Equazioni non lineari del primo ordine e metodo delle caratteristiche

Consideriamo una generica eq. alle der. parz. 1° ordine

$F(\nabla u, u, x) = 0 \qquad F(\vec{p}, u, x): \mathbb{R}^{2n+1} \to \mathbb{R}$

$F(\partial_{x_1} u, \partial_{x_2} u \dots \partial_{x_n} u, u, x)$

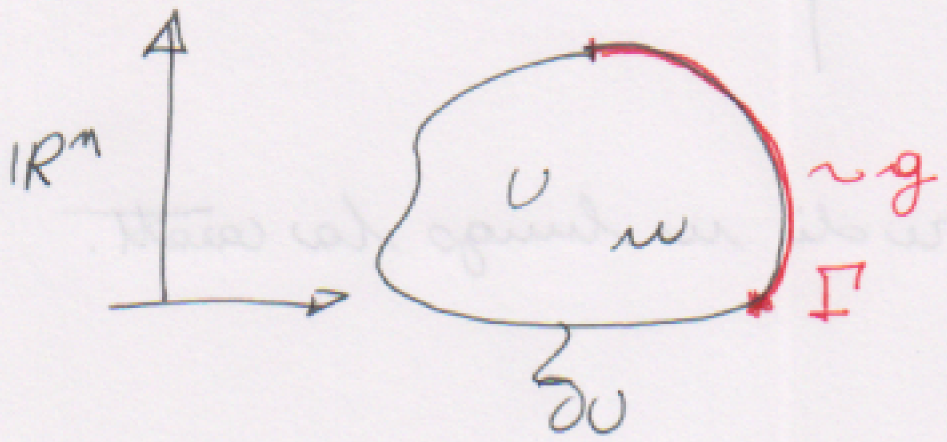
Definita in un dominio $U \subset \mathbb{R}^n$ con bordo ∂U

$F(\nabla u, u, x) = 0 \qquad x \in U$

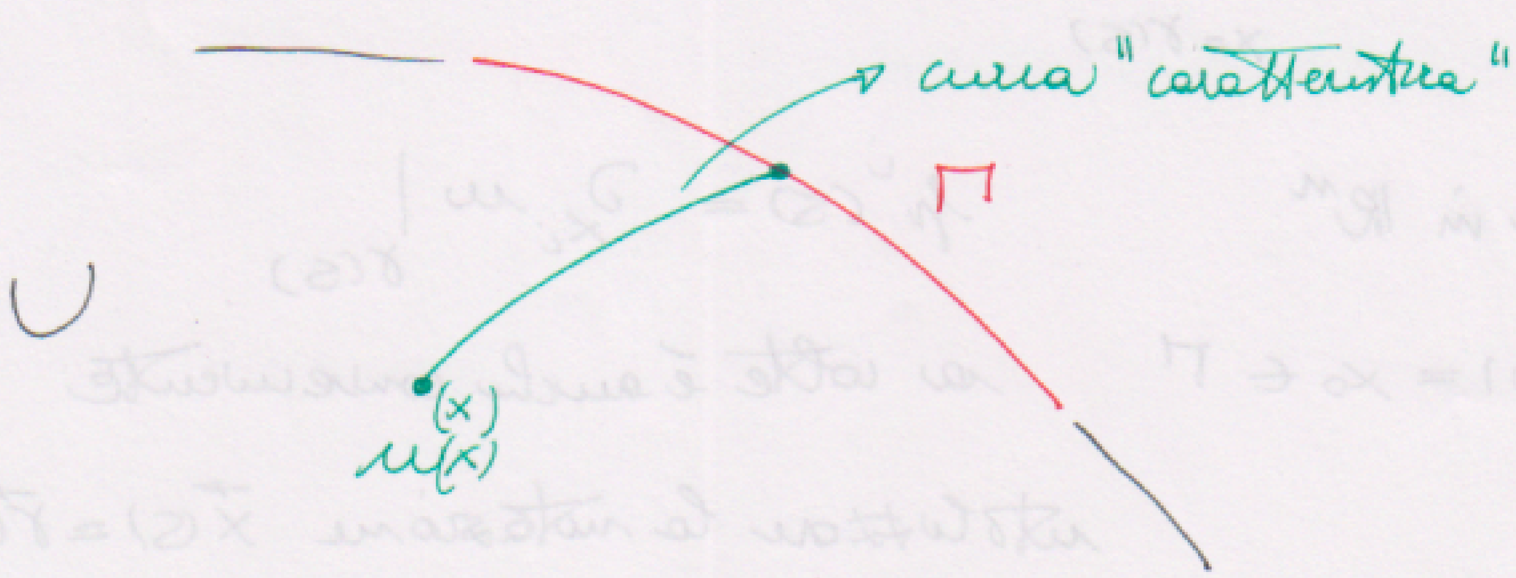
e soggetta alle cond. al contorno

$u(x) = g(x) \qquad x \in \Gamma \subseteq \partial U$

con g assegnata



Metodo delle caratteristiche: idea

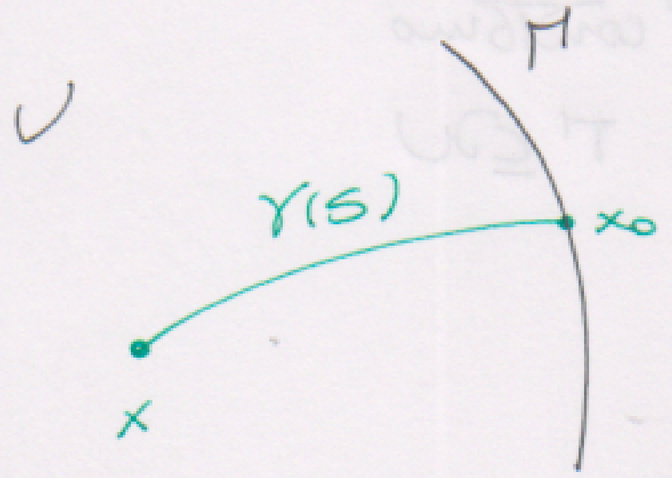


Dato un pt. generico $x \in U$ individuo una curva $\gamma(s)$ che connette x a $x_0 \in \Gamma$ lungo la quale "integro" $u(\gamma(s))$ la soluzione ristretta alla curva t.c. $\gamma(0) = x_0$ e $u(\underset{x_0}{\underset{\Downarrow}{\gamma(0)}}) = g(x_0)$

$g(x_0)$ è l'unico dato che mi serve x poter determinare $u(x)$

Ovviamente a pt. diversi del dominio corrispondono curve caratt. div. e ~~punti~~ $x_0 \in \Gamma$ diversi

Notazioni



$\gamma(s) : \mathbb{R} \to \mathbb{R}^n$

$u(\gamma(s)) = \underset{\text{scalare}}{\underset{\Downarrow}{z(s)}}$: valori di u lungo la caratt.

$\underset{\text{vettore in } \mathbb{R}^n}{\underset{\Downarrow}{\vec{p}(s)}} = \vec{\nabla}_x u \Big|_{x=\gamma(s)} = (\partial_{x_1} u(\gamma(s)), \partial_{x_2} u(\gamma(s)), \dots, \partial_{x_n} u(\gamma(s)))$

$p^i(s) = \partial_{x_i} u \Big|_{\gamma(s)}$

$\vec{\gamma}(0) = x_0 \in \Gamma$

a volte è anche conveniente utilizzare la notazione $\vec{x}(s) = \vec{\gamma}(s)$

L'eq. diff. $F(\nabla u, u, x) = 0$

lungo la curva $\gamma(s)$ si scrive

$$F(\nabla u, u, x)\Big|_{x=\gamma(s)} = F(\vec{p}(s), z(s), \underset{\substack{\downarrow\\ \text{come le } x}}{x(s)}) = 0$$

deriv. risp. a s

$$0 = \frac{d}{ds} F(\vec{p}(s), z(s), \vec{x}(s)) = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \underset{\substack{\| \\ \dot{p}_i}}{\frac{dp_i}{ds}} + \frac{\partial F}{\partial z} \underset{\substack{\| \\ \dot{z}}}{\frac{dz}{ds}} +$$

$$+ \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \underset{\substack{\| \\ \dot{x}_i}}{\frac{dx_i}{ds}} = 0$$

In forma compatta

$$\vec{\nabla}_{\vec{p}} F \cdot \dot{\vec{p}} + \frac{\partial F}{\partial z} \dot{z} + \vec{\nabla}_x F \, \dot{\vec{x}} = 0$$

$$x_i(s) = \gamma_i(s) \Rightarrow \dot{x}_i = \dot{\gamma}_i \quad \dot{\vec{x}} = \dot{\vec{\gamma}}(s)$$

$$\dot{z} = \frac{d}{ds} u(\vec{\gamma}(s)) = \sum_i \frac{\partial u}{\partial x_i}\Big|_{x=\gamma(s)} \dot{\gamma}_i = \vec{p} \cdot \dot{\vec{\gamma}}$$

otteniamo

$$\vec{\nabla}_p F \cdot \dot{\vec{p}} + \dot{\vec{\gamma}} \cdot \vec{p} \cdot \frac{\partial F}{\partial z} + \dot{\vec{\gamma}} \cdot \nabla_x F = 0$$

$$\vec{\nabla}_p F\, \dot{p} + \dot{\gamma} \cdot \left[\frac{\partial F}{\partial z} \vec{p} + \vec{\nabla}_x F \right] = 0$$

Adesso scegliamo la forma della curva γ in modo

che la prec eq. possa essere risolta.

Poniamo $\quad \dot{\gamma} = \nabla_p F(s) \iff \dot{\gamma}_i = \frac{\partial F}{\partial p_i}$

Con questa scelta l'eq. diventa

$$\dot{\gamma} \cdot \underbrace{\left[\dot{p} + \frac{\partial F}{\partial z} \vec{p} + \nabla_x F \right]}_{\overset{\text{"}}{0}} = 0$$

$$\dot{\vec{p}} = - \vec{\nabla}_x F - \vec{p} \frac{\partial F}{\partial z} \iff \dot{p}_i = -\frac{\partial F}{\partial x_i} - p_i \frac{\partial F}{\partial z}$$

Abbiamo ottenuto il sistema delle caratteristiche

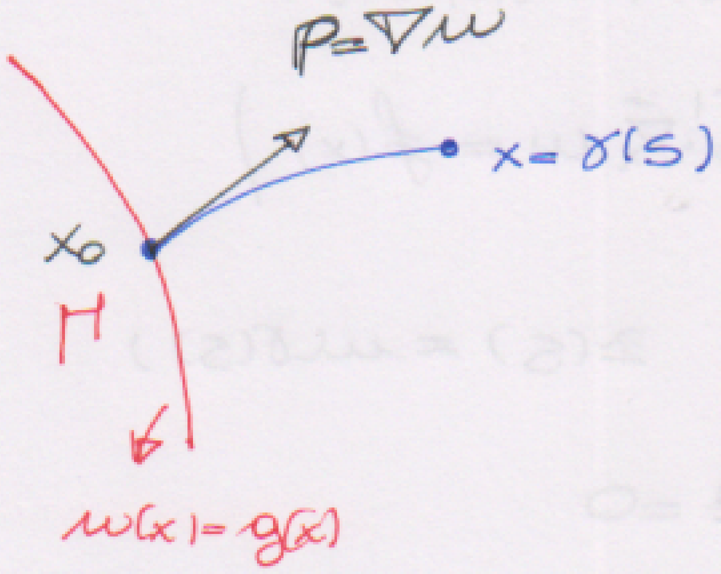
$$\text{S.C.} \begin{cases} \dot{\vec{\gamma}} = \vec{\nabla}_p F \\ \dot{\vec{p}} = -\vec{\nabla}_x F - \vec{p} \frac{\partial F}{\partial z} \\ \dot{z} = \vec{p} \cdot \vec{\nabla}_p F \end{cases} \iff \begin{cases} \dot{\gamma}_i = \frac{\partial F}{\partial p_i} \\ \dot{p}_i = -\frac{\partial F}{\partial x_i} - p_i \frac{\partial F}{\partial z} \\ \dot{z} = \sum_i p_i \frac{\partial F}{\partial p_i} \end{cases}$$

Sistema ODE per le incognite $(\vec{\gamma}, \vec{p}, z) \in \mathbb{R}^{2n+1}$

Date le opportune C.I in $s=0$ può essere integrato

per trovare la soluzione $u(x) = u(\gamma(s)) = z(s)$

Il sistema di equazioni va corredato con opportune C.I. affinché il prob. di Cauchy sia ben definito



Sia $x_0 \in \Gamma$ imponiamo $\vec{\gamma}(s=0) = x_0$

$z(s=0) = u(x_0) = g(x_0)$ e infine

$\vec{p}(s=0) = \vec{\nabla}_x u(x_0) \to$ immag. che sia noto....

Date le C.I. $\vec{\gamma}(0) = \vec{x}_0$, $z(0) = g(x_0)$, $\vec{p}(0) = \vec{\nabla}_x u(x_0)$

possiamo risolvere S.C. e trovare $(\gamma(s), z(s), p(s))$

su $\vec{x} = \gamma(s) \Rightarrow u(\vec{x}) = z(s)$

$\quad\quad\quad\quad \hookrightarrow$ sol. del problema

$$F(\nabla u, u, x) = 0$$

Punto da affrontare

1) Det. le approp. C.I. ($p(0)$)

2) Dato un punto $x \in U$ in cui vogliamo conoscere la soluzione $u(x)$; come trovare una caratt. che partendo da x_0 arriva a x?

ESEMPI

Consideriamo l'eq. del trasporto

$$F(\nabla u, u, x) = \vec{b}\cdot\vec{\nabla}_x u + c(x)\,u(x) = 0$$

(caso particolare $\partial_t u + \vec{b}'\cdot\vec{\nabla}_x u = f(x)$)

posto $\bar{p}(s) = \vec{\nabla} u(\gamma(s))$ $\qquad z(s) = u(\gamma(s))$

$$F(\vec{p}, z, x) = \vec{p}\cdot\vec{b} + c\,z = 0$$

$$\nabla_p F = \vec{b} \iff \frac{\partial F}{\partial p_i} = b_i$$

$$\nabla_x F = \cancel{0}\ \nabla_x c \qquad \frac{\partial F}{\partial z} = c(\gamma(x))$$

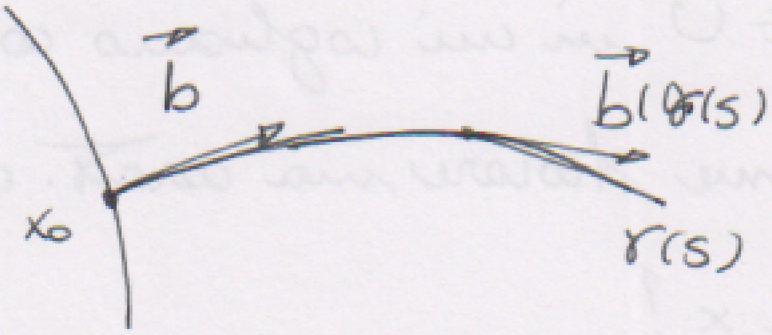
S.C. $\Rightarrow$ 1) $\dot{\gamma} = \nabla_p F = \vec{b}(\gamma(s))$

2) $\dot{z} = \vec{p}\cdot\vec{\nabla}_p F = \vec{p}\cdot\vec{b} = -c\,z$

3) $\dot{p} = -\vec{p}\cdot\dfrac{\partial F}{\partial z} - \nabla_x F = -\vec{p}\cdot c - \nabla_x c$

Notiamo che 1) e 2) sono chiuse

$$\begin{cases} \dot{\gamma}(s) = \vec{b}(\gamma(s)) \\ \dot{z}(s) = -c(\gamma(s))\cdot z(s) \end{cases}$$



caso $\vec{b} = \text{cost} = \vec{b}_0$

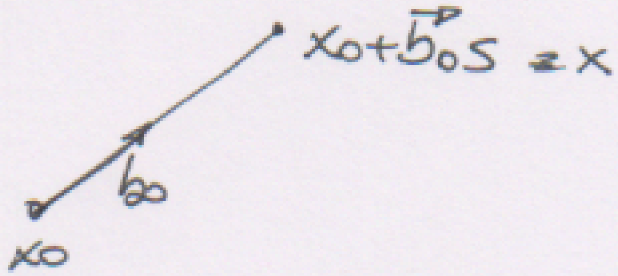
$$\dot{\gamma}(s) = \vec{b}_0 \Rightarrow \gamma(s) = x_0 + \vec{b}_0 s$$

$$\dot{z}(s) = -c(x_0 + \vec{b}_0 s)\, z(s)$$

$$z(s) = z(0)\, e^{-\int_0^s c(x_0 + b_0 s')\, ds'}$$

caso $c = 0 \Rightarrow \dot{z}(s) = 0 \Rightarrow z = z(0) = u(x_0)$

$$z(s) = u(\gamma(s)) = u(x_0 + \vec{b}_0 s) = u(x_0) = u_0(x_0)$$



detto $x = x_0 + \vec{b}_0 s$ $\quad u(x) = u_0(\bar{x} - \bar{b}_0 s)$

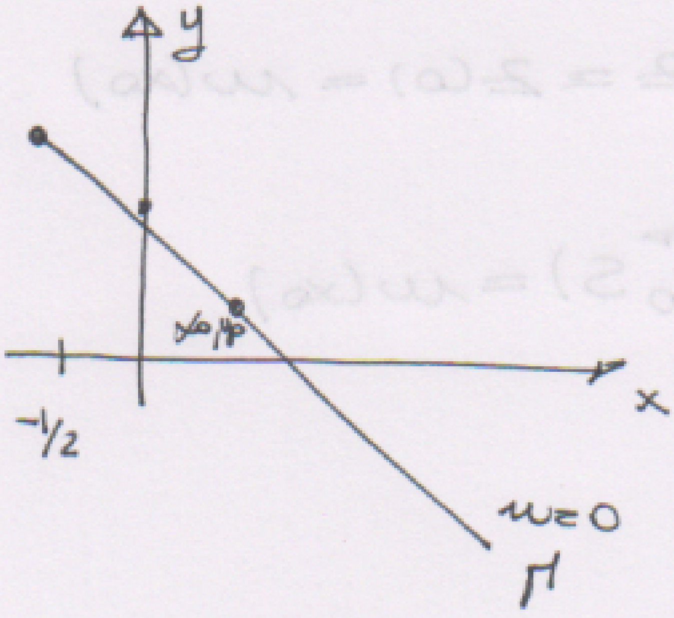
Soluz. di $\vec{b}_0 \cdot \vec{\nabla}_x u = 0$

Es. in $\mathbb{R}^2$

$$\partial_x u + 2x\,\partial_y u = y$$

c.c. $\quad u(x,y)=0 \qquad (x,y)\in\Gamma$

$\Gamma: \ y = 1-x \quad x > -1/2$



$\vec{p} = (\partial_x u, \partial_y u) \qquad F(p,z,x) = p_x + 2x\,p_y - y = 0$

$\nabla_p F = (1, 2x) \qquad \nabla_x F = (2p_y, -1)$

$$\dot{\gamma} = \nabla_p F(\gamma(s)) \iff \begin{cases} \dot{x}(s) = 1 \Rightarrow x = s + x_0 \\ \dot{y}(s) = 2x(s) \Rightarrow \dot{y} = 2(s+x_0) \end{cases}$$

$y_0 = 1-x_0$

$$y = 2\frac{s^2}{2} + 2x_0 s + y_0 = s^2 + 2x_0 s + 1 - x_0$$

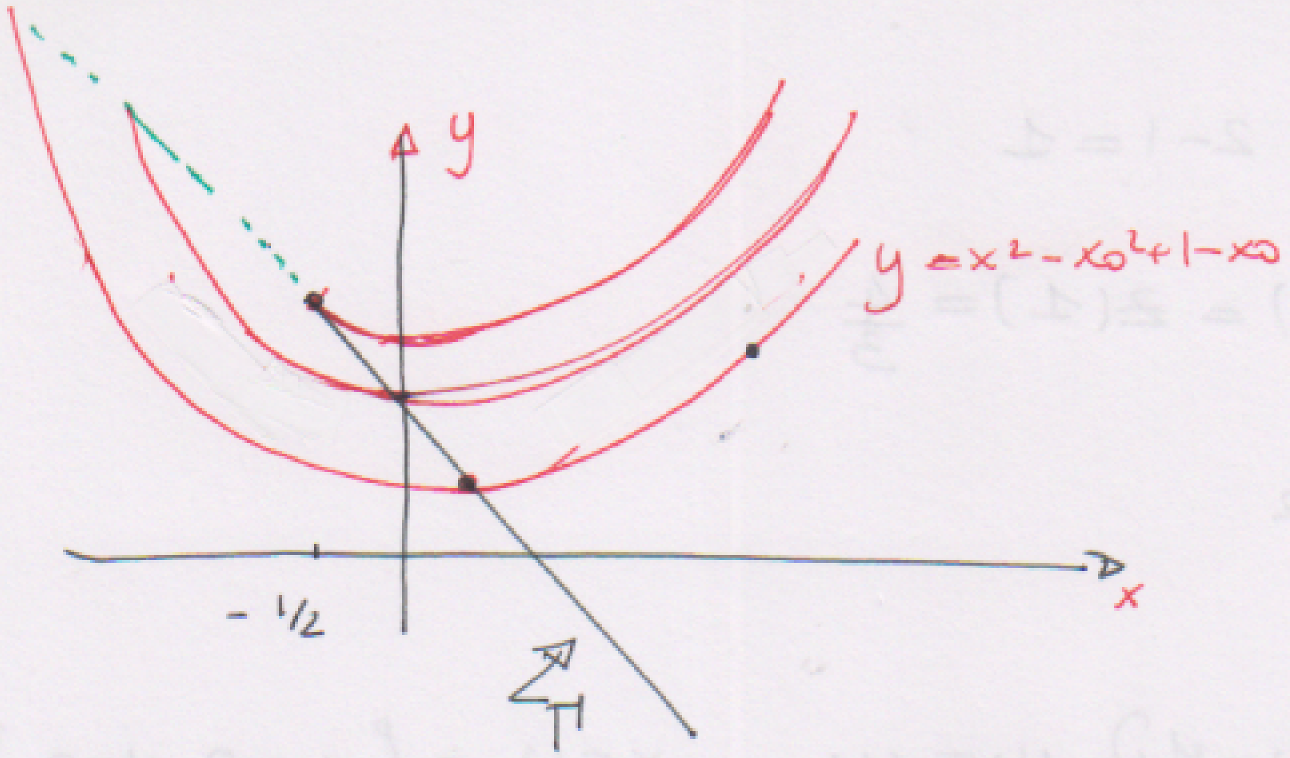
$s = x - x_0$

$$y = s^2 + 2x_0 s + 1 - x_0 = (x-x_0)^2 + 2x_0(x-x_0) + 1 - x_0$$

$$= x^2 + x_0^2 - \cancel{2x_0 x} + \cancel{2x_0 x} - 2x_0^2 + 1 - x_0$$

$$= x^2 - x_0^2 + 1 - x_0$$

caratt. parabole



Nota: le caratt. hanno 1 sola intersezione con Γ

in $x=-1/2$ la caratt è tg a Γ

Eq. per $z(s)$

$$\dot{z}(s) = \vec{p}\,\nabla_p \vec{F} = p_x + 2x\,p_y \overset{eq.}{=} y$$

$$\dot{z} = y(s) = s^2 + x_0 2s + 1 - x_0$$

$$\dot{z} = \frac{s^3}{3} + s^2 x_0 + (1-x_0)s \qquad z(0)=0$$

$u(x(s), y(s)) = z(s)$

Se voglio calcolare il valore di u in un punto (x,y)

devo ricavare x_0 e s in funzione di x e y

$$\begin{cases} -x_0^2 - x_0 + 1 + x^2 - y = 0 \\ s = x - x_0 \end{cases}$$

Es: $(x,y) = (2,+3)$

$-x_0^2 - x_0 + 1 + 4 - 3 = 0 \qquad x_0 = 1$

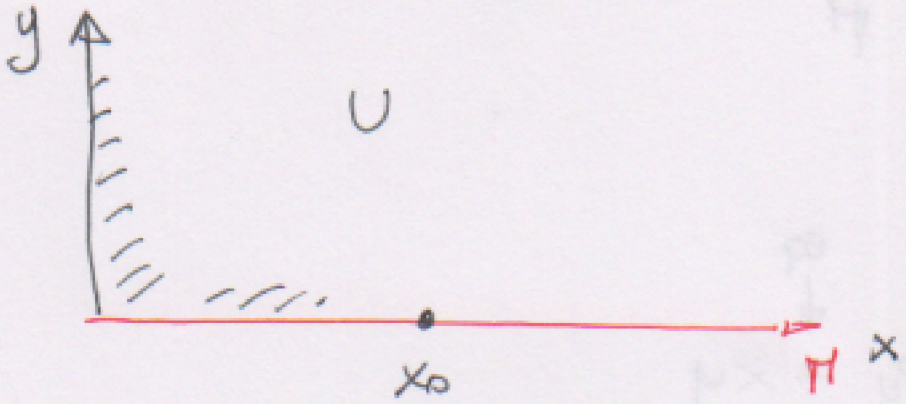
$x_0 = -2 \Rightarrow$ no $x_0 > -1/2$

$$s = x - x_0 = 2 - 1 = 1$$

$$w(x=2,\, y=3) = z(1) = \frac{4}{3}$$

Es. $U \subset \mathbb{R}^2$

$$\begin{cases} x\,\partial_y w - y\,\partial_x w = w & x \in U = \{x>0,\, y>0\} \\ w = g & x \in \Gamma = \{y=0\,;\, x>0\} \end{cases}$$



$$\dot{z} = \vec{p}\,\nabla_p F \qquad F = x p_y - y p_x - z = 0$$

$$\nabla_p F = (-y,\, x)^T \qquad \vec{p}\cdot\vec{\nabla_p} F = -p_x y + x p_y = z$$

$$\dot{z} = z \Rightarrow z = z(0)\, e^s = g(x_0)\, e^s$$

$$\dot{\gamma} = \nabla_p F \Rightarrow \begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \Rightarrow \ddot{x} = -\dot{y} = -x$$

$$\ddot{x} = -x \Rightarrow x = x_0 \cos(s) + v_0 \sin(s)$$

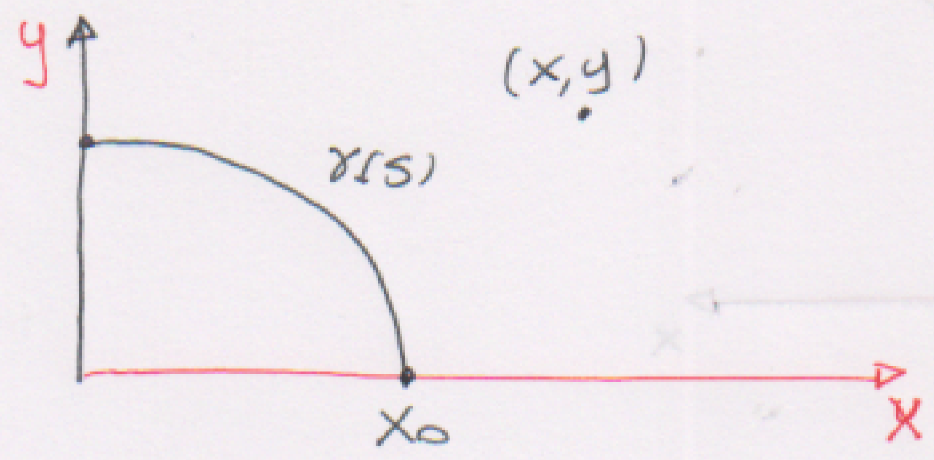
$$y = -\dot{x} = x_0 \sin(s) - v_0 \cos(s)$$

C.I: $x(0) = x_0 \quad y(0) = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x_0 \qquad\qquad -v_0 = 0$$

$\bar{\gamma}(s) = (x_0 \cos(s), x_0 \sin(s)) \rightarrow$ cerchio



Calcoliamo la sol. in un ~~pt~~ (x,y) generico

$$\Rightarrow \begin{cases} x_0 \cos(s) = x \\ x_0 \sin(s) = y \end{cases} \Rightarrow \quad x_0 = \sqrt{x^2+y^2} \qquad \operatorname{tg}(s) = \frac{y}{x} \qquad s = \operatorname{arctg}(y/x)$$

Quindi $\quad u(x,y) = z(x(s), y(s)) =$

$$= g(\sqrt{x^2+y^2})\, e^{\operatorname{arctg}(y/x)}$$

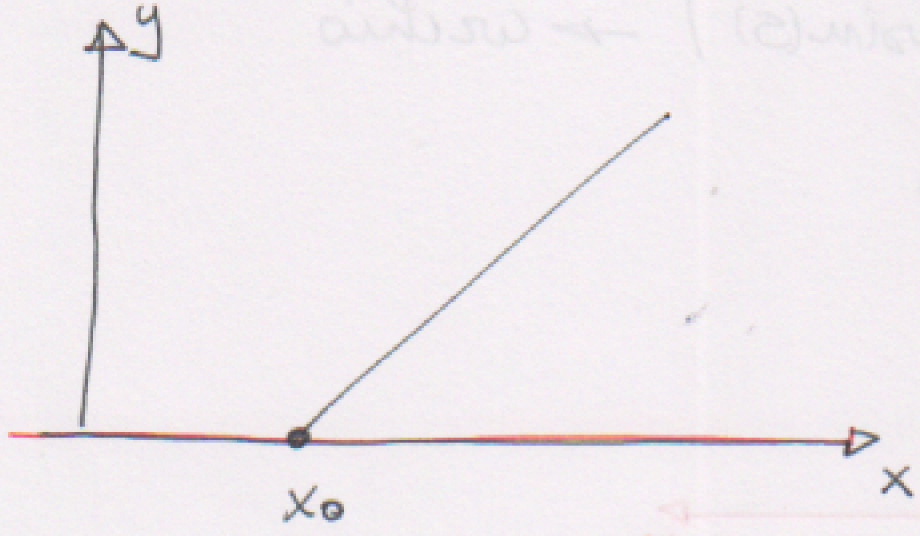
Es.

$$\begin{cases} \partial_x u + \partial_y u = u^2 & y > 0 \\ u = g & y = 0 \end{cases}$$

$$\dot{\gamma} = \nabla_p F \qquad\qquad F = p_x + p_y - z^2 \quad ; F = 0$$

$$\nabla_p F = (1, 1)$$

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \end{cases} \Rightarrow \quad \begin{matrix} x = x_0 + s \\ y = y_0 + s \rightarrow y_0 = 0 \quad y = s \end{matrix}$$



$$\dot{z} = \bar{p}\cdot\bar{\nabla}_p F = p_x + p_y = z^2$$

$$\frac{dz}{ds} = z^2 \to \int ds = \int \frac{dz}{z^2} = -\frac{1}{z} + c$$

$$-\frac{1}{z} = s - c \xRightarrow{s=0} -\frac{1}{z_0} = -c \qquad c = z_0^{-1}$$

$$z = \frac{z_0}{1-sz_0} = \frac{g(x_0)}{1-sg(x_0)}$$

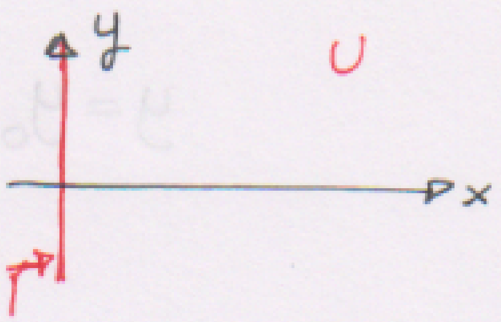
Soluz. valida finché $1-sg(x_0) \neq 0$

Inverto le caratteristiche $\begin{cases} x = s + x_0 \Rightarrow x_0 = x - y \\ y = s \end{cases}$

$$u(x,y) = z(s) = \frac{g(x-y)}{1-yg(x-y)}$$

Es.

$$\begin{cases}\partial_x u \, \partial_y u = u & x > 0 \\ u = y^2 & x = 0\end{cases}$$



$$F = p_x p_y - z$$

$$\dot{p} = -p \frac{\partial F}{\partial z} - \nabla_x F = \vec{p}$$

$$\begin{cases}\dot{p}_x = p_x \\ \dot{p}_y = p_y\end{cases} \Rightarrow \quad \begin{matrix} p_x = \bar{p}_x e^s \\ p_y = \bar{p}_y e^s \end{matrix}$$

$$\dot{z} = \bar{p} \, \bar{\nabla}_p F = (p_x, p_y)(p_y, p_x) = p_x p_y + p_y p_x = 2 p_x p_y$$

$$\dot{z} = 2 \bar{p}_x \bar{p}_y e^{2s} \rightarrow z = \bar{p}_x \bar{p}_y e^{2s} + C$$

$$z(0) = \bar{p}_x \bar{p}_y + C \qquad C = -\bar{p}_x \bar{p}_y + z(0)$$

$$z(s) = \bar{p}_x \bar{p}_y (e^{2s} - 1) + z(0)$$

caratteristiche

$$\dot{\gamma} = \nabla_p F = (p_y, p_x)$$

$$\dot{x} = p_y = \bar{p}_y e^s \Rightarrow x = \bar{p}_y e^s + A$$

$$\dot{y} = p_x = \bar{p}_x e^s \qquad y = \bar{p}_x e^s + B$$

$s=0 \Rightarrow \quad x=0=\bar{p}_y+A \quad \Rightarrow A=-\bar{p}_y$

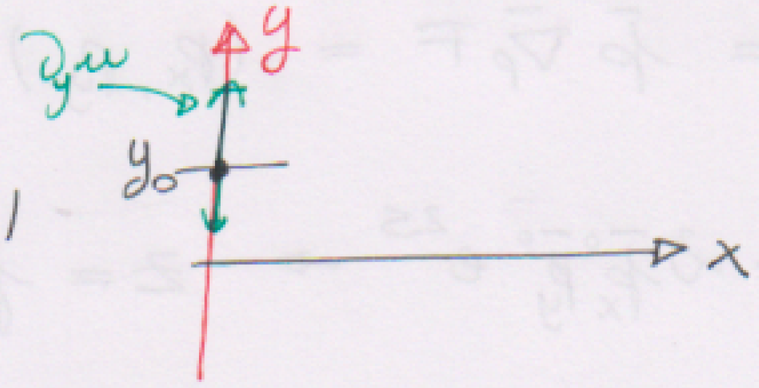
$\qquad\qquad y=y_0=\bar{p}_x+B \quad \Rightarrow B=-\bar{p}_x+y_0$

$$\begin{cases} x=\bar{p}_y\,(e^s-1) \\ y=y_0+\bar{p}_x\,(e^s-1) \end{cases} \xrightarrow{\text{sost. } e^s} \underbrace{y=y_0+\frac{\bar{p}_x}{\bar{p}_y}\,x}_{\text{rette}}$$

Determiniamo $\bar{p}_x$ e $\bar{p}_y$

$\bar{p}_x = p_x(s=0) = \partial_x u\big|_{\gamma(0)=(0,y_0)} = \frac{\partial u}{\partial x}(0,y_0)$ non lo conosco

$\bar{p}_y = p_y(s=0) = \partial_y u\big|_{(0,y_0)} = \partial_y u(0,y_0)$



$\partial_y u(0,y_0)$ richiede solo la conoscenza di u su y $\Rightarrow$ c.c.

$u\big|_{(0,y)}=y^2 \qquad \partial_y u = 2y \Rightarrow \partial_y u\big|_{(0,y_0)} = 2y_0$

$\bar{p}_y = 2y_0$ $\qquad$ Ricordiamo adesso eq. $F(p,z,x)=0$

$p_x\,p_y - z = 0$

calcoliamo in $(x,y)=(0,y_0)$

$\bar{p}_x\,\bar{p}_y - z(0) = 0$

$$\overline{P_x} \cdot 2y_0 - z(0) = 0$$

$$z(0) = u(\gamma(0)) = u(0, y_0) = y_0^2$$

$$2\overline{P_x}\, y_0 - y_0^2 = 0 \qquad \overline{P_x} = \frac{y_0}{2}$$

Abbiamo ottenuto

1) $x = 2y_0(e^s - 1)$

2) $y = y_0 + \frac{y_0}{2}(e^s - 1) = \frac{y_0}{2}(e^s + 1)$ $\Rightarrow$ caratt.

$$z = \frac{y_0}{2} 2y_0 (e^{2s} - 1) + y_0^2 = y_0^2 e^{2s} \Rightarrow \text{soluz.}$$

Inverto 1) e 2) $\Rightarrow y_0 = \frac{4y - x}{4} \qquad e^s = \frac{x + 4y}{4y - x}$

$$u(x,y) = \left(\frac{4y - x}{4}\right)^2 \left(\frac{x + 4y}{4y - x}\right)^2 = \left(\frac{x + 4y}{4}\right)^2$$