Zernike polynomials and atmospheric turbulence*

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This paper discusses some general properties of Zernike polynomials, such as their Fourier transforms, integral representations, and derivatives. A Zernike representation of the Kolmogoroff spectrum of turbulence is given that provides a complete analytical description of the number of independent corrections required in a wave-front compensation system.

INTRODUCTION

The use of Zernike polynomials for describing the classical aberrations of an optical system is well known.[1] Fried[2] used a form of these polynomials to describe the statistical strength of aberrations produced by atmospheric turbulence, while Bradley and Herrmann[3] described atmospheric thermal blooming effects. Bezdid'ko[4] has discussed the advantages of Zernike polynomials in solving many optical problems.

In this paper, a review of Zernike polynomials is undertaken with an emphasis on nomenclature. Some new Zernike polynomial properties such as integral representations and derivatives are discussed. Finally, the work of Fried[2] is extended by developing a Zernike representation of the Kolmogoroff spectrum of turbulence, which permits all the statistical aberration strengths to be calculated analytically.

ZERNIKE POLYNOMIALS

Zernike polynomials are a set of polynomials defined on a unit circle. It is convenient to use polar coordinates so that the polynomials are a product of angular functions and radial polynomials. The angular functions are the basis functions for the two-dimensional rotation group, and the radial polynomials are developed from the well known Jacobi polynomials.[1] The polynomials used in this paper are slightly different than the usual set[1] in that a different normalization is used. The normalization chosen is convenient for statistical analysis. Because of this normalization difference, the polynomials used in this paper are technically a modified set of Zernike polynomials. For convenience, in this paper the modified Zernike polynomials are simply called Zernike polynomials. The polynomials are defined here by

$$\left.\begin{aligned} Z_{\text{even }j} &= \sqrt{n+1}\,R_n^m(r)\sqrt{2}\cos m\theta \\ Z_{\text{odd }j} &= \sqrt{n+1}\,R_n^m(r)\sqrt{2}\sin m\theta \end{aligned}\right\} \quad m \neq 0 \tag{1}$$

$$Z_j = \sqrt{n+1}\,R_n^0(r)\,, \qquad m = 0$$

where

$$R_n^m(r) = \sum_{s=0}^{(n-m)/2} \frac{(-1)^s (n-s)!}{s!\,[(n+m)/2-s]!\,[(n-m)/2-s]!}\, r^{n-2s}\,. \tag{2}$$

The values of n and m are always integral and satisfy $m \le n$, $n - |m| = \text{even}$. The index j is a mode ordering number and is a function of n and m. A convenient ordering of the modes is shown in Table I. The definition in Eq. (1) is convenient because it gives a logical ordering to the modes and allows the modal orthogonality relation to be written

$$\int d^2r\, W(r) Z_j Z_{j'} = \delta_{jj'}\,, \tag{3}$$

where[5]

$$W(r) = 1/\pi \qquad r \le 1$$
$$\quad = 0 \qquad r > 1\,.$$

Typical interest in Zernike polynomials centers around a polynomial expansion of an arbitrary wave front over a circular aperture of arbitrary radius (R). Thus, if $\phi(r, \theta)$ is some arbitrary function, its polynomial expansion over a circle of radius (R) is given by

$$\phi(R\rho, \theta) = \sum_j a_j Z_j(\rho, \theta)\,, \tag{4}$$

with $\rho = r/R$ and the coefficients a_j being given by

$$a_j = \int d^2\rho\, W(\rho)\phi(R\rho, \theta) Z_j(\rho, \theta) \tag{5}$$

or

$$a_j = (1/R^2)\int d^2r\, W(r/R)\phi(r, \theta) Z_j(r/R, \theta)\,. \tag{6}$$

The first few polynomials are shown in Table I along with the classical aberration with which they are associated.[1]

PROPERTIES OF ZERNIKE POLYNOMIALS

Let $Q_j(k, \phi)$ be the Fourier transform of $Z_j(\rho, \theta)$ so that

$$W(\rho) Z_j(\rho, \theta) = \int d^2k\, Q_j(k, \phi) e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\rho}}\,. \tag{7}$$

The transform $Q_j(k, \phi)$ can be written[1] from Eq. (1) as

$$\left.\begin{aligned} Q_{\text{even }j}(k, \phi) &= \\ Q_{\text{odd }j}(k, \phi) &= \\ Q_j(k, \phi) &= \end{aligned}\right\} \sqrt{n+1}\,\frac{J_{n+1}(2\pi k)}{\pi k} \left\{\begin{aligned} &(-1)^{(n-m)/2} i^m \sqrt{2}\cos m\phi\,, \\ &(-1)^{(n-m)/2} i^m \sqrt{2}\sin m\phi\,, \\ &(-1)^{n/2}\,, \quad (m=0) \end{aligned}\right. \tag{8}$$

where $J_l(x)$ is the lth order Bessel function of the first kind. If Eq. (8) is substituted back into Eq. (7), an integral representation for the radial function R_n^m is found to be

$$R_n^m(\rho) = 2\pi(-1)^{(n-m)/2}\int_0^\infty dk\, J_{n+1}(2\pi k) J_m(2\pi k\rho)\,. \tag{9}$$

ZERNIKE DERIVATIVES

The integral representation for the function $R_n^m(\rho)$ provides a good starting point for calculating deriva-

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TABLE I. Zernike polynomials. The modes, Z_j, are ordered such that even j corresponds to the symmetric modes defined by $\cos m\theta$, while odd j corresponds to the antisymmetric modes given by $\sin m\theta$. For a given n, modes with a lower value of m are ordered first.

Radial degree (n)	Azimuthal frequency (m) 0	1	2	3	4	5
0	$Z_1=1$ Constant					
1		$Z_2=2r\cos\theta$ $Z_3=2r\sin\theta$ Tilts (Lateral position)				
2	$Z_4=\sqrt{3}(2r^2-1)$ Defocus (Longitudinal position)		$Z_5=\sqrt{6}r^2\sin 2\theta$ $Z_6=\sqrt{6}r^2\cos 2\theta$ Astigmatism (3rd Order)			
3		$Z_7=\sqrt{8}(3r^3-2r)\sin\theta$ $Z_8=\sqrt{8}(3r^3-2r)\cos\theta$ Coma (3rd order)		$Z_9=\sqrt{8}r^3\sin 3\theta$ $Z_{10}=\sqrt{8}r^3\cos 3\theta$		
4	$Z_{11}=\sqrt{5}(6r^4-6r^2+1)$ 3rd order spherical		$Z_{12}=\sqrt{10}(4r^4-3r^2)\cos 2\theta$ $Z_{13}=\sqrt{10}(4r^4-3r^2)\sin 2\theta$		$Z_{14}=\sqrt{10}r^4\cos 4\theta$ $Z_{15}=\sqrt{10}r^4\sin 4\theta$	
5		$Z_{16}=\sqrt{12}(10r^5-12r^3+3r)\cos\theta$ $Z_{17}=\sqrt{12}(10r^5-12r^3+3r)\sin\theta$		$Z_{18}=\sqrt{12}(5r^5-4r^3)\cos 3\theta$ $Z_{19}=\sqrt{12}(5r^5-4r^3)\sin 3\theta$		$Z_{20}=\sqrt{12}r^5\cos 5\theta$ $Z_{21}=\sqrt{12}r^5\sin 5\theta$
6	$Z_{22}=\sqrt{7}(20r^6-30r^4+12r^2-1)$ 5th order spherical		Z_{23} Z_{24}		Z_{25} Z_{26}	

tives. Derivatives of the radial function can be written

$$(d/d\rho)R_n^m = 2\pi(-1)^{(n-m)/2} \times \int_0^\infty dk\, J_{n+1}(2\pi k)\frac{dJ_m(2\pi k\rho)}{d\rho} . \quad (10)$$

By using the identities

$$dJ_l/dx = \tfrac{1}{2}[J_{l-1}(x) - J_{l+1}(x)] \quad (11)$$

and

$$xJ_{l+1}(x) = 2lJ_l(x) - xJ_{l-1}(x) , \quad (12)$$

Eq. (10) can be written as a recursion relation

$$(d/d\rho)R_n^m = n[R_{n-1}^{m-1} + R_{n-1}^{m+1}] + (d/d\rho)R_{n-2}^m . \quad (13)$$

The recursion relation in Eq. (13) provides the prescription for representing derivatives of Zernike polynomials as a linear combination of Zernike polynomials, i.e.,

$$\nabla Z_j = \sum_{j'} \gamma_{jj'} Z_{j'} . \quad (14)$$

The matrix $\boldsymbol{\gamma}$ is most easily expressed in rectangular coordinates so that

$$\gamma_{jj'}^x = \int d^2\rho\, Z_{j'} \frac{dZ_j}{dx} \quad (15)$$

and

$$\gamma_{jj'}^y = \int d^2\rho\, Z_{j'} \frac{dZ_j}{dy} . \quad (16)$$

The matrix elements $\gamma_{jj'}^x$ and $\gamma_{jj'}^y$ [given by Eqs. (15) and (16) and displayed in Tables II and III] can be constructed with the following rules.

γ^x:

a. All magnitudes are given by
$\sqrt{(n+1)(n'+1)}$ for m and $m' \neq 0$,
$\sqrt{2}\sqrt{(n+1)(n'+1)}$ for m or $m'=0$.

b. The nonzero elements are for j and j' either both even or both odd except for m or $m'=0$. When m or $m'=0$, only even j or j' gives a nonzero result.

c. For a particular m, only $m'=m\pm 1$ gives nonzero matrix elements.

d. All matrix elements are positive.

γ^y:

a. All magnitudes are the same as γ^x.

b. The nonzero elements are for j and j' either even/odd or odd/even except for m or $m'=0$. When m or $m'=0$, only odd j or j' gives a nonzero result.

c. Only $m'=m\pm 1$ gives nonzero results.

d. All m and $m'=0$ elements are (+).
Elements with $m'=m+1$ and odd j are (−).
Elements with $m'=m-1$ and even j are (−).
All other elements are (+).

ATMOSPHERIC STATISTICS

The propagation of a wave through the atmosphere has been well discussed, and the structure function for the phase fluctuations is defined as

$$D_\phi(r) = 2[\langle\phi^2(r_1)\rangle - \langle\phi(r_1)\phi(r_1+r)\rangle] . \quad (17)$$

For Kolmogoroff turbulence $D_\phi(r)$ can be written in

TABLE II. Zernike polynomial derivative matrix: $\gamma^x_{jj'}$.

m	j \ m': j':	0 1	1 2	1 3	0 4	2 5	2 6	1 7	1 8	3 9	3 10	0 11	2 12	2 13	4 14	4 15
0	1															
1	2	2.0														
1	3															
0	4		$2\sqrt{3}$													
2	5			$\sqrt{6}$												
2	6		$\sqrt{6}$													
1	7					$2\sqrt{3}$										
1	8	$\sqrt{8}$			$2\sqrt{6}$		$2\sqrt{3}$									
3	9					$2\sqrt{3}$										
3	10						$2\sqrt{3}$									
0	11		$2\sqrt{5}$						$2\sqrt{10}$							
2	12		$\sqrt{10}$						$2\sqrt{5}$		$2\sqrt{5}$					
2	13			$\sqrt{10}$				$2\sqrt{5}$		$2\sqrt{5}$						
4	14										$2\sqrt{5}$					
4	15									$2\sqrt{5}$						
1	16	$\sqrt{12}$			6.0		$3\sqrt{2}$					$2\sqrt{15}$	$\sqrt{30}$			
1	17					$3\sqrt{2}$								$\sqrt{30}$		
3	18						$3\sqrt{2}$						$\sqrt{30}$			
3	19					$3\sqrt{2}$								$\sqrt{30}$		$\sqrt{30}$
5	20														$\sqrt{30}$	
5	21															$\sqrt{30}$

terms of the correlation length (r_0) introduced by Fried[2] as

$$D_\phi(r) = 6.88(r/r_0)^{5/3}\,. \tag{18}$$

The structure function is related to the Weiner spectrum, $\Phi(\mathbf{k})$, by

$$D_\phi(r) = 2\int d\mathbf{k}\,\Phi(\mathbf{k})[1-\cos(2\pi\mathbf{k}\cdot\mathbf{r})]\,. \tag{19}$$

By using Eq. (18) and the integral

$$\int_0^\infty x^{-P}[1-J_0(bx)]dx = \frac{\pi b^{P-1}}{2^P[\Gamma(P+1)/2]^2\sin[\pi(P-1)/2]}\,, \tag{20}$$

we find that

$$\Phi(k) = (0.023/r_0^{5/3})k^{-11/3}\,, \tag{21}$$

which is the Wiener spectrum of the phase fluctuations due to Kolmogoroff turbulence.

A Zernike representation of this spectrum can be ob-

TABLE III. Zernike polynomial derivative matrix: $\gamma^y_{jj'}$.

m	j \ m': j':	0 1	1 2	1 3	0 4	2 5	2 6	1 7	1 8	3 9	3 10	0 11	2 12	2 13	4 14	4 15
0	1															
1	2															
1	3	2.0														
0	4			$2\sqrt{3}$												
2	5		$\sqrt{6}$													
2	6			$-\sqrt{6}$												
1	7	$\sqrt{8}$			$2\sqrt{6}$		$-2\sqrt{3}$									
1	8					$2\sqrt{3}$										
3	9						$2\sqrt{3}$									
3	10					$-2\sqrt{3}$										
0	11			$2\sqrt{5}$				$2\sqrt{10}$								
2	12			$-\sqrt{10}$				$-2\sqrt{5}$		$2\sqrt{5}$						
2	13		$\sqrt{10}$						$2\sqrt{5}$		$-2\sqrt{5}$					
4	14									$-2\sqrt{5}$						
4	15										$2\sqrt{5}$					
1	16					$3\sqrt{2}$								$\sqrt{30}$		
1	17	$\sqrt{12}$			6.0		$-3\sqrt{2}$					$2\sqrt{15}$	$-\sqrt{30}$			
3	18					$-3\sqrt{2}$								$-\sqrt{30}$		$\sqrt{30}$
3	19						$3\sqrt{2}$						$\sqrt{30}$		$-\sqrt{30}$	
5	20															$-\sqrt{30}$
5	21														$\sqrt{30}$	

tained by evaluating the covariance of the expansion coefficients in Eq. (4). The coefficients a_j can be considered to be Gaussian random variables with zero mean so that the covariance is, from Eq. (5),

$$\langle a_j^* a_{j'} \rangle = \int d\rho \int d\rho' W(\rho) W(\rho') Z_j(\rho, \theta) \times C(R\rho, R\rho') Z_{j'}(\rho', \theta') , \quad (22)$$

where $C(R\rho, R\rho')$ is the phase covariance function

$$C(R\rho, R\rho') = \langle \phi(R\rho) \phi(R\rho') \rangle . \quad (23)$$

Equation (22) can also be written in Fourier space as

$$\langle a_j^* a_{j'} \rangle = \iint d\mathbf{k} d\mathbf{k}' Q_j^*(k) \Phi(k/R, k'/R) Q_{j'}(k') , \quad (24)$$

where

$$\Phi(k/R, k'/R) = 0.023(R/r_0)^{5/3} k^{-11/3} \delta(k - k') .$$

Substituting Eq. (8) into Eq. (24) yields

$$\langle a_j^* a_{j'} \rangle = (0.046/\pi)(R/r_0)^{5/3} [(n+1)(n'+1)]^{1/2} \times (-1)^{(n+n'-2m)/2} \delta_{mm'} \times \int dk\, k^{-8/3} \frac{J_{n+1}(2\pi k) J_{n'+1}(2\pi k)}{k^2} , \quad (25)$$

which is a Zernike matrix representation of the Kolmogoroff phase spectrum. This representation has the advantage that the integrals that appear in Eq. (25) can be evaluated in closed form (see Appendix).

DEGREES OF CORRECTION

If the lowest order aberrations in the random wave front are corrected, one is interested in knowing how much wave-front distortion remains. This question is easily addressed with Zernike polynomials. If the first J modes are corrected, the correction can be written

$$\phi_C = \sum_{j=1}^{J} a_j Z_j . \quad (26)$$

The mean square residual error can be defined as

$$\Delta = \int d\rho\, W(\rho) \langle [\phi(R\rho) - \phi_C(R\rho)]^2 \rangle . \quad (27)$$

Substituting Eq. (26) into (27) and remembering that $\langle a_j \rangle = 0$ yields

$$\Delta_J = \langle \phi^2 \rangle - \sum_{j=1}^{J} \langle |a_j|^2 \rangle , \quad (28)$$

where $\langle \phi^2 \rangle$ is the phase variance, which is infinite for the Kolmogoroff spectrum. This infinity is contained solely in the piston mode (see Appendix) of the spectrum so that Δ_1 is finite. The first few values of Δ_J are shown in Table IV. When J is large ($J > 10$), Eq. (28) can be approximated by[6]

$$\Delta_J \approx 0.2944 J^{-\sqrt{3}/2} (D/r_0)^{5/3} [\mathrm{rad}^2] . \quad (29)$$

Fried[2] calculated the first few values of Δ by a very laborious technique. His results compare with those in Table IV. The advantage of the Fourier representation of the Zernike polynomials is therefore the ease by which all the Δ_J can be calculated.

TABLE IV. Zernike-Kolmogoroff residual errors (Δ_J). (D is the aperture diameter.)

$\Delta_1 = 1.0299\ (D/r_0)^{5/3}$	$\Delta_{12} = 0.0352\ (D/r_0)^{5/3}$
$\Delta_2 = 0.582\ (D/r_0)^{5/3}$	$\Delta_{13} = 0.0328\ (D/r_0)^{5/3}$
$\Delta_3 = 0.134\ (D/r_0)^{5/3}$	$\Delta_{14} = 0.0304\ (D/r_0)^{5/3}$
$\Delta_4 = 0.111\ (D/r_0)^{5/3}$	$\Delta_{15} = 0.0279\ (D/r_0)^{5/3}$
$\Delta_5 = 0.0880\ (D/r_0)^{5/3}$	$\Delta_{16} = 0.0267\ (D/r_0)^{5/3}$
$\Delta_6 = 0.0648\ (D/r_0)^{5/3}$	$\Delta_{17} = 0.0255\ (D/r_0)^{5/3}$
$\Delta_7 = 0.0587\ (D/r_0)^{5/3}$	$\Delta_{18} = 0.0243\ (D/r_0)^{5/3}$
$\Delta_8 = 0.0525\ (D/r_0)^{5/3}$	$\Delta_{19} = 0.0232\ (D/r_0)^{5/3}$
$\Delta_9 = 0.0463\ (D/r_0)^{5/3}$	$\Delta_{20} = 0.0220\ (D/r_0)^{5/3}$
$\Delta_{10} = 0.0401\ (D/r_0)^{5/3}$	$\Delta_{21} = 0.0208\ (D/r_0)^{5/3}$
$\Delta_{11} = 0.0377\ (D/r_0)^{5/3}$	
$\Delta_J \sim 0.2944 J^{-\sqrt{3}/2} (D/r_0)^{5/3}$ (For large J)	

CONCLUSION

The properties of Zernike polynomials have been reviewed. In particular, rules for computing the derivatives of these polynomials as a linear combination of the polynomials themselves have been given.

Derivatives of Zernike polynomials can be useful whenever the gradient of a wave front is required. Wave-front gradients occur in some geometrical optics problems as well as direct measurements in an electronic Hartmann Test.[7]

An application of Zernike polynomials to the problem of atmospheric wave-front correction is discussed. It is found that the Zernike polynomials permit an analytic evaluation of the residual wave-front error for any number of independent corrections.

In general, the optimum correction would be obtained from a set of orthonormal functions that make the matrix defined by Eq. (22) a diagonal matrix. These functions are eigenfunctions of the covariance matrix and constitute the basis for a Karhunen-Loève expansion of the wave front. For the Kolmogoroff spectrum of turbulence, the Karhunen-Loève functions are not analytic functions. The advantage of Zernike polynomials as a basis is not only that results can be obtained in closed form, but also that the first few modes represent the classical aberrations familiar to opticians. Comparison of the Zernike with a Karhunen-Loève expansion[8] suggests that the Zernike expansion is near optimum.

APPENDIX

Evaluation of the integral $I_{nn'}$

In this section the integral in Eq. (25), $I_{nn'}$, is evaluated:

$$I_{nn'} = \int_0^\infty dk\, k^{-8/3} \frac{J_{n+1}(k) J_{n'+1}(k)}{k^2} . \quad (A1)$$

This integral is tabulated in most standard integral table handbooks of Bessel function integrals:

$$I_{nn'} = \frac{\Gamma(\frac{14}{3})\Gamma[(n+n'-\frac{14}{3}+3)/2]}{2^{14/3}\Gamma[(-n+n'+\frac{14}{3}+1)/2]\Gamma[(n-n'+\frac{14}{3}+1)/2]\Gamma[(n+n'+\frac{14}{3}+3)/2]} \quad \text{(A2)}$$

for n, $n' \neq 0$, where $\Gamma(x)$ is the gamma function.

The piston integral I_{00}, as indicated in the text, diverges. The singularity in the piston integral exactly matches the singularity in the variance. This indicates that the piston corrected variance Δ_1 is finite. To evaluate Δ_1 an integral of the form

$$\int_0^\infty \{1-[4J_1^2(x)]/x^2\}x^{-P}dx \quad \text{(A3)}$$

needs to be evaluated. To evaluate the integral in Eq. (A3), consider

$$\int_0^\infty dx\, \frac{|J_1(x)|^2}{x^q} = \frac{\Gamma[(3-q)/2]\Gamma(q)}{2^q\Gamma[(q+1)/2]\Gamma[(q+1)/2]\Gamma[(q+3)/2]} = F_1(q) \quad 0<q<3 . \quad \text{(A4)}$$

By analytic continuation of $F_1(q)$ to the domain $3<q<5$, the integral in Eq. (A3) can be written

$$\int_0^\infty \left(1-\frac{4J_1^2(x)}{x^2}\right)x^{-P}dx = \frac{\pi\Gamma(P+2)}{2^P\{\Gamma[(P+3)/2]\}^2\Gamma[(P+5)/2]\Gamma[(1+P/2]\sin[(\pi/2)(P-1)]} .$$

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[5]Defining the aperture weight function $W(r)$ as shown allows the aperture weighted variance, σ^2, of a phase function, ϕ, to be written as

$$\sigma^2 = \int d^2r W(r)\phi^2(r) .$$

[6]Although Eq. (29) has not been proven to be an asymptotic form of Eq. (28), a graph of Eq. (28) yields a linear log plot for large J. Equation (29) represents a fit to such a plot.

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Sums of independent lognormally distributed random variables

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The probability-density function of the sum of lognormally distributed random variables is studied by a method that involves the calculation of the Fourier transform of the characteristic function; this method is exact. When the number of terms in the sum is large, we employ an asymptotic series in N^{-1}, where N is the number of terms, developed by Cramer. This method is employed in order to show that the permanence of the lognormal probability-density function is a consequence of the fact that the skewness coefficient of the lognormal variables is nonzero. Finally, a simplified proof, by use of the Carleman criterion, is presented to show that the lognormal is not uniquely determined by its moments.

The lognormal probability distribution permeates much of the current literature on optical propagation through the turbulent atmosphere and has been the source of a good deal of controversy in various contexts of the general problem.

The purpose of the present paper is to report on three aspects of the lognormal probability-density function that are pertinent to the atmospheric-propagation problem. The three topics are as follows.

(i) An expression is developed for the characteristic function of the lognormal probability-density function. This expression is used to calculate the probability-density function of sums of lognormally distributed random variables.

(ii) The permanence of the lognormal probability-density function is investigated by use of an adaptation

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