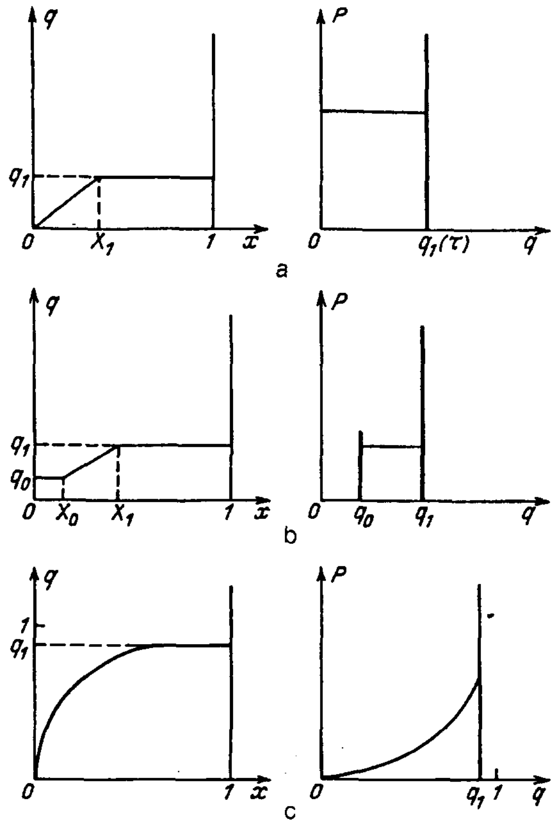

FIG. 16. The qualitative behavior of the functions $q(x)$ and $P(q)$: (a) in zero external magnetic field and for $(1-T)\ll 1$; (b) for $0<h<h_c(T)$ and $(1-T)\ll 1$; (c) for $h=0$ and $T\ll 1$.

The calculations of this function in terms of the replica symmetry breaking scheme are quite similar to those for the function $P(q)$ (Sec. 7.3). In terms of the replica matrix Q_{ab} the result obtained is similar to that of (7.18):

$$P(q_1,q_2,q_3)=\lim_{n\to 0}\frac{1}{n(n-1)(n-2)}\times\sum_{a\neq b\neq c}\delta(Q_{ab}-q_1)\delta(Q_{ac}-q_2)\delta(Q_{bc}-q_3). \quad (8.2)$$

The crucial property of this function which will be derived below is in the following. For the RSB solution this function is identically equal to zero if all three overlaps q_1, q_2 and q_3 are different, and it is not equal to zero only if at least two of the three overlaps are equal and their value is not bigger than the third one.

Let us perform these simple calculations. In terms of the Fourier transform of the function $P(q_1,q_2,q_3)$:

$$g(y_1,y_2,y_3)=\int dq_1dq_2dq_3P(q_1,q_2,q_3)\times\exp(iq_1y_1+iq_2y_2+iq_3y_3) \quad (8.3)$$

Eq. (8.2) becomes:

$$g(y_1,y_2,y_3)=\lim_{n\to 0}\frac{1}{n(n-1)(n-2)}\times\sum_{a\neq b\neq c}\exp(iQ_{ab}y_1+iQ_{ac}y_2+iQ_{bc}y_3)=\lim_{n\to 0}\frac{1}{n(n-1)(n-2)}\mathrm{Tr}[\hat{A}(y_1)\hat{A}(y_2)\hat{A}(y_3)] \quad (8.4)$$

where

$$A_{ab}(y)=\begin{cases}\exp(iQ_{ab}y); & a\neq b\\ 0; & a=b\end{cases}. \quad (8.5)$$

Let us substitute now the RSB solution for the matrix Q_{ab} into Eq. (8.4). According to the general RSB scheme the matrix Q_{ab} turns into the function $q(x)$ in the limit $n\to 0$. One can easily prove that in this limit the replica matrix $A_{ab}(y)$ turns into the function $A(x;y)$:

$$A(x;\ y)=\exp(iq(x)y). \quad (8.6)$$

Simple calculations (see also Eq. 7.29) give:

$$\lim_{n\to 0}\frac{1}{n(n-1)(n-2)}\mathrm{Tr}[\hat{A}(y_1)\hat{A}(y_2)\hat{A}(y_3)]=\frac{1}{2}\int_0^1 dx\Big[xA(x;\ y_1)A(x;\ y_2)A(x;\ y_3)+A(x;\ y_1)\int_0^x dzA(z;\ y_2)A(z;\ y_3)+A(x;\ y_2)\int_0^x dzA(z;\ y_1)A(z;\ y_3)+A(x;\ y_3)\int_0^x dzA(z;\ y_1)A(z;\ y_2)\Big]. \quad (8.7)$$

Accordingly, for the function $P(q_1,\ q_2,\ q_3)$

$$P(q_1,q_2,q_3)=\int dy_1dy_2dy_3g(y_1,\ y_2,\ y_3)\times\exp(-iq_1y_1-iq_2y_2-iq_3y_3) \quad (8.8)$$

one gets:

$$P(q_1,q_2,q_3)=\frac{1}{2}\int_0^1 dx\Big[x\delta(q(x)-q_1)\delta(q(x)-q_2)\times\delta(q(x)-q_3)+\delta(q(x)-q_1)\times\int_0^x dz\delta(q(z)-q_2)\delta(q(z)-q_3)+\delta(q(x)-q_2)\int_0^x dz\delta(q(z)-q_1)\times\delta(q(z)-q_3)+\delta(q(x)-q_3)\times\int_0^x dz\delta(q(z)-q_1)\delta(q(z)-q_2). \quad (8.9)$$

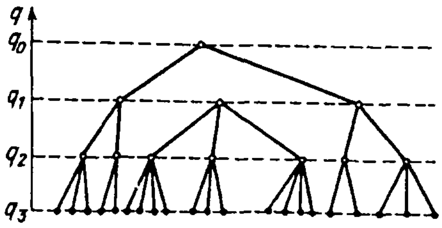

FIG. 17. The ultrametric tree of the spin-glass states.

Replacing integration over x by integration over q and taking into account that $dx(q)/dq=P(q)$ one finally obtains the following result:

$$\begin{aligned}P(q_1,q_2,q_3)=&\tfrac{1}{2}P(q_1)x(q_1)\delta(q_1-q_2)\delta(q_1-q_3)\\&+\tfrac{1}{2}P(q_1)P(q_2)\theta(q_1-q_2)\delta(q_2-q_3)\\&+\tfrac{1}{2}P(q_2)P(q_3)\theta(q_2-q_3)\delta(q_3-q_1)\\&+\tfrac{1}{2}P(q_3)P(q_1)\theta(q_3-q_1)\delta(q_1-q_2).\end{aligned} \quad (8.10)$$

One can easily see from this equation that the property of the function $P(q_1,q_2,q_3)$ which has been stated above: it is not equal to zero only if at least two of the three overlaps are equal and their value is not bigger than the third one. Note that the result (8.10) is just a direct consequence of the block-like hierarchical structure of the replica matrix Q_{ab} and nothing else.

Now we have arrived to the concept of the ultrametricity. A space is said to be ultrametric if its metric has the property that for any three points a, b and c of this space the following inequality is satisfied:

$$d_{ab}\leqslant\max(d_{ac},\quad d_{bc}). \quad (8.11)$$

This property is essentially different from the triangle inequality of the "ordinary" spaces:

$$d_{ab}\leqslant d_{ac}+d_{bc}. \quad (8.12)$$

One of the simplest ways to describe ultrametric space is in terms of the hierarchical tree (Fig. 17). The ultrametric space is associated with the set of the endpoints of the tree. The distance between any two points of this space is defined such that it depends only on the number of "generations" in the "vertical" direction to that level of the tree at which these two points have a common ancestor. One can easily check that with this definition of the distance the set of the endpoints of the tree is ultrametric.

A detailed description of ultrametric spaces (rather from the point of view of physics than pure mathematics) the reader could find in the excellent review of Ref. 6. Here we are going to concentrate mainly on general qualitative properties of ultrametricity which are directly connected with the physics of the spin-glass state.

8.2. The tree of states

Now, keeping in mind that the space of the states of the spin-glass has a tree-like structure, let us try to construct this kind of a space in more general terms. It may help to understand what kinds of degrees of freedom are involved in such structures in general. Later on the concrete parameters of the tree of states which describes the considered spin-glass model will be specified.

Consider the following discrete stochastic process which is assumed to take place *independently* at each site i of the lattice.

1. At the first step one generates with the probability $P_0(y)n_1$ random numbers y^{α_1} ($\alpha_1=1,2,...,n_1$), which belong to the interval $[-1,+1]$.

2. At the second step for each of the numbers y^{α_1} one generates with the conditional probability $P_1(y^{\alpha_1}|y)n_2$ random numbers $y^{\alpha_1\alpha_2}$ ($\alpha_2=1,2,...,n_2$), belonging to the same interval $[-1,+1]$.

3. At the third step, again, for each of the numbers $y^{\alpha_1\alpha_2}$ one generates with the conditional probability $P_2(y^{\alpha_1\alpha_2}|y)n_3$ random numbers $y^{\alpha_1\alpha_2\alpha_3}$ ($\alpha_3=1,2,...,n_3$), belonging to the same interval $[-1,+1]$.

.....

Continue this process up to the Lth step. As a result one gets the hierarchical tree of $n_1n_2...n_L$ numbers in the interval $[-1,+1]$ which are described by the set of probability functions

$$P_{l-1}(y^{\alpha_1\cdots\alpha_{l-1}}|y^{\alpha_1\cdots\alpha_l}) \quad (l=1,2,...,L). \quad (8.13)$$

This stochastic (Markov) process takes place independently at each site of the lattice. Then, for each set of the obtained numbers at each site i define the corresponding spin state as follows:

$$\sigma_i^{\alpha_1\ldots\alpha_L}=\mathrm{sign}(y_i^{\alpha_1\ldots\alpha_L}). \quad (8.14)$$

As a result of the above construction one obtains the set of $n_1n_2...n_L$ spin states which are labeled by $\alpha_1...\alpha_L$ which are a sort of hierarchical "addresses" of the states. The "address" of a specific state describes its genealogical "history." Looking at the "addresses" of two arbitrary states one can immediately tell to what extent these states are close in their genealogy: the longer the coinciding initial part of their "addresses," the closer "relatives" they are.

Simple probabilistic arguments show that the overlap between any two spin states depends only on the degree of their "relationship," i.e., the number of generations which separates them from the closest common ancestor. Consider two spin states which have the following "addresses:"

$$\alpha_1\alpha_2...\alpha_l\alpha_{l+1}\alpha_{l+2}...\alpha_L$$

and

$$\alpha_1\alpha_2...\alpha_l\beta_{l+1}\beta_{l+2}...\beta_L.$$

The two "addresses" begin to differ starting from the generation number l. Since the stochastic processes generating the states have been defined to be independent at each site, for the overlap between these two states

$$q^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}_{\alpha_1...\alpha_l\beta_{l+1}...\beta_L}=\frac{1}{N}\sum_i^N \sigma_i^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}\sigma_i^{\alpha_1...\alpha_l\beta_{l+1}...\beta_L} \tag{8.15}$$

in the thermodynamic limit $N\to\infty$ one gets:

$$\begin{aligned} q^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}_{\alpha_1...\alpha_l\beta_{l+1}...\beta_L} &= \int_{-1}^{+1} dy_1...dy_l P_0(y_1)P_1(y_1|y_2)...P_{l-1}(y_{l-1}|y_l) \\ &\times\left[\int_{-1}^{+1} dy_{l+1}...dy_L P_l(y_l|y_{l+1})P_{l+1}(y_{l+1}|y_{l+2})...\right. \\ &\left.\times P_{L-1}(y_{L-1}|y_L)\mathrm{sign}(y_L)\right]^2 \equiv q_l. \end{aligned} \tag{8.16}$$

Therefore, the overlap depends only on the number l of the level of the tree at which the two states were separated in their genealogical history, and does not depend on the concrete "addresses" of these states. One can easily see that it automatically means that the considered set of the states is ultrametric.

Note, that this is a general property of the considered stochastic evolution process, and it remains true for any choice of the probability distribution functions (8.13) which describe the concrete tree of states. A general reason for that is very simple. The considered procedure of construction of the tree is by definition the random branching process which takes place in the infinite-dimensional (in the limit $N\to\infty$) space. It is clear that in the infinite-dimensional space the branches once separated never comes close again. Therefore, it is of no surprise that ultrametricity is just a routine property which is observed in Nature very often. The examples are the space of biological species, the hierarchical state structures of disordered human societies, etc.

Consider the above hierarchical tree of states in greater detail. The equations for the overlaps between the states (8.15) and (8.16) could be formulated also in terms of the so-called ancestor states $m^{\alpha_1...\alpha_l}$:

$$q_l=\frac{1}{N}\sum_i^N (m_i^{\alpha_1...\alpha_l})^2 \tag{8.17}$$

where the site magnetizations at the level l of the tree are defined as follows:

$$m_i^{\alpha_1...\alpha_l}=\langle\sigma_i^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}\rangle_{(\alpha_{l+1}...\alpha_L)}\equiv m_l(y_i^{\alpha_1...\alpha_l}). \tag{8.18}$$

Here $\langle...\rangle_{(\alpha_{l+1}...\alpha_L)}$ denotes averaging over all the descendant states (branches) of the tree outgoing from the branch $\alpha_1...\alpha_l$ at the level number l. By definition:

$$\begin{aligned} m_l(y_i^{\alpha_1...\alpha_l}) &= \int_{-1}^{+1} dy_{l+1}...dy_L P_l(y_i^{\alpha_1...\alpha_l}|y_{l+1})P_{l+1} \\ &\times(y_{l+1}|y_{l+2})...P_{L-1}(y_{L-1}|y_L)\mathrm{sign}(y_L). \end{aligned} \tag{8.19}$$

This equation for the function $m_l(y)$ could also be written in the following recurrent form:

$$m_l(y)=\int_{-1}^{+1} dy'\mathbf{P}_{ll'}(y|y')m_{l'}(y') \tag{8.20}$$

where

$$\begin{aligned} \mathbf{P}_{ll'}(y|y') &= \int_{-1}^{+1} dy_{l+1}...dy_{l'-1}P_l(y|y_{l+1}) \\ &\times P_{l+1}(y_{l+1}|y_{l+2})...P_{l'-1}(y_{l'-1}|y'). \end{aligned} \tag{8.21}$$

Therefore, all the concrete properties of the tree of states, and in particular the values of the overlaps $\{q_l\}$, are fully determined by the set of the probability functions (8.13) or (8.21). To describe a concrete spin-glass system all these functions have to be calculated, or at least concrete algorithms for their calculations must be specified. In particular, this can be done for the SK model of spin glass. Unfortunately, the actual calculations for this model are rather painful and cumbersome. The reader interested in the details may refer to the original papers of Refs. 16 and 17, while here I shall present only the results.

The ultrametric tree of states which describes the spin-glass phase of the SK model is defined by the random branching process discussed above in which the continuous limit $L\to\infty$ must be taken. In this limit instead of the integers l which define just the discrete level of the hierarchy it is more convenient to describe the tree in terms of the selfoverlaps $\{q_l\}$ of the ancestor states of the corresponding hierarchical level. In the limit $L\to\infty$ the discrete parameters $\{q_l\}$ turn into the continuous variable $0\leqslant q\leqslant 1$. Note also that in the limit $N\to\infty$ all the branching ratios n_q of the tree also diverge at each level.

Instead of the functions (8.13) which are essentially discrete, in the continuous limit it is more natural to describe the tree in terms of the functions (8.21) which define the evolution of the tree from a level q to another level q'. It can be proved (and this proof requires considerable effort) that in the continuous limit these functions are defined by the nonlinear equation of diffusion in "time" q:

$$-\frac{\partial}{\partial q}\mathbf{P}=\frac{1}{2}\frac{\partial^2}{\partial y^2}\mathbf{P}+x(q)m_q(y)\frac{\partial}{\partial y}\mathbf{P} \tag{8.22}$$

with the initial condition:

$$\lim_{q\to q'}\mathbf{P}_{qq'}(y|y')=\delta(y-y'). \tag{8.23}$$

Here $x(q)$ is the inverse function to that of $q(x)$ which is given by the RSB solution (Chap. 6), and the function $m_q(y)$ is the continuous limit of the discrete function (8.20) which defines the distribution of the site magnetizations in the ancestor states at the level q of the tree. Note that this function (which has a clear physical meaning) coincides with the function $m_q(y)$ introduced strictly formally in Chap. 6 (Eqs. (6.19), (6.20)) for the interpretation of the replica symmetry breaking solution in terms of nonlinear differential equations. To prove that, one can easily deduce from Eqs. (8.20) and (8.22) that the function $m_q(y)$ satisfies the equation:

$$-\frac{\partial}{\partial q} m_q(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2} m_q(y) + x(q) m_q(y) \frac{\partial}{\partial y} m_q(y) \tag{8.24}$$

which coincides with Eq. (6.20).

8.3. Summary

Now, to see the complete picture, let us assemble in order all the facts obtained for the spin-glass model with long range interactions.

1) Going through the formal replica calculations of the free energy (Chap. 6) one could represent it in terms of the functional $F[\hat{Q}]$ which depends on the $n \times n$ replica matrix $\hat{Q}$. In the thermodynamic limit the main contribution to the free energy comes from the matrices $\hat{Q}^*$ which correspond to the extrema of this functional, and the physical free energy (as well as any other physical quantity) is obtained in the limit $n \to 0$. In this limit the extrema matrices $\hat{Q}^*$ are defined by the infinite set of parameters which could be described in terms of the continuous function $q(x)$ within the interval $0 \leqslant x \leqslant 1$. Formally, this function could be calculated for any temperature in the low-temperature region, and near the phase transition point it can be obtained explicitly (Fig. 16).

2) On the other hand, in terms of pure physical speculations (Chap. 7) one could define as the order parameter the probability distribution function $P(q)$ which gives the probability to find a pair of the pure (spin-glass) states having the mutual overlap equal to q. Following the same replica symmetry breaking scheme one can show that the function $q(x)$ uniquely defines the distribution function $P(q)$: $P(q) = \mathrm{d}x(q)/\mathrm{d}q$, where $x(q)$ is the inverse function to $q(x)$. The obtained replica symmetry breaking solution for $q(x)$ and correspondingly for $P(q)$ shows that in the low-temperature spin-glass state in a certain (depending on the temperature) interval of values of q there exists a continuous spectrum of overlaps among the pure states.

3) Next, one can introduce the joint distribution function $P(q_1,q_2,q_3)$ which gives the probability that an arbitrary set of three pure states would have their mutual pair overlaps equal to q_1, q_2, and q_3. Following the same replica symmetry breaking scheme this function can be calculated to show that the space of the pure states of a spin-glass has an ultrametric topology.

4) To describe this space of spin-glass states one can construct the ultrametric space in general terms. It is obtained as the result of the random branching process which is defined by a set of probability functions. For the concrete model under consideration these functions could be formally calculated (more precisely, one can obtain the nonlinear evolution equations which define these functions). This way one can see that the structure of the space of the spin-glass states can be described in terms of the hierarchical tree which appears as the result of the random branching process in the space of states.

As the result of all these calculations the following physical picture of the spin-glass phase is obtained.

Just below T_c the space of states is divided into numerous pure states (valleys). These states are described by the average site magnetizations m_i. The configurations of m_i's are different in different states. However, the value of the selfoverlaps:

$$q(T) = \sum_i^N m_i^2 \tag{8.25}$$

appears to be the same in all the states. The value of q is just some function of temperature (near T_c it can be calculated explicitly).

Besides, the mutual pair overlaps of the pure states appear to cover continuously the whole interval $0 \leqslant q^{\alpha\beta} \leqslant q(T)$. (In the presence of an external magnetic field h this interval starts not from zero, but from some finite value: $q_0(h,T) \leqslant q^{\alpha\beta} \leqslant q_1(h,T)$, where $q_0(h,T) \to 0$ for $h \to 0$.) The distribution of the values of $q^{\alpha\beta}$ is described by some probability function $P(q)$ which depends on the temperature (and the magnetic field). The structure of the space of these spin-glass states is described by the ultrametric hierarchical tree discussed above.

If the temperature is slightly decreased $T \to T' = T - \delta T$, each of the pure states is divided into numerous new ones which could be called the descendant states. These states are characterized by a new value of the selfoverlap $q(T') > q(T)$. Correspondingly, the interval of the mutual pair overlaps of the states is getting longer: $0 \leqslant q^{\alpha\beta} \leqslant q(T')$.

At a further decrease of the temperature each of the pure states is (continuously) divided into new and new descendant pure states. This branching process which goes on down to zero temperature $(q(T \to 0) \to 1)$ is fully described by the evolution equation (8.22). This tree of states has the property of self-similarity (scaling), and at any given temperature the natural scale in the space of states is given by the value $q(T)$.

On the other hand, if the temperature is increasing, the opposite process of merging of the families of pure states goes on. Therefore, over the entire temperature interval $0 \leqslant T \leqslant T_c$ a continuous sequence of phase transitions takes place.

9. SOME FANTASIES

9.1. Scaling in the space of states

Consider now the qualitative physical picture of the spin-glass phase from the point of view of its possible generalization to more realistic spin-glass systems with finite-range interactions.

Let us *assume* that on a qualitative level the tree-like hierarchical structure of the spin-glass states remains valid also in spin-glass systems with finite-range interactions. There are only two relevant arguments in favor of this assumption: recent experiments on real spin glasses (Chap. 4) and aesthetic attractiveness of such kind of a structure. The problem then is in the following: is it possible to construct a physical theory of spin-glasses based on this assumption, which would make it possible to make calculations of real *observable* thermodynamics, to make explanations of actual experiments, to make predictions for concrete spin-glass systems, etc.? In other words, is it possible, instead of drawing abstract trees and presenting hand-waving-arguments, to construct a real *science*? This question remains open.

A possible approach could be the following. According to what we have seen in the previous Chapters, at a given temperature T below T_c the spin-glass system remains in one of the pure states, which in terms of the hierarchical tree corresponds to one of the "ancestor" states at the level (scale) $q(T)$ of the tree. All these states could be obtained in the horizontal cross section of the tree at the level $q(T)$. In a real experiment the system once trapped in one of these states (valleys) is not able to go over to other states, since they are separated by infinite free energy barriers. Therefore, the real physics observed in the actual experiment must be defined only by this particular limited part of the phase space. Correspondingly, to calculate the actual oservable physics one has to limit the summations (in the partition function) to this particular part of the space of states. Most probably, the result of such calculations has to be independent of the concrete valley, since results of the experiments were demonstrated to be reproducible. In this sense, the statistical properties of the valleys must be equivalent.

How could the actual calculations be done? According to what we have assumed, inside the valley at a given temperature a whole "mini-tree" of states is hidden (this tree could be developed by lowering the temperature down to zero). Therefore, one could *assume* that it is these states which give the leading contribution to the thermodynamics inside the valley. In other words, in the calculation of the limited (corresponding to one valley) partition function one could restrict the summation by taking into account only the descendant states belonging to the "mini-tree."

Actually, all the above speculations are just an attempt to formulate the intuitive guess for the traditional and principal problem of statistical mechanics: what are the *relevant* degrees of freedom in the system under consideration. In the case that these degrees of freedom are guessed correctly, all the remaining problems are just technical. Note, that in any nontrivial statistical mechanical problem one never takes into account all the degrees of freedom.

Thus, in the approach under consideration, it is assumed that the leading contribution to the observable thermodynamics at a given temperature T comes from the states of the ultrametric tree descending down from the level $q(T)$. In other words, it is assumed that all the relevant degrees of freedom of the spin glass could be classified in terms of the hierarchical tree. At the level of the present "semi-philosophical" discussion, however, it is difficult to conclude, whether this assumption is correct or not. Let us see what could come out of it.

To make things more simple, consider the discrete version of the hierarchical tree (Sec. 8.2). Then, the free energy $f_{\alpha_1\alpha_2...\alpha_l}$ of the pure state at the level $l(T)$ (corresponding to the scale $q_l=q(T)$) could be represented as follows:

$$\exp(-\beta f_{\alpha_1\alpha_2...\alpha_l}) = \sum_{\alpha_{l+1}\alpha_{l+2}...\alpha_L} \exp(-\beta H[\sigma^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}]). \qquad (9.1)$$

Obviously, the most natural way of performing such kind of summation is the iteration prosess, i.e., the summation step by step going from one (lower) level of the tree to the next (higher) one. This procedure could also be called as a sort of renormalization group in the space of states. Summing over the families of the states at the level L one gets a new effective (renormalized) Hamiltonian depending on the states of the level $L-1$. Next, one sums over the families of states at the level $L-1$ and gets a new renormalized Hamiltonian corresponding to the level $L-2$, and so on. At some intermediate level k $(l \leqslant k \leqslant L)$ this one-step transition could be represented as follows:

$$\exp(-\beta H_k[m^{\alpha_1\alpha_2...\alpha_k}]) = \sum_{\alpha_{k+1}} \exp(-\beta H_{k+1}[m^{\alpha_1...\alpha_k\alpha_{k+1}}]). \qquad (9.2)$$

If the corresponding change of scale in the space of states is assumed to be small: $\delta q = q_{k+1} - q_k \ll 1$ then the changes of the parameters of the Hamiltonian after this one-step transition must also be small in δq. As a result one might be able to derive a sort of evolution equations for the parameters of the Hamiltonian and to obtain their dependence on scale in the space of states.

In the traditional renormalization group approach one performs the summation over the "fast" degrees of freedom corresponding to a small spatial scale, and gets a new Hamiltonian with the renormalized parameters corresponding to a larger spatial scale. Then, looking at the asymptotic behavior of the renormalized Hamiltonian at large scales, one could see what the thermodynamic state of the system is. Usually, the result essentially depends on the dimensionality of the space, the temperature and the other parameters involved. In the framework of this procedure one could also calculate the observable thermodynamical quantities.

The idea of the present approach is similar to that of the traditional renormalization group, and the only difference is that the scaling is assumed to take place not in real space (apparently, it doesn't exist there), but in the space of states. In reality, however, the situation in spin glasses appears to be much more sophisticated. Actual calculations show that the renormalized spin-glass Hamiltonian appears to be dependent on an infinite number of parameters, and all these parameters (as well as the original parameters J_{ij}) are random quenched quantities. Unfortunately these calculations are rather cumbersome and the results obtained could not be clearly interpreted yet (for details see Ref. 18)

Nevertheless, in some cases certain concrete conclusions can be derived from this approach. First of all, if the temperature is higher than a certain critical temperature T_c, then all the effective (renormalized) interactions in the Hamiltonian can be proved to tend to zero in the limit of the largest scale $q \to 0$ (note, that here the microscopic scale corresponds to $q=1$, and the macroscopic one corresponds to $q=0$). It indicates that in this case the system is in the paramagnetic state. On the other hand, if $T<T_c$, then a certain characteristic scale $q(T)>0$ comes into play, such that as $q \to q(T)$ some parameters of the renormalized Hamiltonian diverge. Presumably, this indicates that at the scale $q(T)$ the states become "frozen." In any case the renormalization procedure can not go beyond the scale $q(T)$, and it is just that situation which was assumed to take place based on a general qualitative picture of the spin-glass phase. The quantity calculated in this way must be interpreted as the free energy of the pure states at the temperature T. Besides, the dependence of the characteristic values of the effective renormalized interactions on the scale in the interval $q(T)<q\leqslant 1$ could be interpreted as the dependence of the finite free energy barriers (separating the metastable states inside the valley) on the scale in phase space and, correspondingly, on the temperature (Sec. 4.3).

It should be stressed here that the states we are dealing with in this renormalization group approach *are not* the states which are obtained from direct free energy calculations of the SK model. In the SK model at a given temperature one calculates the ultrametric tree of states which is defined at scales $0\leqslant q\leqslant q(T)$. Here instead, one deals with the states at scales $q(T)<q\leqslant 1$, and the idea is to obtain the pure states at the scale $q(T)$ summing over the states starting from the microscale at $q=1$ and approaching the scale $q(T)$ from "below."

Unfortunately, this renormalization group scheme is still far from providing reliable algorithms of calculations of the observable physics. And for that reason it is difficult to conclude at the present stage to what extent the physical assumptions involved in it are correct. It is not impossible, of course, that the technical problems which block the actual calculation are not *technical* at all, but rather *real* ones. Then it would mean that new physical ideas are needed.

9.2. Phenomenological dynamics

To conclude this Chapter, consider a purely phenomenological approach which could qualitatively illustrate the relaxation processes in the spin-glass phase.[19]

Assume that in the low temperature spin-glass phase the free energy landscape is of the type shown in Fig. 2: big wells contain a lot of smaller ones, each of the smaller wells contains many even smaller ones, and so on. Such kind of landscape could be characterized by the typical value of the potential barrier $\Delta(q)$ separating the wells (the states) at the scale q. Assuming that this potential relief has the scaling property (which is a quite natural property of all such fractal-like structures), the dependence of the typical value of the potential barrier on the scale could be assumed to be of the following simple scaling form:

$$\Delta(q)=\Delta_0(q-q(T))^{-\nu}; \quad (q>q(T); \quad \nu>0). \tag{9.3}$$

Here $q(T)$ (according to a general spin-glass philosophy) is the value of the selfoverlap of the pure states at the temperature T, which, on the other hand, is the characteristic scale (scale of the valleys) at which the barriers separating the states become infinite. Note also, that this type of scaling is in agreement with the qualitative results obtained in experiments (Sec. 4.3).

Consider now what kind of relaxation properties could be derived from the above representation. The characteristic time needed to overcome the barrier Δ is

$$\tau(\Delta)\sim\tau_0\exp\left(\frac{\Delta}{T}\right) \tag{9.4}$$

where τ_0 is some microscopic time. Then, the spectrum of the relaxation times inside one valley can be represented as follows:

$$\tau(q)\sim\tau_0\exp(\beta\Delta_0(q-q(T))^{-\nu}). \tag{9.5}$$

For the relaxation of, e.g., the order parameter

$$q(t)=\frac{1}{N}\sum_i\langle\sigma_i(0)\sigma_i(t)\rangle \tag{9.6}$$

one can easily derive the following simple estimate:

$$q(t)\sim\int_{q(T)}^{1}\mathrm{d}q\; q\exp\left(-\frac{t}{\tau(q)}\right). \tag{9.7}$$

Using (9.5), one gets:

$$q(t)\sim\int_{q(T)}^{1}\mathrm{d}q\exp\left[\ln(q)-\frac{t}{\tau_0}\times\exp(-\beta\Delta_0(q-q(T))^{-\nu})\right]. \tag{9.8}$$

In the limit of large times $t\gg\tau_0$ the saddle-point estimate gives the following result:

$$q(t)\sim q(T)+\left[\frac{\beta\Delta_0}{\ln(t/\tau_0)}\right]^{1/\nu} \tag{9.9}$$

Therefore at large times the order parameter approaches its equilibrium value $q(T)$ logarithmically slowly. It is obvious, that the relaxation behavior of other observable quantities would be of the same type.

In the framework of such kind of phenomenology two possible scenarios of transition into the spin-glass phase could be considered. The first, and the most natural one, is that the spin-glass structure "grows" from inside the paramagnetic phase when the temperature approaches T_c from above. Near T_c at $T>T_c$ the energy barriers grow to some large but finite value Δ_∞ as the scale of the phase space increases. This situation can be described in the way similar to Eq. (9.3):

$$\Delta_{T>T_c}(q)=\Delta_0(q+\tilde{q}(T))^{-\nu'}; \quad (\nu'>0). \tag{9.10}$$

Here $\tilde{q}(T)$ is a sort of disorder parameter which defines the limiting value of the energy barriers:

$$\Delta_\infty=\Delta_0(\tilde{q}(T))^{-\nu'}. \tag{9.11}$$

The exponent ν' at $T>T_c$ could in principle be different from that of ν at $T<T_c$. Correspondingly, the spectrum of the relaxation times will be limited in this case, and the maximum relaxation time will be:

$$\tau_\infty \sim \tau_0 \exp(\beta\Delta_\infty). \tag{9.12}$$

As a result, the slow logarithmic relaxation of the order parameter

$$q(t) \sim \left[\frac{\beta\Delta_0}{\ln(t/\tau_0)}\right]^{1/\nu'} - \tilde{q}(T) \tag{9.13}$$

would take place only within the limited time interval: $\tau_0 \ll t \ll \tau_\infty$. At the largest times $t \gg \tau_\infty$ the relaxation must become of the ordinary exponential type:

$$q(t) \sim \exp\left(-\frac{t}{\tau_\infty}\right). \tag{9.14}$$

As the temperature T approaches T_c from above the disorder parameter $\tilde{q}(T)$ must go to zero. Assuming scaling behavior (which is typical for all second-order phase transitions) one could expect that

$$\tilde{q}(T) \sim \left(\frac{T}{T_c}-1\right)^{\alpha} \tag{9.15}$$

for $(T/T_c-1) \ll 1$, where $\alpha>0$ is some critical exponent. Correspondingly, for the maximum energy barrier and for the maximum relaxation time one gets:

$$\Delta_\infty(T) \sim \Delta_0\left(\frac{T}{T_c}-1\right)^{-\alpha/\nu'} \to \infty, \tag{9.16}$$

$$\tau_\infty(T) \sim \tau_0 \exp\left[\beta\Delta_0\left(\frac{T}{T_c}-1\right)^{-\alpha/\nu'}\right] \to \infty. \tag{9.17}$$

However, the other scenario of the spin-glass phase transition could also be assumed, in which it is the paramagnetic phase which "grows" from the spin-glass one as the temperature approaches the glass transition temperature T_g from below. In this scenario, as $T \to T_g$, the values of the barriers decrease at any given scale q, although the spectrum of the barriers remains divergent as $q \to q(T)$ anyway. This situation could be modeled by the ansatz (9.3) in which the exponent $\nu \to 0$ as $T \to T_g$:

$$\nu(T \to T_g) \sim \left(1-\frac{T}{T_g}\right)^{\delta} \tag{9.18}$$

where $\delta>0$. Note that there are no reasons to assume that the spin-glass order parameter $q(T)$ and the exponent ν turn to zero at the same temperature. For that reason two critical temperatures are introduced: one is T_c at which $q(T)$ turns to zero, and the other one is T_g at which the exponent ν turns to zero.

If $T_c<T_g$, then, as the temperature increases, the spin-glass order parameter turns to zero first, and it is T_c which would be the point of the phase transition into the paramagnetic phase in the usual thermodynamic sense ($q \neq 0$ at $T<T_c$ and $q=0$ at $T>T_c$). Nevertheless, although in the temperature interval $T_c<T<T_g$ the spin-glass order parameter is zero, the exponent ν is still nonzero, and therefore the relaxation properties of the system in this temperature interval would not be paramagnetic.

The relaxation behavior of the order parameter in this case can be easily estimated from the integral (9.8) in which one has to take $q(T)=0$, $t \gg \tau_0$ and $\nu \ll 1$. As the temperature T_g is approached from below the relaxation would be paramagnetic $\sim \exp(-t/\tau_0)$ only at times which are not very large: $\tau_0 \ll t \ll \tau^*(T)$. The saddle-point estimate of the integral in Eq. (9.8) gives:

$$\tau^*(T) \sim \frac{1}{\nu} \sim \left(1-\frac{T}{T_g}\right)^{-\delta} \tag{9.19}$$

so that $\tau^*(T) \to \infty$ as $T \to T_g$. However, at the largest times $t \gg \tau^*(T)$ the relaxation becomes logarithmically slow (spin-glass-like):

$$q(t) \sim [\ln(\nu t)]^{-1/\nu}. \tag{9.20}$$

As $T \to T_g$, the times at which this type of the relaxation could be observed are shifting to infinity.

10. CONCLUDING REMARKS

The physics of the spin-glass state discussed in the present review is just an attempt to give a rather qualitative (and as simple as possible) description of a new area of statistical mechanics which deals with systems in which quenched disorder is the dominant factor. As for the real spin-glass materials, this qualitative physics is still hypothetical rather than well-grounded.

It should be stressed that the phenomenon of the spin-glass state appears to be a quite general one. At present it is observed in all sorts of systems, which are very far from the original magnets with random interactions. In particular, it appears to be the crucial point for statistical modeling of biological evolution, for statistical memory models (neural networks), and for optimization problems, etc. (Presumably, the problem of $1/f$-noise is also connected with some sort of spin-glass effects). Unfortunately, although the original problem of spin glasses has successfully developed into an entire branched tree of problems (this phenomenon seems to be its intrinsic property), a general understanding of the physics of the spin-glass state is still far from being complete.

The problem is that if the physics of the spin-glass state claims to be really a *new* physics (which it really does), it must be adequately formulated. It must be something more than just drawing branching trees and fractal landscapes in infinite-dimensional spaces. It must be a system of selfconsistent algorithms which would make it possible, at least in principle, to calculate observable quantities and to make predictions. Then it will become a *science*. Until now, however, it is just a little bit above the level of the so-called "descriptive zoology." For the time being it is not so bad, of course, but this situation can not be called satisfactory either.

[1)]This term seems to be quite adequate in its meaning, since the triangle discussed above might as well be interpreted as the famous love triangle. In addition, the existence of frustrations in spin glasses destroys any hope, as we will see later, for finding a simple solution of the problem. This term has been introduced by G. Toulouse.[3]

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The English text was provided by the Author. It differs in many respects from the Russian text published in Usp. Fiz. Nauk and should be regarded as a translation combined with a revision. The Translation Editor made only stylistic changes.—Translation Editor