From Continuum Mechanics to Finite Element Analysis

For an Interaction of Physicists and Engineers

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Outline

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Primary and Derived Quantities and their Relationships

- Primary quantities
  - geometry (point, line, length, ...)
  - time
  - force (typically *static* and *dynamic*)
  - mass
  - temperature
  - body
- Derived quantities
  - velocity, acceleration
  - work

Two kinds of equations

- applicable to all bodies (balance equations)
  - geometric balance, or congruence
  - mechanical balance, or equilibrium
  - thermal balance (OT)
- describing the particular essence of bodies (constitutive equations, e.g. $f = k\delta$)

Both kinds are usually differential equations

they are to be carefully written

Assumptions

The *Continuous* Model

Objection	Answer
As the body is builds of atoms or particles it isn't regular	Only the continuous model can solve systematically the problem

Moreover, the continuous model [1], [2]:

- can separate the mechanical effects from other effects (e.g. magnetic, thermal)
- gives good results at a macroscopic level
- uses the classical differential/integral calculus

Outline

Definitions

Definition (Configuration)

The place of a body in a 3D space

Definition (Deformation state)

$X \in V \rightarrow X^{\star} \in V^{\star}$ according to Eq. 1

$$
\begin{aligned}
x_1 &= x_1\left(X_1, X_2, X_3\right) \\
x_2 &= x_2\left(X_1, X_2, X_3\right) \qquad (1) \\
x_3 &= x_3\left(X_1, X_2, X_3\right)
\end{aligned}
$$

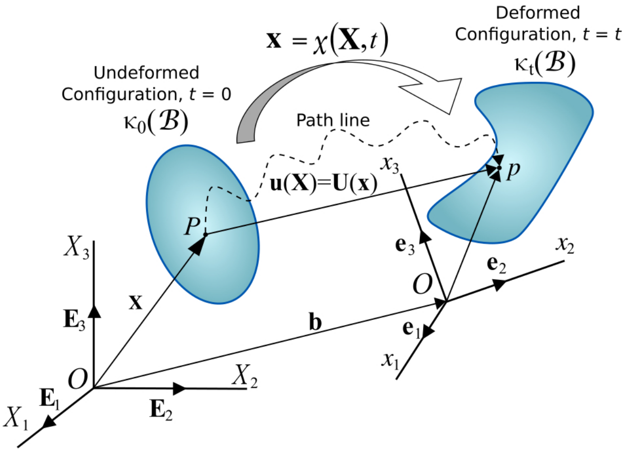


Figure 1 Credits: Wikipedia

Assumptions and Restrictions

Fact

The functions must be sufficiently regular, i.e. must be continuous along with their partial derivatives, so that e.g. no break is allowed.

Fact

Eq. 1 must be locally invertible (Eq. 2) $\rightsquigarrow$ 3D $\rightarrow$ 2D *isn't allowed.*

Fact

The deformations $u_1 = x_1 - X_1,\ u_2 = x_2 - X_2,\ u_3 = x_3 - X_3$ *must be small, i.e.* $\frac{\partial u_1}{\partial X_1}, \frac{\partial u_2}{\partial X_1}, \ldots, \frac{\partial u_3}{\partial X_3} \ll 1$, *so that the linear classic theory is applicable.*

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \neq 0 \quad (2)$$

From Motion to Deformation I

Definition

As stresses depend on deformations instead of motions, we focus on the gradient of the deformation (Eq. 3)

$$\boldsymbol{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{3}$$

Defining $P' = X_1 + dX_1, X_2 + dX_2, X_3 + dX_3$ and $P'^{\star} = x_1 + dx_1, x_2 + dx_2, x_3 + dx_3$ as two points in the undeformed and deformed states, respectively, we want to calculate $\overrightarrow{P^{\star}P'^{\star}} = [dx_1; dx_2; dx_3]$ as a function of $\overrightarrow{PP'} = [dX_1; dX_2; dX_3]$

From Motion to Deformation II

Recalling Eq. 1 we have:

$$\begin{aligned} dx_1 &= \frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3 \\ dx_2 &= \frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3 \\ dx_3 &= \frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3 \end{aligned} \tag{4}$$

or, because of Eq. 3:

$$\overrightarrow{dx} = \boldsymbol{F} \overrightarrow{dX} \tag{5}$$

An Example I

Example

What if $x_1 = aX_1^2 + bX_2, x_2 = aX_2, x_3 = aX_3$?

An Example II

From Eq. 4:

$$\begin{aligned} dx_1 &= 2a\,X_1 dX_1 + b\,dX_2 \\ dx_2 &= a\,dX_2 \\ dx_3 &= a\,dX_3 \end{aligned} \tag{6}$$

Fulfills the requests?

- is *regular* because is continuous with its derivatives
- is locally invertible (Eq. 2) if $|J| = 2a^3 X_1 \neq 0$, i.e. if $a \neq 0$ and $X_1 \neq 0$
- produces small deformations if $a \ll 1$

An Example III

Let's compute $\left|\overrightarrow{P^\star P'^\star}\right|^2 = f\left(\left|\overrightarrow{PP'}\right|^2\right)$

$$\left|\overrightarrow{PP'}\right|^2 = dX_1^2 + dX_2^2 + dX_3^2, \quad \left|\overrightarrow{P^\star P'^\star}\right|^2 = \left(\frac{\partial x_1}{\partial X_1} dX_1 + \dots\right)^2 + \left(\frac{\partial x_2}{\partial X_1} dX_1 + \dots\right)^2 + \left(\frac{\partial x_3}{\partial X_1} dX_1 + \dots\right)^2$$

$$\Downarrow$$

$$\left|\overrightarrow{P^\star P'^\star}\right|^2 = \sum_{i,j,r=1}^{3} \frac{\partial x_r}{\partial X_i} \frac{\partial x_r}{\partial X_j} dX_i dX_j \tag{7}$$

Eq. 6 gives $dx_1^2 + dx_2^2 + dx_3^2 = 4a^2 dX_1^2 + 4ab^2 dX_1 dX_2 + b^2 dX_2^2 + a^2 dX_2^2 + a^2 dX_3^2$

The Deformation Symmetry I

Recalling the displacement definition $u_1 = x_1 - X_1, u_2 = x_2 - X_2, u_3 = x_3 - X_3$, Eq. 7 becomes

$$\left|\overrightarrow{P^\star P'^\star}\right|^2 = \sum_{i,j,r=1}^{3} \left(\frac{\partial X_r}{\partial X_i} + \frac{\partial u_r}{\partial X_i}\right)\left(\frac{\partial X_r}{\partial X_j} + \frac{\partial u_r}{\partial X_j}\right) dX_i dX_j \tag{8}$$

or, neglecting the second order terms $\dfrac{\partial u_r}{\partial X_i}\dfrac{\partial u_r}{\partial X_j}$ and defining $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$:

$$\left|\overrightarrow{P^\star P'^\star}\right|^2 = \sum_{r=1}^{3} dx_r dx_r = \sum_{i,j=1}^{3} \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) dX_i dX_j \tag{9}$$

where $\delta_{ij} dX_i dX_j = dX_1^2 + dX_2^2 + dX_3^2 = \left|\overrightarrow{PP'}\right|^2$.

The Deformation Symmetry II

Theorem

Eq. 9 defines the length of $\overrightarrow{P^\star P'^\star}$ *in terms of* $\overrightarrow{PP'}$*. Such variations depend only on the symmetric components* $\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i}$*, defined as* $2\varepsilon_{ij} = 2\varepsilon_{ji} \rightsquigarrow$ *the symmetric tensor* $\boldsymbol{\varepsilon}$ *completely defines the deformations.*

The Physical Meaning of ε I

Example

If $PP' \parallel X_1$ and $P = [0;0;0]$, from Eqs. 7 and 9, as $|PP'|^2 = dX_1^2$, we have:

$$|P^\star P'^\star|^2 = \left[\left(1 + \frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] dX_1^2 \approx \left[1 + 2\frac{\partial u_1}{\partial X_1}\right] dX_1^2$$

$$|P^\star P'^\star| = \sqrt{1 + 2\frac{\partial u_1}{\partial X_1}} dX_1 \approx \left(1 + \frac{\partial u_1}{\partial X_1}\right) dX_1 = (1 + \varepsilon_{11})\, dX_1$$

Fact

$\varepsilon_{ii}\ (i = 1, 2, 3)$ *gives length variations of elements parallel to axes.*

The Physical Meaning of $\boldsymbol{\varepsilon}$ II

Example

If $PP' \parallel X_1$, $PP'' \parallel X_2$, and $P = [0; 0; 0]$, we have:

$$|PP'| = \left[\left(1 + \frac{\partial u_1}{\partial X_1}\right) dX_1; \frac{\partial u_2}{\partial X_1} dX_1; \frac{\partial u_3}{\partial X_1} dX_1\right]$$

$$|PP''| = \left[\frac{\partial u_1}{\partial X_2} dX_2; \left(1 + \frac{\partial u_2}{\partial X_2}\right) dX_2; \frac{\partial u_3}{\partial X_2} dX_2\right]$$

$$\cos(\gamma^\star) = \frac{\overrightarrow{PP'} \cdot \overrightarrow{PP''}}{|PP'||PP''|} = \frac{\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \dots\right) dX_1 dX_2}{(1 + \varepsilon_{11})\, dX_1\, (1 + \varepsilon_{22})\, dX_2} \approx 2\varepsilon_{12}$$

Fact

ε_{ij} $(i = 1, 2, 3)$ *gives angular variations of elements parallel to axes.*

The Physical Meaning of ε III

Example

If $\overrightarrow{PP'} = [dX_1; 0; 0]$, $\overrightarrow{PP''} = [0; dX_2; 0]$ and $P = [0; 0; 0]$, the undeformed area $dA = dX_1 dX_2$ and the deformed area $dA^\star = \overrightarrow{P^\star P'^\star} \times \overrightarrow{P^\star P''^\star}$, by dropping the infinitesimal terms of higher order, are related by $dA^\star = dA(1 + \varepsilon_{11} + \varepsilon_{22})$.

Fact

$\varepsilon_{11} + \varepsilon_{22}$, $\varepsilon_{22} + \varepsilon_{33}$, *and* $\varepsilon_{11} + \varepsilon_{33}$ *give the coefficients of surface expansion.*

Example

If $\overrightarrow{PP'} = [dX_1; 0; 0]$, $\overrightarrow{PP''} = [0; dX_2; 0]$, $\overrightarrow{PP'''} = [0; 0; dX_3]$, and $P = [0; 0; 0]$, the undeformed volume $dV = dX_1 dX_2 dX_3$ and the deformed volume $dV^\star = |\overrightarrow{P^\star P'^\star}||\overrightarrow{P^\star P''^\star}||\overrightarrow{P^\star P'''^\star}|$, by dropping the infinitesimal terms of higher order, are related by $dV^\star = dV(1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$.

Fact

$\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ *gives the coefficient of volume expansion.*

Principal Strains

Problem

What if the axes are rotated?

If $u_3 = 0 \rightsquigarrow u_1 = u_1 n_{11} + u_2 n_{12}$ and $\varepsilon'_{11} = \frac{\partial u'_1}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial u'_1}{\partial x_2}\frac{\partial x_2}{\partial x'_1} = \frac{\partial u'_1}{\partial x_1} n_{11} + \frac{\partial u'_1}{\partial x_2} n_{21}$.

In the most general case we have $u'_h = \sum_{i=1}^{3} u_i \, n_{ih}$ and $\varepsilon'_{hk} = \sum_{i,j=1}^{3} \varepsilon_{i,j} \, n_{ih} \, n_{ik}$.

Is there any direction along which the deformation is maximum?

Theorem

$\exists$ 3 *axes* $X_1^\star$, $X_2^\star$, $X_3^\star$, *mutually orthogonal, along which the deformation tensor is diagonal, according to Eq. 10. They are computed by solving* $\det|\boldsymbol{\varepsilon} - \lambda \, \mathbf{I}| = 0$.

$$\boldsymbol{\varepsilon}^\star = \begin{bmatrix} \varepsilon_{11}^\star & 0 & 0 \\ 0 & \varepsilon_{22}^\star & 0 \\ 0 & 0 & \varepsilon_{33}^\star \end{bmatrix} \tag{10}$$

The Congruence Equation

$u_1 = u \quad u_2 = v \quad u_3 = w \quad \boldsymbol{u} = [u; v; w] \quad X_1 = x \quad X_2 = y \quad X_3 = z$

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} \\
\varepsilon_{xy} = \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\gamma_{xy}}{2} \\
\varepsilon_{xz} = \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{\gamma_{xz}}{2} \\
\varepsilon_{yz} = \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{\gamma_{yz}}{2}
\end{aligned} \tag{11}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{12}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}\left((\boldsymbol{\nabla u}) + (\boldsymbol{\nabla u})^T\right) \tag{13}$$

Outline

Force Classification

Basic Definitions

Forces are basic quantities, defined by vectors, classified as

- external (known)
  1. contact-less forces (e.g. gravitational, magnetic) $[N \times m^{-3}]$
  2. contact forces $[N \times m^{-2}]$
- internal (unknown)
  3. contact-less forces $[N \times m^{-3}]$
  4. contact forces $[N \times m^{-2}]$

Problem

As forces # 3 are mostly negligible, the core problem of continuum mechanics is computing forces # 4 as a function of the given *external forces # 1 and # 2.*

Assumption

As we assume small displacement and deformations, all the upcoming computations are carried out in the undeformed configuration.

The Basic Assumptions

In order to characterize the internal contact forces, we need to define the equilibrium with the following assumptions

Euler's principle

An arbitrary subset V' of the volume V can be independently analyzed provided that all the forces applied by $V - V'$ on V' are considered.

Cauchy's principle

1. Internal contact forces are pressures.
2. The contact pressure $\vec{t}$ depends only on the point coordinates and the normal $\vec{n}$, i.e. $\vec{t} = \vec{t}(X_1, X_2, X_3, \vec{n})$.

Cauchy's Theorem

Theorem

If all kinds of forces are equilibrated, $\vec{t}$ depends linearly on $\vec{n}$, according to $\boldsymbol{t} = \boldsymbol{\sigma n}$ in Eq. 14, where $\sigma_{11}, \sigma_{12}, \ldots, \sigma_{33}$ are suitable coefficients.

$$\begin{aligned} t_1 &= \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 \\ t_2 &= \sigma_{21}\, n_1 + \sigma_{22}\, n_2 + \sigma_{23}\, n_3 \\ t_3 &= \sigma_{31}\, n_1 + \sigma_{32}\, n_2 + \sigma_{33}\, n_3 \end{aligned} \tag{14}$$

Cauchy's Theorem Consequences: the Translational Equilibrium I

As, given σ_{11}, σ_{12}, ..., σ_{33}, $\forall P \in V$, $\vec{t}$ can be calculated, we must relate σ_{ij} with the (known) data: the surface forces F_k and the volume f_k.
In a volume V' bounded by the surface S' the translational equilibrium equation is $\int_{S'} \vec{t}\, dS + \int_{V'} \vec{f}\, dV = 0$, which, according to Cauchy (Eq. 14), can be expressed as

$$
\begin{aligned}
\int_{S'} (\sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3)\, dS + \int_{V'} f_1 dV = 0 \\
\int_{S'} (\sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3)\, dS + \int_{V'} f_2 dV = 0 \\
\int_{S'} (\sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3)\, dS + \int_{V'} f_3 dV = 0
\end{aligned}
\tag{15}
$$

Cauchy's Theorem Consequences: the Translational Equilibrium II

According to Gauss's theorem, $\int_V (\boldsymbol{\nabla} \cdot \mathbf{G}) dV = \oint_S (\mathbf{G} \cdot \mathbf{n}) dS$, Eq. 15 can be rewritten as

$$\begin{aligned}
\int_{V'} \left(\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{12}}{\partial X_2} + \frac{\partial \sigma_{13}}{\partial X_3} + f_1 \right) dV = 0 \\
\int_{V'} \left(\frac{\partial \sigma_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \frac{\partial \sigma_{23}}{\partial X_3} + f_2 \right) dV = 0 \\
\int_{V'} \left(\frac{\partial \sigma_{31}}{\partial X_1} + \frac{\partial \sigma_{32}}{\partial X_2} + \frac{\partial \sigma_{33}}{\partial X_3} + f_3 \right) dV = 0
\end{aligned} \tag{16}$$

As V' is arbitrary, Eq. 16 is satisfied if the integrands of the integrals are null, i.e.:

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial X_i} + f_j = 0 \, ; \quad (i = 1, 2, 3) \tag{17}$$

Cauchy's Theorem Consequences: the Rotational Equilibrium

In a volume V' bounded by the surface S' the rotational equilibrium equation is $\int_{S'} \overrightarrow{OP'} \times \vec{t}\, dS + \int_{V'} \overrightarrow{OP} \times \vec{f}\, dV = 0$, which, according to Cauchy (Eq. 14), can be expressed as

$$\begin{aligned}
&\int_S \left[x'_2(n_1\sigma_{31} + n_2\sigma_{32} + n_3\sigma_{33}) - x'_3(n_1\sigma_{21} + n_2\sigma_{22} + n_3\sigma_{23})\right] dS + \int_V (f_3X_2 - f_2X_3)\, dV \\
&\int_S \left[x'_3(n_1\sigma_{11} + n_2\sigma_{12} + n_3\sigma_{13}) - x'_1(n_1\sigma_{31} + n_2\sigma_{32} + n_3\sigma_{33})\right] dS + \int_V (f_1X_3 - f_3X_1)\, dV \\
&\int_S \left[x'_1(n_1\sigma_{21} + n_2\sigma_{22} + n_3\sigma_{23}) - x'_2(n_1\sigma_{11} + n_2\sigma_{12} + n_3\sigma_{13})\right] dS + \int_V (f_2X_1 - f_1X_2)\, dV
\end{aligned} \tag{18}$$

Applying the Gauss theorem to Eq. 18, substituting Eq. 17, and developing the integral calculation, yields, because of the arbitrarity of V:

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{13} = \sigma_{31}, \quad \sigma_{23} = \sigma_{32} \quad \rightsquigarrow \quad \sigma_{ij} = \sigma_{ji} \tag{19}$$

Fact

According to the translational and rotational equilibrium equations, the stress state in a point is completely defined by the 6 quantities σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{13}, and σ_{23} $\rightsquigarrow$ $\boldsymbol{\sigma}$ *in Eq. 14 is symmetric.*

Cauchy's Theorem Outcome

Fact

- *analogy to* $\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$
- *Cauchy's equations are valid for* f_i *continuous as well for* $\frac{\partial \sigma_{ij}}{\partial X_i}$ *continuous*
- f_i *include the inertia forces*

Theorem

As with the deformation tensor, $\exists$ 3 *axes* $X_1^\star$, $X_2^\star$, $X_3^\star$, *mutually orthogonal, along which the stress tensor is diagonal (Eq. 20). They are computed by solving* $\det|\boldsymbol{\sigma} - \lambda\,\mathbf{I}| = 0$.

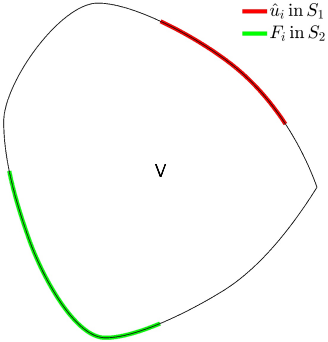
$$\boldsymbol{\sigma}^\star = \begin{bmatrix} \sigma_{11}^\star & 0 & 0 \\ 0 & \sigma_{22}^\star & 0 \\ 0 & 0 & \sigma_{33}^\star \end{bmatrix} \tag{20}$$

Outline

Do We Need More Equations? I

The problem statement

- data
  - $\vec{f}$ in V: body forces
  - $\vec{F}$ in S_2: surface forces
  - $\vec{\hat{u}}$ in S_1: surface displacements
- unknowns
  - $\vec{u}$ in V: volume displacements
  - ε_{ij} in V: volume deformations
  - σ_{ij} in V: volume stresses
- available equations (field equations)
  - $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ in V
  - $\sum_{i=1}^{3} = \frac{\partial \sigma_{ij}}{\partial X_i} + f_i = 0$ in V
- boundary conditions (needed to solve the PDE)
  - $\vec{u}_i = \vec{\hat{u}}_i$ in S_1
  - $t_i = F_i$ in S_2



Do We Need More Equations? II

The general solution

Problem

$3\,\vec{u} + 6\,\varepsilon_{ij} + 6\,\sigma_{ij} = 15\,\text{unknowns}$

$6\,\varepsilon_{ij} + 3\,\sigma_{ij} = 9\,\text{equations}$

The problem isn't solvable, unless we introduce further equations, able to link σ_{ij} *with the kinematic variables.*

Fact

We define as constitutive relationship *an equation able to* connect σ_{ij} *with displacements, deformations, coordinates, and time (displacements and deformations depend on the* history *of the material, like fatigue and work-hardening), i.e.:*
$\sigma_{ij} = f\,(u_1(\tau), u_2(\tau), u_3(\tau), \varepsilon_{ij}(\tau), X_1, X_2, X_3, t)$.

Do We Need More Equations? III

Assumptions and restrictions

The 3 axioms (Noll, 1954)

determinism the stress state of P at t is affected only by the motion history of all the points of the body for $-\infty < \tau < t$

local action the motion history of points at finite distances from P doesn't affect the constitutive relationship of P

material regardlessness the material response isn't affected by the coordinate system, i.e. the constitutive relationships don't depend on rigid rotations of the coordinate system

The 2 experimental deductions

t isn't an explicit variable

elastic materials[1]: $\sigma_{ij} = f\left(\varepsilon_{ij}, X_1, X_2, X_3\right)$

[1] Structural materials are mostly elastic; besides, many materials behave elastically if *smoothly* loaded.

A First Class of Materials I

The Cauchy's definition

A further simplification derives from assuming a linear σ–ε relationship (Cauchy, 1829), with the 36 constants c_{ijkl} in Eq. 21, so that $\sigma_{ij} = c_{ijkl}\epsilon_{kl}$. Such constants are in principle dependent only on X_h and are with a good level of approximation not dependent of the time derivative of load/unload.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & & \cdots \\ \vdots & & & \\ \vdots & & \ddots & \\ \vdots & & & \\ c_{3111} & \cdots & & c_{3131}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{bmatrix} \tag{21}$$

A First Class of Materials II

A subset of Cauchy's definition

Additionally simplifying:

- if $\dfrac{\partial c_{ijkl}}{\partial X_h} = 0$ the material is homogeneous
- given a coordinate system,

  if $u_1 \neq 0$ and $u_2 = u_3 = 0$, so that $\dfrac{\partial u_1}{\partial X_1} \neq 0 \rightsquigarrow \sigma_{11} = c_{1111}\varepsilon_{11}$,

  if $u_2 \neq 0$ and $u_1 = u_3 = 0$, so that $\dfrac{\partial u_2}{\partial X_2} \neq 0 \rightsquigarrow \sigma_{22} = c_{2222}\varepsilon_{22}$,

  and so on.

  Thus, generalizing, we have only 4 constants, i.e. the material is isotropic.

A Second Class of Materials I

Some energy considerations

The energy balance equation in a body of volume V, subject to a body force f and a surface force F in a given coordinate system, is

$$\frac{d}{dt}\int_V (W+K)\,dV = \int_V \sum_j f_j \dot{u}_j\,dV + \int_S F_j \dot{u}_j\,dS + \int_V Q\,dV \tag{22}$$

where W, K, and Q are the internal, kinetic, and heat energies, respectively. If the displacements are small (i.e. V doesn't change), considering Eq. 17, Eq. 22 becomes

$$\int_V (\dot{W}+\dot{K})\,dV = \int_V \sum_j \left(-\sum_i \frac{\partial \sigma_{ij}}{\partial X_i}\dot{u}_j\right)dV + \int_S F_j \dot{u}_j\,dS + \int_V Q\,dV \tag{23}$$

A Second Class of Materials II

Simplifying the energy equations

According to Gauss's theorem, because, from Eq. 14, $\int_S \sum_j (-\sigma_{ij} n_i)\, \dot{u}_j \, dS = \int_S \sum_j F_j \, \dot{u}_j \, dS$, naming ρ the material density, Eq. 22 becomes

$$\int_V (\dot{W} + \rho \sum_j \dot{u}_j \ddot{u}_j)\, dV = \int_V \sum_{ij} \left(\frac{\partial \sigma_{ij}}{\partial X_i} \dot{u}_j \right) dV + \int_V Q \, dV \tag{24}$$

If the process is slow ($\dot{u} \approx 0$) and adiabatic ($Q = 0$), recalling that $\varepsilon_{ij} = \dfrac{\partial u_j}{\partial X_i}$, Eq. 24 gives

$$\int_V \dot{W} \, dV = \int_V \sum_{ij} \sigma_{ij} \dot{\varepsilon}_{ij} \, dV \tag{25}$$

A Second Class of Materials III

The Green's materials

Supposing $W = W(\varepsilon_{ij})$, we have $\dot{W} = \frac{\partial W}{\partial \varepsilon_{11}}\dot{\varepsilon}_{11} + \frac{\partial W}{\partial \varepsilon_{22}}\dot{\varepsilon}_{22} + \ldots + \frac{\partial W}{\partial \varepsilon_{23}}\dot{\varepsilon}_{23}$, and, from Eq. 25,

$$\int_V \left[\left(\frac{\partial W}{\partial \varepsilon_{11}} - \sigma_{11}\right)\dot{\varepsilon}_{11} + \left(\frac{\partial W}{\partial \varepsilon_{22}} - \sigma_{22}\right)\dot{\varepsilon}_{22} + \ldots + \left(\frac{\partial W}{\partial \varepsilon_{23}} - \sigma_{23}\right)\dot{\varepsilon}_{23}\right] dV = 0 \quad (26)$$

As V in Eq. 26 is arbitrary and if $\varepsilon_{11} \neq 0,\ \varepsilon_{22}, \ldots, \varepsilon_{23} = 0 \rightsquigarrow \sigma_{11} = \frac{\partial W}{\partial \varepsilon_{11}}$, we have

$$\sigma_{11} = \frac{\partial W}{\partial \varepsilon_{11}},\ \sigma_{22} = \frac{\partial W}{\partial \varepsilon_{22}}, \ldots, \sigma_{23} = \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{23}} + \frac{\partial W}{\partial \varepsilon_{32}}\right), \ldots \quad (27)$$

Fact

W is a potential function of σ_{ij}.

Comparing Cauchy's and Green's Materials I

As Cauchy's materials are linear, according to $\sigma_{ij} = c_{ijhk}\varepsilon_{hk}$, and Green's materials depend on the potential W, according to Eq. 27, we can state:

Fact (a Green's material is also a Cauchy's material if)

- *W is a quadratic function of ε_{ij}*
- $c_{ijhk} = c_{hkij}$ [2]

[2] because $\dfrac{\partial \sigma_{11}}{\partial \varepsilon_{22}} = \dfrac{\partial^2 W}{\partial \epsilon_{22} \partial \epsilon_{11}} = \dfrac{\partial^2 W}{\partial \epsilon_{11} \partial \epsilon_{22}} = \dfrac{\partial \sigma_{22}}{\partial \varepsilon_{11}}$, where the first term is c_{1122} and the last term is c_{2211}.

Comparing Cauchy's and Green's Materials II

If a material is homogeneous and isotropic

homogeneous W doesn't depend on X_i

isotropic W depends only on the three invariants of the deformation[3]:

- $I = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$
- $II = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{33} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2$
- $III = \det \boldsymbol{\varepsilon}$

Fact (homogeneous and isotropic materials)

Cauchy's material $\cap$ Green's material $\rightsquigarrow W = W(I, II, III)$, *i.e., excluding non-rational forms and the cubic term* III,

$$W = A(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 + \frac{B}{2}(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{33} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2).$$

[3] Analytically, I, II, and III don't depend on the axis directions, but only on the coordinates of the point.

The Constitutive Relationship I

As a consequence, because of Eq. 27, the explicit form of the constitutive relationship is

$$\begin{aligned} \sigma_{11} &= 2A(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + B(\varepsilon_{22}+\varepsilon_{33}) \\ \sigma_{22} &= 2A(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + B(\varepsilon_{11}+\varepsilon_{33}) \\ \sigma_{33} &= 2A(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + B(\varepsilon_{11}+\varepsilon_{22}) \\ \sigma_{12} &= -B\varepsilon_{12} \\ \sigma_{13} &= -B\varepsilon_{13} \\ \sigma_{23} &= -B\varepsilon_{23} \end{aligned} \tag{28}$$

Naming $\mu = -\dfrac{B}{2}$ and $\lambda = 2A + B$, Eq. 28 may be rewritten in the compact form[4]

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right) \tag{29}$$

Inverting the Lamé equations (Eq. 29) gives:

[4] Eq. 29 is matricially expressed as $\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$.

The Constitutive Relationship II

$$\begin{aligned}
\varepsilon_{12} &= \frac{1}{2\mu}\sigma_{12} \\
\varepsilon_{13} &= \frac{1}{2\mu}\sigma_{13} \\
\varepsilon_{23} &= \frac{1}{2\mu}\sigma_{23} \\
\varepsilon_{11} &= \frac{1}{2\mu}\left[\sigma_{11} - \frac{\lambda}{2\mu + 3\lambda}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{22} &= \frac{1}{2\mu}\left[\sigma_{22} - \frac{\lambda}{2\mu + 3\lambda}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{33} &= \frac{1}{2\mu}\left[\sigma_{33} - \frac{\lambda}{2\mu + 3\lambda}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right]
\end{aligned} \tag{30}$$

The Constitutive Relationship III

Naming

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \qquad \text{(Young's modulus)}$$

$$\nu = \frac{\lambda}{2(\mu + \lambda)} \qquad \text{(Poisson's modulus)}$$

$$G = \mu = \frac{E}{2(1 + \nu)} \qquad \text{(Shear modulus)}$$

Eq. 30 gives Eq. 31

$$\begin{aligned}
\varepsilon_{12} &= \frac{1}{2G}\sigma_{12} \\
\varepsilon_{13} &= \frac{1}{2G}\sigma_{13} \\
\varepsilon_{23} &= \frac{1}{2G}\sigma_{23} \\
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right] \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]
\end{aligned} \tag{31}$$

The Constitutive Relationship IV

Fact (The physical meaning of μ and λ)

The Lamé equations are invertible only if $\mu \neq 0$ and $2\mu + 3\lambda \neq 0$. Such a condition comes from the experience — there isn't any material which exhibits $\mu \leq 0$ and $2\mu + 3\lambda \leq 0$.

Fact (The physical meaning of ν and E)

In a straight bar $A \times h$ subject to $F = pA$, $\sigma_{11} = \ldots \sigma_{23} = 0$ and $\sigma_{33} = p$. Thus, $\varepsilon_{33} = \dfrac{p}{E}$ and $\varepsilon_{11} = \varepsilon_{22} = -\nu \dfrac{p}{E}$.

Solving the Problem I

The Necessary and the Sufficient Condition

Problem

Given the body forces $\vec{f}$ in V, the surface forces $\vec{F}$ in S_2, and the surface displacements $\hat{\vec{u}}$ in S_1, find the 15 unknowns u_i, σ_{ij}, and ε_{ij}.

6 *congruence* equations	$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} \frac{\partial u_j}{\partial X_i}\right)$	in V		
3 *Cauchy's* equations	$\sum_j^3 = \frac{\partial \sigma_{ij}}{\partial X_j} + f_i$	in V	+	BC
6 *constitutive* equations	$\sigma_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)$	in V	+	BC

Fact

Although there are mathematical proofs that the system of 15 linear equations with 15 unknowns has a unique solution, we can further simplify the problem.

Solving the Problem II

The Necessary and the Sufficient Condition

The solutions must:

- exist (*x=1 and 2x=1 isn't allowed*)
- be unique (*x+y=1 isn't allowed*)
- be stable (*small data perturbations* $\rightsquigarrow$ *small result perturbations*)

Outline

The Virtual Work Principle

Theorem (VW)

Given a body of volume V subject to a body force f_i, on which boundary S the forces F_i are applied, if f_i, F_i, and σ_{ij} are equilibrated, i.e. Cauchy's equations are satisfied, the internal work W_i and the external work W_e are equal, provided that the displacement field $u_i^\star$ is regular, $|u_i^\star| \ll D$, and $\left|\frac{\partial u_i^\star}{\partial X_j}\right| \ll 1$, as stated in Eq. 32[5]*, where $\varepsilon_{ij}^\star = \frac{1}{2}\left(\frac{\partial u_i^\star}{\partial X_j} + \frac{\partial u_j^\star}{\partial X_i}\right)$.*

$$W_i = \int_V \sum_{i,j} \sigma_{ij}\varepsilon_{ij}^\star \, dV = \int_V \sum_i f_i \, u_i^\star \, dV + \int_S \sum_i F_i \, u_i^\star \, dS = W_e \tag{32}$$

[5] See Appendix 10 for a demonstration of Eq. 32 and a relevant consequence.

Consequences of the Virtual Work Principle

Fact

As any constitutive relationship is involved in the VW principle, it is valid also for non-elastic bodies.

Lemma (Clapeyron's Theorem)

If V is an elastic body according to Lamé and $u_i^\star = u_i$, $\varepsilon_{ij}^\star = \varepsilon_{ij}$, with u_i small, as $\frac{1}{2}\int_V \sum_{ij} \varepsilon_{ij}\sigma_{ij}\, dV = \int_V W\, dV$ because of Eq. 27, the external work is equal to two times the internal energy.

Lemma (Betti's Theorem)

In a linear elastic body subject to two sets of forces f_i', F_i' and f_i'', F_i'', the work done by the first set through the displacements produced by the second set is equal to the work done by the second set through the displacements produced by the first set.

Outline

The Basic Assumptions

Geometrical and Physical Constraints

The beam theory (A. de Saint-Venant, 1855) deals with prismatic bodies, on which we make the following assumptions

- geometrical
  - *slender* prismatic bodies ($D \ll l$)
  - closed (Appendix 14) cross sections
- static
  - body force $= 0$
  - surface forces only at ends
  - translational and rotational equilibrium
- constitutive: material isotropic and homogeneous according to Lamé

Outline

The Axial Problem I

The data

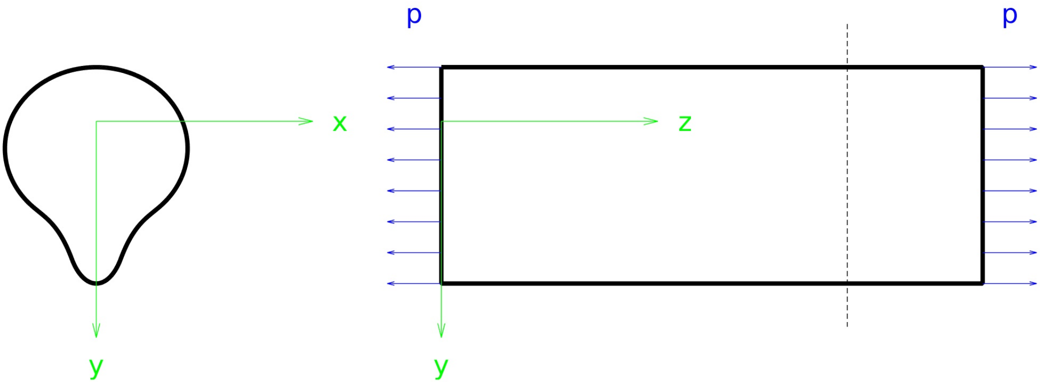


Figure 2

The Axial Problem II

The data

We choose $X_2 \parallel Z$ (the axis of the cylindrical body passing through the cross-section centroid), and X_1 and X_2 as $-x$ and y (the principal axes of the cross section, Fig. 2), respectively, and naming $-u_1 \equiv u$, $u_2 \equiv v$, $u_3 \equiv w$. Besides the constitutive and congruence equations, we have the equations Eq. 33 in V with the boundary conditions of Eq. 34 in S_1 (the lateral surfaces) and Eq. 35 in S_2 (the bases), where $\vec{n} = [n_x; n_y; 0]$ and $\vec{n} = [0; 0; 1]$, respectively.

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \qquad (33)$$

$$\begin{aligned} \sigma_{xx} n_x + \sigma_{xy} n_y &= 0 \\ \sigma_{xy} n_x + \sigma_{yy} n_y &= 0 \\ \sigma_{xz} n_x + \sigma_{zy} n_y &= 0 \end{aligned} \qquad (34)$$

$$\begin{aligned} \sigma_{zx} &= 0 \\ \sigma_{zy} &= 0 \\ \sigma_{zz} &= p \end{aligned} \qquad (35)$$

The Axial Problem III

The solution

The heuristic solution

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} + \sigma_{yz} = 0, \; \sigma_{zz} = p$$

fulfills the boundary conditions and the Eq. 33, and is unique. Moreover, according to the Lamé equations (Eq. 30), we have

$$\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0, \quad \varepsilon_{xx} = \varepsilon_{yy} = -\nu \frac{\sigma_{zz}}{E}, \quad \varepsilon_{zz} = \frac{\sigma_{zz}}{E}$$

Fact (De Saint-Venant Principle)

"The difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load" [1], i.e. the solution is valid if $\int_A p \, dA = N = p\,A \;@\; z > D$.

Outline

The Bending Problem I

The Data

The data

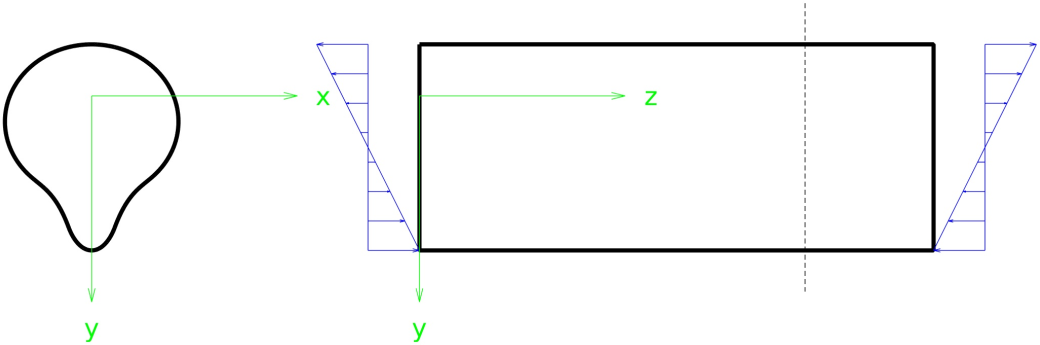


Figure 3

The Bending Problem II

The Data

The data

If $F_x = F_y = 0$ and $F_z = b'y$, with $b' = const$ (Fig. 3), the bending moment M_x is defined as $\int_A F_z y\, dA = b' J_x$, as $\int_A y^2\, dA = J_x$, so that $b' = \dfrac{M_x}{J_x}$. Besides the constitutive and congruence equations, we have the equations Eq. 36 in V with the boundary conditions of Eq. 37 in S_1 (the lateral surfaces) and Eq. 38 in S_2 (the bases), where $\vec{n} = [n_x; n_y; 0]$ and $\vec{n} = [0; 0; 1]$, respectively.

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \tag{36}$$

$$\begin{aligned} \sigma_{xx} n_x + \sigma_{xy} n_y &= 0 \\ \sigma_{xy} n_x + \sigma_{yy} n_y &= 0 \\ \sigma_{xz} n_x + \sigma_{zy} n_y &= 0 \end{aligned} \tag{37}$$

$$\begin{aligned} \sigma_{zx} &= F_x = 0 \\ \sigma_{zy} &= F_y = 0 \\ \sigma_{zz} &= F_z = \frac{M_x}{J_x} y \end{aligned} \tag{38}$$

The Bending Problem III

The Data

The solution

The heuristic solution $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} + \sigma_{yz} = 0,\ \sigma_{zz} = \dfrac{M_x}{J_x} y$ fulfills the boundary conditions and the Eq. 36, and is unique.

Fact (Consequences)

1. *The result is valid for* $\forall F$
2. *If* $y = 0 \rightsquigarrow \sigma_{zz} = 0$
3. σ_{zz} *is maximum if* y *is maximum*
4. *Given a material,* $|\sigma_{zz}|_{max}$ *decreases with* J_x

Outline

The Torsion Problem I

In order to solve the most general case, we define

$$\begin{aligned} u &= -\alpha yz \\ v &= \alpha xz \\ w &= \alpha\,\varphi(x,y) \end{aligned} \tag{39}$$

where the constant α and the function $\varphi(x,y)$ are unknowns. Substituting Eq. 39 in Eq. 11 gives

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0;\ \varepsilon_{xz} = \frac{\alpha}{2}\left(-y + \frac{\partial\varphi}{\partial x}\right);\ \varepsilon_{yz} = \frac{\alpha}{2}\left(x + \frac{\partial\varphi}{\partial y}\right) \tag{40}$$

Substituting Eq. 40 in Eq. 29 (the constitutive equations) gives

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0;\ \sigma_{xz} = \mu\alpha\left(-y + \frac{\partial\varphi}{\partial x}\right);\ \sigma_{yz} = \mu\alpha\left(x + \frac{\partial\varphi}{\partial y}\right) \tag{41}$$

The Torsion Problem II

Substituting Eq. 41 in Eq. 17 gives

$$\mu\alpha\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)=0 \rightsquigarrow \boldsymbol{\nabla}^2\varphi=0 \tag{42}$$

i.e. φ is harmonic in A (the cross section). Substituting Eq. 40 in the third boundary condition (Eq. 37) on C (the boundary of A) gives

$$\mu\alpha\left(-yn_x+\frac{\partial\varphi}{\partial x}n_x+xn_y+\frac{\partial\varphi}{\partial y}n_y\right)=0 \tag{43}$$

As a consequence, φ solves the Neumann problem[6] (if A has a single contour C $\exists$ a solution φ, except for a constant):

$$\begin{cases}\Delta\varphi=0 & \text{on } A\\ \dfrac{\partial\varphi}{\partial x}n_x+\dfrac{\partial\varphi}{\partial y}n_y=yn_x+xn_y & \text{on } C\end{cases} \tag{44}$$

[6] Δ in Eq. 44 denotes the Laplace operator, or Laplacian, $\Delta=\boldsymbol{\nabla}^2$.

The Torsion Problem III

Fact (Unspecific cross section)

While α *in Eq. 42 can be computed as* $\alpha = \dfrac{M_z q}{\mu J_p}$*, where* $J_p = \displaystyle\int_A (x^2 + y^2)\, dA$*,*
$q = \dfrac{J_p}{J_p + \int_A \left(\frac{\partial\varphi}{\partial y} x - \frac{\partial\varphi}{\partial x} y\right)\, dA}$*, and* $M_z = \displaystyle\int_A (\sigma_{yz} x + \sigma_{xz} y)\, dA$*, the* warping function $\varphi(x, y) \neq 0$ *has an analytic solution only if* A *is an equilateral triangle or an ellipse.*

Fact (Circular cross section)

If A *is a circle of radius* R*, on* C *we have* $n_x = \dfrac{x}{R}$*,* $n_y = \dfrac{y}{R}$*, so that the solution of Eq. 43 is* $\varphi = 0$*, i.e. the deformed cross section is a plane.*

Fact (Thin wall tube)

According to Bredt (1896), in a thin tube of thickness h *and cross section* Ω_m*, as* $M_z \vec{k} = \displaystyle\oint_C \overrightarrow{OP} \times \tau\, h\, \vec{t}\, ds$ *and* $\tau \approx const$*, we have* $\tau = \dfrac{M_z}{2h\Omega_m}$.

Outline

The Shear Problem I

The problem statement

The shear is to be associated with a moment, in order to satisfy the rotational equilibrium. This implies that, $\forall$ section, if $\exists\, T_y \rightsquigarrow M_z = -T_y(l-z)$

The two solutions

Rigorous We set ($\rightsquigarrow$ DSV Principle) $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$, and we find σ_{yz}, σ_{xz}, σ_{zz}, via the equilibrium, congruence, and constitutive equations, as well the boundary conditions $\rightsquigarrow$ *complicated solutions*.

Approximate According to Grashof (~1880), as $D \ll l$ and T_y and M_z must coexist the possible, unknown σ_{xz} and σ_{yz} are to be associated with $\sigma_{zz} = -\dfrac{T_y(l-z)y}{J_x}$, being σ_{zz} the greatest contribution to the stress state.

The Shear Problem II

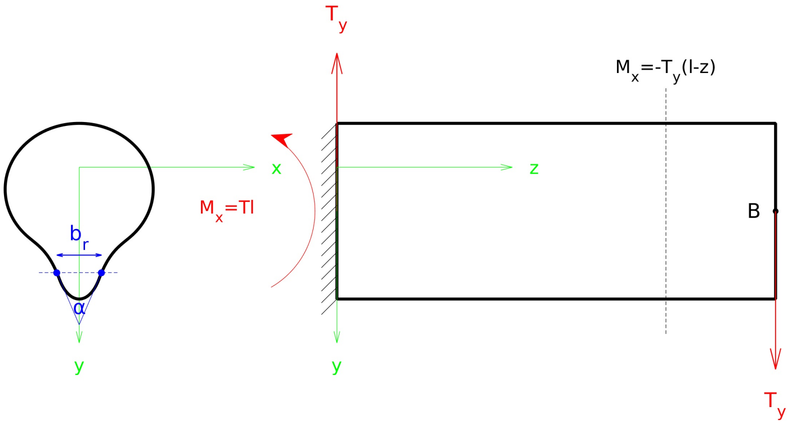


Figure 4

The Shear Problem III

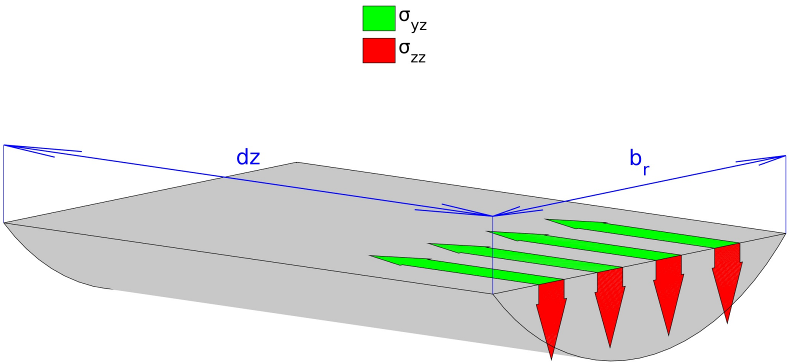


Figure 5

The Shear Problem IV

The approximate solution

We find a solution with the equilibrium equations — avoiding to verify the congruence — on a cross section symmetric wrt the y axis, i.e. $\dfrac{\partial \sigma_{ij}}{\partial x} = 0$ (Fig. 4). The z equilibrium equation of the portion in Fig. 5 is $\int_{A'} \left(\sigma_{zz} + \dfrac{\partial \sigma_{zz}}{\partial z} dz \right) dA - \int_{A'} \sigma_{zz}\, dA - \sigma_{zy} b_r dz = 0$ so that $\int_{A'} \dfrac{\partial \sigma_{zz}}{\partial z} dA = \sigma_{zy} b_r$. As $\sigma_{zz} = -\dfrac{T_y (l-z) y}{J_x}$, we have $\dfrac{\partial \sigma_{zz}}{\partial z} = \dfrac{T_y y}{J_x}$, which, substituted in the previous expression of σ_{zy} and defining $m_r = \int_{A'} y dA$, gives

$$\sigma_{zy} = \frac{T_y m_r}{J_x b_r} \tag{45}$$

Example (rectangular cross section $b \times h$)

Eq. 45 gives $\sigma_{zy} = 12 \dfrac{T_y \frac{1}{2} \left(\frac{h^2}{4} - y^2 \right)}{h^3 b}$ $\rightsquigarrow$ $\sigma_{zy_{max}} = \dfrac{3}{2} \dfrac{T_y}{hb}$ @ $y = 0$.

The Shear Problem V

The shear effects

Let's compute the y displacement of point B in Fig. 4 caused by the shear. Enforcing the VW principle (Eq. 32), we have $W_e = T_y v_{T_B} = \int_V 2\,\sigma_{zy}\varepsilon_{zy}\,dV = \frac{1}{\mu}\int_V \sigma_{zy}^2\,dV$, from which, defining $\chi = \frac{A}{J_x^2}\int_A \left(\frac{m_r}{b_r}\right)^2 dA$, v_{T_B} is defined in Eq. 46[7]. Besides, the bending, again because of Eq. 32, gives $W_e = T_y v_{M_B} = \int_V 2\,\sigma_{zz}\varepsilon_{zz}\,dV = \frac{1}{E}\int_V \sigma_{zz}^2\,dV$, so that v_{M_B} is defined in Eq. 47 and $\beta = \frac{v_{T_B}}{v_{M_B}}$ in Eq. 48.

$$v_{T_B} = \chi\frac{T_y l}{\mu A} \qquad (46) \qquad\qquad v_{M_B} = \frac{T_y l^3}{3EJ_x} \qquad (47) \qquad\qquad \frac{v_{T_B}}{v_{M_B}} = \frac{6\chi(1+\nu)J}{Al^2} \qquad (48)$$

Fact

The bending deformation is $\gg$ than the shear one in a slender *beam, i.e. when* $\frac{J}{Al^2}s \ll 1$ *(Eq. 48).*

[7] See ▸ Appendix 11 for some considerations about χ in Eq. 46.

Outline

Definitions

Assumptions and Restrictions

Definition (Beam assembly)

A *beam assembly* is a combination of beams mutually connected. When all of them $\in x = const$, according to the de Saint-Venant's assumption (σ_{zz}, σ_{zx}, $\sigma_{zy} \neq 0$ and $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$) the stress state is a function of N, M_x, M_y, T_x, T_y, and M_z.

Getting rid of limitations

	theory	*real world*	*approximation*
geometrical	prismatic beam straight axis	tapered arc	$\cup$ slender beams $R \gg l$
mechanical	loads on ends no mass forces	lumped forces distributed weight	volume forces
constitutive	homogeneity isotropy	polymers reinforced concrete	mean moduli

The ODE Re-definition

In the (very common) case of plane assemblies we deal only with deformations in the yz plane (so that $u = 0$, $v \neq 0$, $w \neq 0$), and the loads (per unit length) are $q \parallel z$ and $p \parallel y$. With these assumptions, the stress/deformations state is obtained from $M(z)$, $N(z)$, and $T(z)$, so that we are dealing with ODE's.

The ODE's of the beam assembly

Given $p(z)$ and $q(z)$, as well the boundary conditions, we redefine the ODE's in terms of the unknowns $M(z)$, $N(z)$, $T(z)$, $v(z)$, and $w(z)$. To that, we have to find a suitable form of the following equations:

- congruence
- equilibrium
- constitutive

The Congruence Equations

As v is affected by the sum of the shear and moment effects, we define $dv = (\varphi + \varphi_m)dz$, where $\varphi_m = \arctan \frac{dv}{dz}$ is the angle between the tangents to the mean line in the undeformed and deformed states and φ is the rotation angle of the cross section due the shear. Because of we assume small deformations and displacements, we define the total rotation as $\gamma = \frac{dv}{dz} + \varphi$. As $ds = Rd\psi$, where R is the radius of curvature, where $d\psi = d\left(\arctan \frac{dv}{dz}\right) = \frac{1}{1+v'^2}v''dz$ and $ds = \left(dv^2 + dz^2\right)^{\frac{1}{2}} = dz\left(1+v'\right)^{\frac{1}{2}}$, again under the previous assumptions, we define the curvature[8] $\kappa = \frac{1}{R} = \frac{d\psi}{ds}$ as $-v''$. Consequently, the deformation is identified by means of Eq. 49.

$$\varepsilon = \frac{dw}{dz} \qquad \kappa = -\frac{d^2 v}{dz^2} \qquad \gamma = \frac{dv}{dz} + \varphi \tag{49}$$

[8] See ▸ Appendix 13 for some additional considerations about κ.

The Equilibrium Equations

The equilibrium equations of the portion of beam of length l from 1 to 2 for $0 \leq z \leq l$ ($z = 0\,@\,1$ and $z = l\,@\,2$) are

$$\parallel z \quad -N_1+N_2+\int_1^2 q\,dz = 0 \qquad \parallel y \quad -T_1+T_2+\int_1^2 p\,dz = 0 \qquad \circlearrowleft x \quad -M_1-T_1 l+M_2+\int_1^2 p(l-z)\,dz = 0 \quad (50)$$

If $M(z)$, $N(z)$, and $T(z)$ are continuous functions and their first derivatives are also continuous (i.e. there's no concentrated load), we have $N_2 - N_1 = \int_1^2 \frac{dN}{dz}\,dz$, $T_2 - T_1 = \int_1^2 \frac{dT}{dz}\,dz$, and $M_2 - M_1 = \int_1^2 \frac{dM}{dz}\,dz$, which, substituted in Eq. 50, give, as the integration limits are arbitrary:

$$\frac{dN}{dz} + q = 0 \quad (51a) \qquad \frac{dT}{dz} + p = 0 \quad (51b) \qquad \frac{dM}{dz} - T = 0 \quad (51c)$$

Eq. 51a and Eq. 51b are the translation equilibrium equations along z and y, respectively, and Eq. 51c[9] is the rotational equilibrium equation around x.

[9] Eq. 51c is obtained integrating by parts $\int_1^2 p(l-z)\,dz$, substituting $pdz = -dT$, according to Eq. 51b, and $g = l - z \rightsquigarrow dg = -dz$, so that $\int_1^2 p(l-z)\,dz = -T_1 l - \int_2^1 T\,(-dz) = -T_1 l - \int_1^2 T\,dz$.

The Constitutive Equations I

Recalling the Lamé equations (Eq. 31), the internal work can be expressed as $\int_V \left(\frac{\sigma_{zz}^2}{E} + \frac{1}{\mu}\sigma_{zy}^2 \right) dV$ where $\sigma_{zz} = \frac{N}{A} + \frac{M}{J}y$ (Eqs. 35 and 38), $\sigma_{zy} = \frac{Tm_r}{Jb_r}$ (Eq. 45). Thus, it is:

$$W_i = \int_1^2 dz \left[\int_A \frac{1}{E} \left(\frac{N^2}{A^2} + \frac{2NM}{AJ}y + \frac{M^2y^2}{J^2} \right) + \frac{T^2m_r^2}{\mu J^2 b_r^2} \right] dA \tag{52}$$

Choosing the origin of the xy plane coincident with the neutral axis of the cross section, so that $\int_A y\, dA = 0$, Eq. 52, recalling the definition of χ, can be written as

$$W_i = \int_1^2 \left[\frac{N^2}{EA} + \frac{M^2}{EJ} + \chi \frac{T^2}{\mu A} \right] dz \tag{53}$$

The Constitutive Equations II

As we have assumed the continuity, the internal energy can be expressed also as

$$W_i = \int_1^2 Md\varphi + Tdv + Ndz = \int_1^2 (M\kappa + T\gamma + Nd\varepsilon)\, dz \tag{54}$$

Equalling the expressions of W_i in Eqs. 53 and 54, as the integration limits are arbitrary, we obtain $\frac{N^2}{EA} + \frac{M^2}{EJ} + \chi\frac{T^2}{\mu A} = N\varepsilon + M\kappa + T\gamma$, so that we can define the axial, bending, and shear stiffnesses, respectively, as in Eq. 55.

$$\varepsilon = \frac{N}{EA} \qquad \kappa = \frac{M}{EJ} \qquad \gamma = \chi\frac{T}{\mu A} \tag{55}$$

Fact

The constitutive equations of the beam assembly are unconcatenated.

The Differential Equations of Beams

Combining ▸ Eqs. 49, 55, and 51 allows to define the following three ODE's [3] in w, v, and φ, each one with its suitable boundary conditions:

$$\frac{d}{dz}\left(EA\frac{dw}{dz}\right)+q=0 \qquad (56)$$

$$\frac{d}{dz}\left[\frac{\mu A}{\chi}\left(\frac{dv}{dz}+\varphi\right)\right]+p=0 \quad \frac{d}{dz}\left(EJ\frac{d\varphi}{dz}\right)=T \qquad (57)$$

Although the two statements in Eq. 57 are combined, Bernoulli proposed their decoupling supposing $\gamma=0$ in Eq. 49, i.e. neglecting the rotation due to the shear, as shown in Fig. 6, so that $\varphi = -\frac{dv}{dz} = -v'$, $\varphi' = v''$.

Thus, Eq. 57 is replaced by Eq. 58:

$$\frac{d^2}{dz^2}\left(EJv''\right) = -T' = p \qquad (58)$$

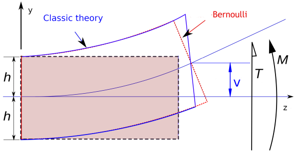


Figure 6

Fact

Eqs. 56 and 58 are the ODE's of any plane system of beams.

Solving the Beam ODE: Axial

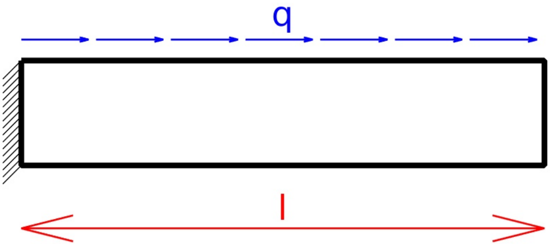


Figure 7

Example (Fig. 7)

If $q = const$ and $p = 0$, as $EA = const$ from the integration of Eq. 56 with the boundary conditions $w(0) = 0$ and $N(l) = EAw'(l) = 0$ we have $w(z) = \frac{q}{EA} z \left(l - \frac{z}{2} \right)$ and $N(z) = q\,(l - z)$.

Solving the Beam ODE: Bending (1)

Figure 8

Example (Fig. 8)

If $p = 0$ and $q = 0$, as $EJ = const$ from the four times integration of Eq. 58 with the boundary conditions $v(0) = v'(0) = M(l) = 0$ and $T(l) = P$ we have
$w(z) = \frac{P}{EJ}\left(-\frac{z^3}{3} + \frac{z^2}{2}l\right)$ and $M(z) = P\,(l - z)$.

Solving the Beam ODE: Bending (2)

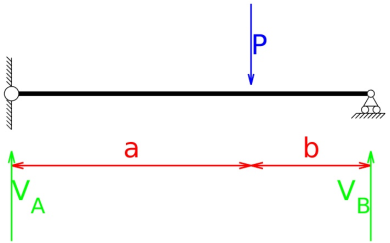


Figure 9

Example (Fig. 9)

Considering two beams loaded at their ends, with v_1 and v_2 the v in a and b, respectively, we have the eight equations $EJ\dfrac{d^4v_1}{dz^4} = 0 \cup EJ\dfrac{d^4v_2}{dz^4} = 0$ and the eight boundary conditions $v_1(0) = M_1(0) = v_2(l) = M_2(l) = 0$, $v_1(a) = v_2(a)$, $v_1'(a) = v_2'(a)$, $-EJv_1''(a) = M_1(a) = M_2(a) = -EJv_2''(a)$, $-T_1 + P + T_2 = 0$. A (unique) solution $\exists$.

The Limitations of the Beams ODE's

Fact

Analytically solving the beam problem *by means of ODE, as in the previous examples, may become extremely complicated, unless numerical methods are adopted. Nevertheless, the virtual work principle allows to solve a statically indeterminate problem with* few *unknowns.*

Outline

Assumption and Restrictions

- From continuum mechanics
  - small displacements and small deformations
  - homogeneous, isotropic, Green's material, i.e. there exists a potential function by which stresses and strains can be represented
  - the material is a Cauchy continuum, i.e. the stress tensor is symmetric
- From thin shell theory [4]
  - two geometrical dimensions are prevalent with respect to the third one, i.e. the plate is *thin*
  - stress $\perp$ to middle plane is 0, i.e. stress diffusivity isn't considered
  - only forces acting perpendicularly to the middle plane are considered
  - a generic straight segment, initially $\perp$ to the middle plane, after the deformation it is still straight — not necessarily, after the deformation, it is still perpendicular to the deformed middle plane
  - the displacement $\perp$ to the middle plane depends only on in-plane coordinates

Outline

The General Scheme

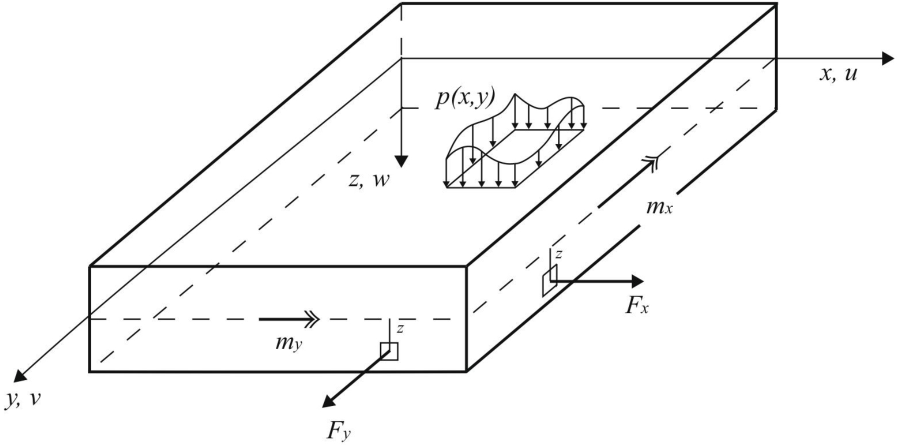


Figure 10 Credits: [5]

Displacement Definitions

Naming $\boldsymbol{s}(x, y, z) = [u; v; w]$ the local displacement where $u = -z\varphi_x(x, y)$, $v = -z\varphi_y(x, y)$, and $w(x, y)$ are the displacements in the x, y, and z directions, and $\varphi_x(x, y)$ and $\varphi_y(x, y)$ are are the rotations around the y axis occurring in the xz plane and the x axis occurring in the yz plane, respectively, (we name $\varphi_x(x, y)$, $\varphi_y(x, y)$, and $w(x, y)$ the generalized displacements, as in Eq. 59), we have $\boldsymbol{s} = \boldsymbol{n}\boldsymbol{U}$, where $\boldsymbol{n}$ is defined in Eq. 60.

$$\boldsymbol{U} = \begin{bmatrix} \varphi_x(x, y) \\ \varphi_y(x, y) \\ w(x, y) \end{bmatrix} \qquad (59)$$

$$\boldsymbol{n} = \begin{bmatrix} -z & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (60)$$

Strain Definitions I

The strain components are

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \end{bmatrix} = \begin{bmatrix} -z\dfrac{\partial \varphi_x}{\partial x} \\ -z\dfrac{\partial \varphi_y}{\partial y} \\ 0 \\ -z\dfrac{\partial \varphi_x}{\partial y} - z\dfrac{\partial \varphi_y}{\partial x} \\ -\varphi_x + \dfrac{\partial w}{\partial x} \\ -\varphi_y + \dfrac{\partial w}{\partial y} \end{bmatrix} \tag{61}$$

Strain Definitions II

Naming χ_i the generalized curvatures and t_i the shear angular deformations (Eq. 62), and defining $\boldsymbol{b}$ as in Eq. 63, Eq. 61 can be rewritten as $\boldsymbol{\varepsilon} = \boldsymbol{b}\,\boldsymbol{q}$, where $\boldsymbol{q}$ is the vector of the generalized strains.

$$\boldsymbol{q} = \begin{bmatrix} -z\dfrac{\partial \varphi_x}{\partial x} \\ -z\dfrac{\partial \varphi_y}{\partial y} \\ 0 \\ -z\dfrac{\partial \varphi_x}{\partial y} - z\dfrac{\partial \varphi_y}{\partial x} \\ -\varphi_x + \dfrac{\partial w}{\partial x} \\ -\varphi_y + \dfrac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \chi_x \\ \chi_y \\ 0 \\ \chi_{xy} \\ t_x \\ t_y \end{bmatrix} \quad (62)$$

$$\boldsymbol{b} = \begin{bmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (63)$$

Load Definitions I

Naming $\boldsymbol{F} = [F_x; F_y; F_z]$ the external force per unit area, the external specific work per unit area is defined by Eq. 64, where $\boldsymbol{P}$ is the generalized load (per unit area) and $\delta\hat{s}_i$ is the virtual displacement field.

$$\frac{dW_e}{dA} = \int_{-h/2}^{h/2} \sum_i F_i \delta\hat{s}_i \, dz = \boldsymbol{P}^T \delta\hat{\boldsymbol{U}} = \int_{-h/2}^{h/2} \delta\hat{\boldsymbol{s}}^T \boldsymbol{F} \, dz \tag{64}$$

As $\boldsymbol{s} = \boldsymbol{n}\boldsymbol{U}$, so that $\delta\hat{\boldsymbol{s}}^T = \delta\hat{\boldsymbol{U}}^T \boldsymbol{n}^T$, from Eq. 64 we have

$$\frac{dW_e}{dA} = \delta\hat{\boldsymbol{U}}^T \int_{-h/2}^{h/2} \boldsymbol{n}^T \boldsymbol{F} \, dz = \boldsymbol{P}^T \delta\hat{\boldsymbol{U}} = \delta\hat{\boldsymbol{U}}^T \boldsymbol{P} \tag{65}$$

Thus, from Eq. 65 we have

$$\boldsymbol{P} = \int_{-h/2}^{h/2} \boldsymbol{n}^T \boldsymbol{F} \, dz \tag{66}$$

Load Definitions II

Substituting the Eq. 60 into Eq. 66 and recalling the definition of $\boldsymbol{F}$, we get

$$\boldsymbol{P} = \int_{-h/2}^{h/2} \begin{bmatrix} -zF_x \\ -zF_y \\ F_z \end{bmatrix} dz = \begin{bmatrix} m_x(x,y) \\ m_y(x,y) \\ p(x,y) \end{bmatrix} \tag{67}$$

Fact

$p(x,y)$ is a force per unit area, and $m_x(x,y)$ and $m_y(x,y)$ are moments per unit area.

Fact

There isn't any explicit information about the points where the generalized loads are acting — we know only that they are applied onto the middle plane, as shown in Fig. 10.

Stress Definitions I

Naming $\delta\hat{\boldsymbol{\varepsilon}}$ the virtual local strains and $\boldsymbol{\sigma}$ the local stresses, the internal specific work per unit area, according to the VW principle, is, recalling that $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon} = \boldsymbol{b}\,\boldsymbol{q}$,

$$\frac{dW_i}{dA} = \int\limits_{-h/2}^{h/2} \delta\hat{\boldsymbol{\varepsilon}}\boldsymbol{\sigma}\, dz = \delta\boldsymbol{q}^T \int\limits_{-h/2}^{h/2} \boldsymbol{b}\boldsymbol{\sigma}\, dz = \delta\boldsymbol{q}^T\boldsymbol{Q} \tag{68}$$

where $\boldsymbol{Q}$ is the generalized stress vector. As a consequence, $\boldsymbol{Q} = \int_{-h/2}^{h/2} \boldsymbol{b}\boldsymbol{\sigma}\, dz$.

Stress Definitions II

Performing the computation, we obtain:

$$\boldsymbol{Q} = \int_{-h/2}^{h/2} \begin{bmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} dz = \int_{-h/2}^{h/2} \begin{bmatrix} z\sigma_{xx} \\ z\sigma_{yy} \\ 0 \\ z\tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} dz = \begin{bmatrix} M_x \\ M_y \\ 0 \\ M_{xy} \\ V_x \\ V_y \end{bmatrix} \tag{69}$$

Fact

The generalized moments M_x, M_y, and M_{xy} are moments per unit length, while the generalized shears V_x and V_y are forces per unit length.

Fact

Both the local and the generalized stresses, along with their directions, are shown in Fig. 11.

Stress Definitions III

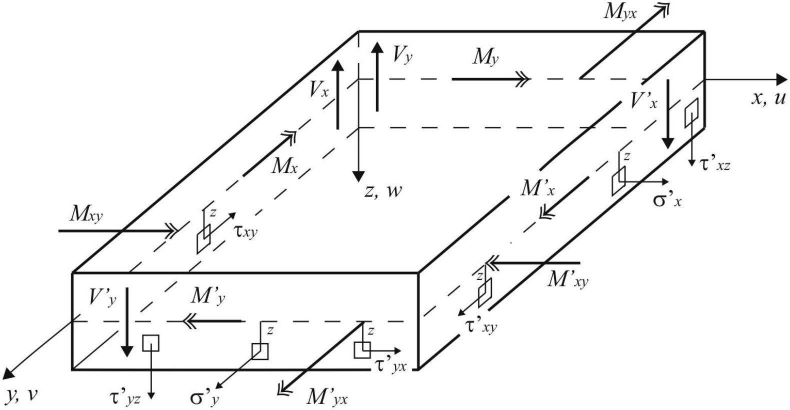


Figure 11 Credits: [5]

The Plate Equations I

The equation of the rotational equilibrium wrt x of the $dx \times dy$ plate of Fig. 11 is

$$V_y' dxdy = M_y' dx + M_y dx + M_{xy}' dy + p(x,y)dxdy\frac{dy}{2} + m_y dxdy = 0 \tag{70}$$

Dropping out the terms of higher order and because of $G' = G + \frac{\partial G}{\partial \alpha}d\alpha$, we obtain

$$V_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - m_y \tag{71}$$

Similarly, the rotational equilibrium wrt the y axis gives

$$V_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - m_x \tag{72}$$

and the translational equilibrium wrt the z axis gives

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + p(x,y) = 0 \tag{73}$$

Finally, substituting Eq. 71 and Eq. 72 into Eq. 73 we obtain

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x(,y) + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} = 0 \tag{74}$$

The Plate Equations II

Recalling the local constitutive relationship[10], redefined as $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$, where

$$\boldsymbol{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{75}$$

and the strain definition of Eq. 61, we can write the specific energy per unit area as

$$\frac{d\Omega}{dA} = \frac{1}{2}\int_{-h/2}^{h/2} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dz = \frac{1}{2}\boldsymbol{q}^T \int_{-h/2}^{h/2} \boldsymbol{b}^T \boldsymbol{D} \boldsymbol{b} \boldsymbol{q}\, dz = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{D}^{\star} \boldsymbol{q} \tag{76}$$

[10] Cf. Eq. 29

The Plate Equations III

Recalling Eq. 63, as $\int_{-h/2}^{h/2} z^2\, dz = \frac{h^3}{12}$ and $\int_{-h/2}^{h/2} 1\, dz = h$, the generalized stiffness matrix $\boldsymbol{D}^{\star}=\frac{1}{2}\int_{-h/2}^{h/2} \boldsymbol{b}^T \boldsymbol{D} \boldsymbol{b}\, dz$ in Eq. 76 can be defined as:

$$\boldsymbol{D}^{\star} = \frac{E}{1-\nu^2}\begin{bmatrix} \frac{h^3}{12} & \nu & 0 & 0 & 0 & 0 \\ \nu & \frac{h^3}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h\frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & h\frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & h\frac{1-\nu}{2} \end{bmatrix} \tag{77}$$

The generalized stiffness matrix $\boldsymbol{D}^{\star}$ in Eq. 77 relates the generalized stresses $\boldsymbol{Q}$ (Eq. 69) to the generalized strains $\boldsymbol{q}$ (Eq. 62) as in Eq. 78

$$\boldsymbol{Q} = \boldsymbol{D}^{\star}\boldsymbol{q} \tag{78}$$

The Plate Equations IV

Expanding the Eq. 78 gives rise to the following relationships

$$\begin{aligned} M_x &= D(\chi_x + \nu\chi_y) \\ M_y &= D(\chi_y + \nu\chi_x) \\ M_{xy} &= \frac{h^3}{12}G\chi_{xy} \\ V_x &= Ght_x \\ V_y &= Ght_y \end{aligned} \tag{79}$$

where $D = \dfrac{Eh^3}{1-\nu^2}$ is the plate flexural rigidity and $G = \dfrac{E}{1+\nu}$ is the plate shear modulus.

Outline

The Simplified Equations of a $a \times b \times h$ Plate

Kirchhoff's (~1880) hypothesis:

- $h < min(a, b)/5$
- $w_{max} < h/5$

$$\gamma_{xz} = \gamma_{yz} = 0 \rightsquigarrow \varphi_x = \frac{\partial w}{\partial x},\ \varphi_y = \frac{\partial w}{\partial y}$$

Fact

Under the Kirchhoff's hypothesis the rotation of the generic straight segment is exactly equal to the one of the middle plane, i.e. there are no angular deformations. Therefore the plate model can be reformulated in this simplified case, obtaining, from Eq. 74, the more undemanding form in Eq. 80.

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \nabla^4 w(x, y) = \frac{-p(x, y)}{D} \qquad (80)$$

Example

If the dimensions of the rectangular plate are a and b, the plate is simply supported at $y = 0$ and $y = b$, and $p = p_0 = const$, the (Navier) solution of Eq. 80 is

$$w(x, y) = \frac{16 p_0}{\pi^6 D} \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{mn(\frac{m^2}{a^2} + \frac{n^2}{b^2})} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Outline

The Simplified Equations of a Circular Plate I

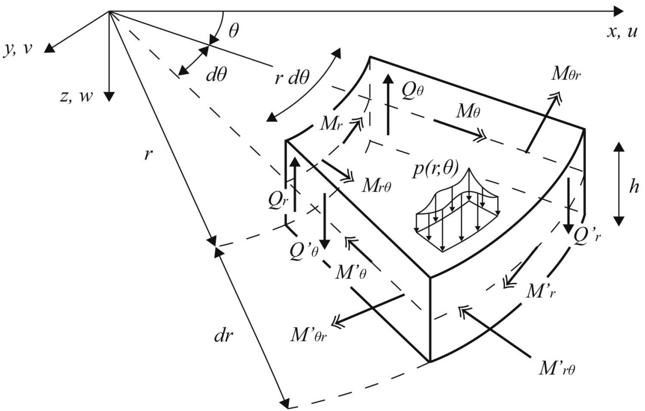


Figure 12

The Simplified Equations of a Circular Plate II

When dealing with an axially symmetric geometry (Fig. 12), substituting $x = r\cos\theta$ and $y = r\sin\theta$ into the previous equations, as $\dfrac{\partial}{\partial\theta} = 0$, leads to

$$\nabla_r^4 w(r,\theta) \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = \frac{-p(r,\theta)}{D} \tag{81}$$

If also the load axially symmetric, Eq. 81 can be rewritten as

$$\nabla_r^4 w(r) \equiv \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{-p(r)}{D} \tag{82}$$

Example

If $p(r) = p_0 = const$ the solution of Eq. 82 is $w(r) = C_1 \log r + C_2 r^2 \log r + C_3 r^2 + C_4 + \dfrac{p_0 r^4}{64D}$ where the constants of integration C_1, C_2, C_3, and C_4 are found using the boundary conditions at $r = a$ and the conditions that w, $\dfrac{dw}{dr}$, M_r and Q_r must be finite at $r = 0$.

Outline

Outline

Definition

FEM

The Finite Element Method is a numerical technique to find approximate solutions of PDE. Originated from the need of solving complex elasticity and structural analysis problems in engineering, FEM:

- helps in producing stiffness and strength visualizations
- helps to minimize material weight and its cost of the structures
- allows for detailed visualization and indicates the distribution of stresses and strains inside the body of a structure
- as a powerful yet complex tool requires training and education to properly interpret the results
- allows entire designs to be constructed, refined and optimized before the design is manufactured
- decreases the time to take products from concept to the construction

The Numerical Methods

Approximate Solutions instead of Analytical Solutions

- Finite Difference Method
- Finite Volume Method
- Finite Element Method
- Boundary Element Method
- Meshless Method

Each method has advantages and limitations . . .

. . . however, it is possible to solve various problems by *finite element method*, even with highly complex geometry and loading conditions, with the restriction that there is always some numerical errors. Therefore, effective and reliable use of this method requires a solid understanding of its limitations.

The General Description I

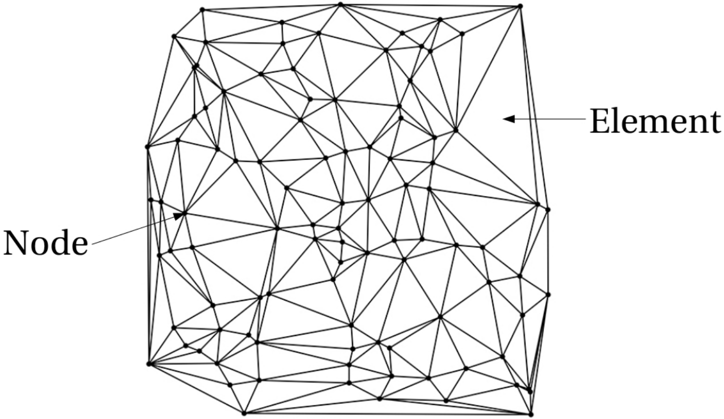


Figure 13

The General Description II

Any continuum/domain is divided into small pieces, called *Finite Elements* (Fig. 13).

- the original domain is considered as an assemblage of number of such small *elements*, connected through joints called *nodes*
- the elements are attached to the adjacent elements only at the nodal points
- each node has (not more than) 6 degrees-of-freedom (*dof*), namely u, v, w and $\theta_x, \theta_y, \theta_z$
- each element has a simple spatial variation (field quantity), described by polynomial terms
- each element contains the material and geometrical properties — the material properties inside an element are assumed to be constant
- the physical object is modeled by choosing appropriate elements (beam, plate, solid)
- all elements are then assembled to obtain the solution of the entire domain/structure under certain loading conditions
- nodes are assigned at a certain density throughout the continuum depending on the estimated stress levels of a particular domain, i.e. regions which will receive large amounts of stress variation usually have a higher node density than those which experience little or no stress

The Background

- The Finite Element Model (FEM) is a computer model of a continuum, with infinite particles and continuous variation of material properties, that is stressed and analyzed for specific results
- Therefore, it is to be simplified as an assemblage of substructures, components and members
- A discretization process is necessary to convert whole structure to an assemblage of members/elements (*mesh*), in order to determine its responses
- On the basis of assumptions, the appropriate constitutive model can be constructed.
- For the linear-elastic-static analysis of structures, the final form of equation will be made in the form of $\boldsymbol{F} = \boldsymbol{K}\boldsymbol{d}$ where $\boldsymbol{F}$, $\boldsymbol{K}$ and $\boldsymbol{d}$ are the nodal loads, global stiffness and nodal displacements respectively

The Method

Classical Actual structure

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f = 0$$

(PDE)

↓

Assumptions Equilibrium, congruence, constitutive

$$\int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV = \int_V \delta \boldsymbol{d}^T \boldsymbol{f} \, dV + \int_S \delta \boldsymbol{d}^T \boldsymbol{F} \, dS$$

(Principle of virtual works)

↓

FEM Structural model

$$\boldsymbol{F} = \boldsymbol{K} \boldsymbol{d}$$

(Algebraic equations)

Basic Steps in FEA

1. **Discretization of the continuum** The continuum is divided into a number of elements by imaginary lines or surfaces. The interconnected elements may have different sizes and shapes.
2. **Identification of variables** The elements are assumed to be connected at their intersecting points, the nodes. At each node, unknown displacements are to be prescribed.
3. **Choice of approximating functions** Displacement function is the starting point of the mathematical analysis. This represents the variation of the displacement within the element. The displacement function may be approximated in the form a linear function or a higher-order function. A convenient way to express it is by polynomial expressions. The shape or geometry of the element may also be approximated.

Basic Steps in FEA

4. **Formation of the element stiffness matrix** After continuum is discretized with desired element shapes, THE INDIVIDUAL ELEMENT STIFFNESS MATRIX IS FORMULATED. Basically it is a minimization procedure whatever may be the approach adopted. For certain elements, the form involves a great deal of sophistication. The geometry of the element is defined in reference to the global frame. Coordinate transformation must be done for elements where it is necessary.
5. **Formation of overall stiffness matrix** After the element stiffness matrices in global coordinates are formed, they are assembled to FORM THE OVERALL STIFFNESS MATRIX. The assembly is done through the nodes which are common to adjacent elements. The overall stiffness matrix is symmetric and banded.
6. **Formation of the element loading matrix** THE LOADING INSIDE AN ELEMENT IS TRANSFERRED AT THE NODES and a consistent element matrix is formed. The loading forms an essential parameter in any structural engineering problem.

Basic Steps in FEA

7. **Formation of the overall loading matrix** Like the overall stiffness matrix, the element loading matrices are assembled to FORM THE OVERALL LOADING MATRIX. This matrix has one column per loading case.
8. **Incorporation of boundary conditions** THE BOUNDARY RESTRAINT CONDITIONS ARE TO BE IMPOSED IN THE STIFFNESS MATRIX. Various techniques can satisfy the boundary conditions. One is the size of the stiffness matrix may be reduced or condensed in its final form. To ease computer programming aspect and to elegantly incorporate the boundary conditions, the size of overall matrix is kept the same.
9. **Solution of simultaneous equations** The unknown nodal displacements are calculated by the MULTIPLICATION OF FORCE VECTOR WITH THE INVERSE OF STIFFNESS MATRIX.
10. **Calculation of stresses or stress-resultants** NODAL DISPLACEMENTS ARE UTILIZED FOR THE CALCULATION OF THE STRESSES for all elements of the continuum or for some of them. The results, in terms of displacements and stress/strain, may also be obtained by graphical means.

The Shape Function

In FEA the variations of displacement u at any point inside the element are expressed by its nodal displacement u_i by means of $u = \sum_i N_i u_i$, where the interpolation function N_i is the *shape function* — generally a n degree polynomial which provides a single-valued and continuous field.

- displacement must be compatible between adjacent elements
- must be continuous within the elements: this can be ensured by choosing a suitable polynomial
- must be capable of rigid body displacements of the element: the constant terms used in the polynomial ensure this condition

Element Classification

- One dimensional elements
  - 2 nodes
  - 3 nodes
- Two dimensional elements
  - 3 node triangle
  - 6 node triangle
  - 4 node rectangle
  - 8 node rectangle
- Three dimensional elements
  - tetrahedron
  - brick
  - hexahedron

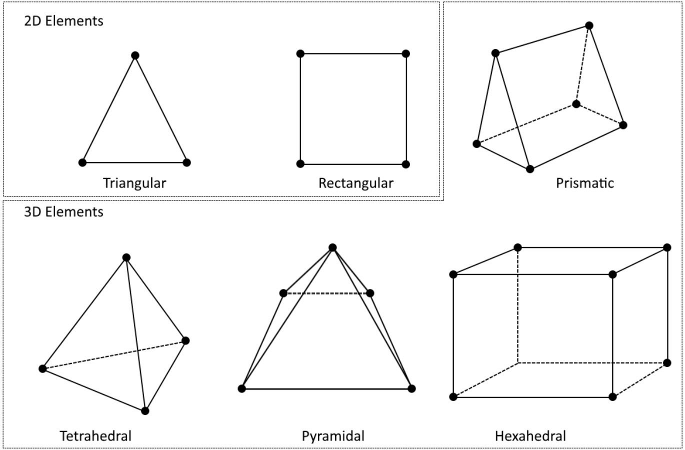


Figure 14

Outline

The Element Stiffness Matrix

According to Eq. 32, $\delta W_e = \delta W_i$, i.e. the work done by external forces due to the virtual displacement $\delta \boldsymbol{d}$ of a structure in equilibrium is equal to the work done by the internal forces for the virtual internal displacement $\delta \boldsymbol{\varepsilon}$, as in Eq. 83

$$\int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV = \int_V \delta \boldsymbol{d}^T \boldsymbol{f} \, dV + \int_S \delta \boldsymbol{d}^T \boldsymbol{F} \, dS \tag{83}$$

Defining $\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{d}$ (Eq. 12) and recalling the constitutive equation $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$, we have $W_i = \frac{1}{2} \int_V \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV = \frac{1}{2} \int_V \boldsymbol{d}^T \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\boldsymbol{d} \, dV = \frac{1}{2} \boldsymbol{d}^T \boldsymbol{d} \int_V \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} \, dV$, and, differentiating wrt $\delta \boldsymbol{d}$,

$$\delta W_i = \boldsymbol{d} \;\; \delta \boldsymbol{d} \int_V \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} \, dV \tag{84}$$

According to $\boldsymbol{F} = \dfrac{\partial W_i}{\partial \boldsymbol{d}}$ (VW principle), from Eq. 84 the element stiffness matrix is

$$\boldsymbol{k} = \int_V \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} \, dV \tag{85}$$

Fact (Properties of element stiffness matrix)

- *symmetric and square*
- *all diagonal elements are positive*

Outline

The Global Stiffness Matrix

A structural system is an assemblage of number of elements, interconnected together to form the whole structure. Therefore, the element stiffness of all the elements are first to be calculated and then systematically assembled together.

Process of assembling the global stiffness matrix $\boldsymbol{K}$ via the local stiffness matrix $\boldsymbol{k}$

a. initialize the $n \times n$ $\boldsymbol{K}$ as zero
b. compute individual element properties and calculate $\boldsymbol{k}$ of that element
c. add $\boldsymbol{k}$ to $\boldsymbol{K}$ using proper locations
d. repeat steps (b) and (c) till all $\boldsymbol{k}$ are placed globally

- the stiffness at the joint (node) i out of n (the total number of *dof*) is obtained by adding the stiffness of all elements *meeting* at joint i
- the *dof* of the structure are numbered from 1 to n
- the $\boldsymbol{k}$ of each element is placed in its proper position in the $\boldsymbol{K}$

A Simple Case of Assembling the Global Stiffness Matrix I

The equilibrium of a linear spring of constant k, at which ends act the forces f_1 and f_2, can be expressed as $\boldsymbol{k}_e \boldsymbol{u} = \boldsymbol{f}$, that is:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{86}$$

where $\boldsymbol{k}_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ is the element stiffness matrix in the element coordinate system (or local system), $\boldsymbol{u}$ is the column vector of nodal displacements u_1 and u_2, and $\boldsymbol{f}$ is the column vector of element nodal forces.

- the element stiffness matrix for the linear spring element is a 2 by 2 matrix
  - the element exhibits two nodal displacements (*dof*)
  - the two displacements are not independent, i.e. the body is continuous and elastic
- the matrix is symmetric, as a consequence of the symmetry of the forces (equal and opposite to ensure the equilibrium)
- the matrix is singular and therefore not invertible, as a consequence of the incompleteness of the problem (boundary conditions are required)

A Simple Case of Assembling the Global Stiffness Matrix II

For a system with 3 nodes and 2 springs of stiffness k_1 and k_2, respectively, the equations for each spring in matrix form are[11]

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix} \qquad \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} \tag{87}$$

As $u_1^{(1)} = U_1 \quad u_2^{(1)} = u_1^{(2)} = U_2 \quad u_2^{(2)} = U_3$, we can expand Eq. 87 as in Eq. 88

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 0 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} \tag{88}$$

[11] The notation $f_i^{(j)}$ represents the force exerted on element j at node i.

A Simple Case of Assembling the Global Stiffness Matrix III

Summing member by member the terms of Eq. 88 and defining the nodal forces as $f_1^{(1)} = F_1$, $f_2^{(1)} + f_2^{(2)} = F_2$, and $f_3^{(2)} = F_3$ the final form of Eq. 88 is

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \tag{89}$$

$$\text{where} \quad \boldsymbol{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

is the system (global) stiffness matrix, $\boldsymbol{U} = [U_1; U_2; U_3]$, and $\boldsymbol{F} = [F_1; F_2; F_3]$.

Features of the system stiffness matrix

Symmetry

As is the element stiffness matrix

Superposition

The individual element stiffness matrices are superposed with proper assignment of element nodal displacements and associated stiffness coefficients to system nodal displacements

Singularity

No constraints are applied to prevent rigid body motion of the system

The Solution of the Simple Case

Applying (at least) one boundary condition the solution is given by Eq. 90, provided that the row i and the column i are eliminated from $\boldsymbol{K}$ in Eq. 89, being i the index of the constrained displacement.

$$\boldsymbol{U} = \boldsymbol{K}^{-1}\boldsymbol{F} \tag{90}$$

For instance, if $U_1 = 0 \rightsquigarrow i = 1$ the solution of Eq. 89 is given by Eq. 91: the matrix equations governing the unknown displacements are obtained by simply striking out the first row and column of $\boldsymbol{K}$, since the constrained displacement is zero (homogeneous[12]).

$$U_2 = \frac{F_2 + F_3}{k_1} \qquad U_3 = \frac{F_2}{k_1} + \frac{F_3(k_1 + k_2)}{k_1 k_2} \tag{91}$$

[12] If the displacement boundary condition is not equal to zero (non homogeneous) the method cannot be applied — the matrices need to be manipulated differently (partitioning).

Assembly of the Global Equation System I

The aim of assembly is to form the global equation system $\boldsymbol{KQ} = \boldsymbol{F}$ using the element equations $\boldsymbol{k}_i\boldsymbol{q}_i = \boldsymbol{f}_i$. The total potential energy for the body is sum of the element potential energies π_i

$$\Pi = \sum_i \pi_i = \sum_i \frac{1}{2}\boldsymbol{q}_i^T\boldsymbol{k}_i\boldsymbol{q}_i - \sum_i \boldsymbol{q}_i^T\boldsymbol{f}_i \tag{92}$$

where $\boldsymbol{k}_i$, $\boldsymbol{q}_i$, and $\boldsymbol{f}_i$ are the stiffness matrix, the displacement vector and the load vector of the element i, respectively. Introducing the matrices defined in Eq. 93

$$\boldsymbol{Q}_d = \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \dots\end{bmatrix} \quad \boldsymbol{F}_d = \begin{bmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \dots\end{bmatrix} \quad \boldsymbol{K}_d = \begin{bmatrix}\boldsymbol{k}_1 & 0 & 0\\ 0 & \boldsymbol{k}_2 & 0\\ 0 & 0 & \dots\end{bmatrix} \tag{93}$$

we have to find a matrix $\boldsymbol{A}$ such that $\boldsymbol{Q}_d = \boldsymbol{AQ}$ and $\boldsymbol{F}_d = \boldsymbol{AF}$.

Assembly of the Global Equation System II

Eq. 92 may be rewritten as

$$\Pi = \frac{1}{2}\boldsymbol{Q}_d^T\boldsymbol{K}_d\boldsymbol{Q}_d - \boldsymbol{Q}_d^T\boldsymbol{F}_d = \frac{1}{2}\boldsymbol{Q}^T\boldsymbol{A}^T\boldsymbol{K}_d\boldsymbol{A}\boldsymbol{Q} - \boldsymbol{Q}^T\boldsymbol{A}^T\boldsymbol{F}_d \tag{94}$$

According to the Galerkin's method[13], i.e. using the condition of minimum of the total potential energy $\dfrac{\partial\Pi}{\partial\boldsymbol{Q}} = 0$, the global equation system is

$$\boldsymbol{A}^T\boldsymbol{K}_d\boldsymbol{A}\boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{F}_d = 0 \tag{95}$$

Eq. 95 shows that the algorithms of assembly the global stiffness matrix and the global load vector are

$$\boldsymbol{K} = \boldsymbol{A}^T\boldsymbol{K}_d\boldsymbol{A} \qquad \boldsymbol{F} = \boldsymbol{A}^T\boldsymbol{F}_d \tag{96}$$

[13] In elasticity problems, Galerkin's method turns out to be the principle of virtual work.

Assembly of the Global Equation System III

Example

Write the matrix $\boldsymbol{A}$ which relates local (element) and global (domain) node numbers for the finite element mesh in Fig. 15. In order to make the matrix representation compact, we assume that each node has one *dof* — in 3D solid mechanics problems each node has three *dof*.

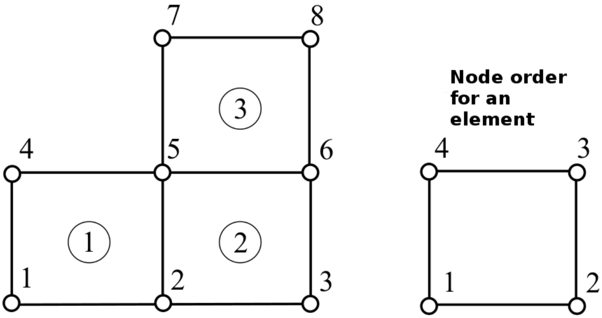


Figure 15

Assembly of the Global Equation System IV

$\boldsymbol{A}$ relates element and global nodal values in the following way: $\boldsymbol{Q}_d = \boldsymbol{A}\boldsymbol{Q}$, where $\boldsymbol{Q}$ is a global vector of nodal values and $\boldsymbol{Q}_d$ is the vector containing the element vectors. The explicit rewriting of the above relation looks as in Eq. 97.

$$\begin{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_5 \\ Q_4 \end{bmatrix} \\ \begin{bmatrix} Q_1 \\ Q_2 \\ Q_5 \\ Q_4 \end{bmatrix} \\ \begin{bmatrix} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix} \tag{97}$$

Assembly of the Global Equation System V

Fact

$\boldsymbol{A}$ in Eq. 96 is the matrix providing transformation from global to local enumeration. Because of the fraction of entries $\neq 0$ in $\boldsymbol{A}$ is very small, $\boldsymbol{A}$ is never used explicitly in actual computer codes. Suitable algorithms are implemented to sort the enumeration in order to minimize the band *in Fig. 16.*

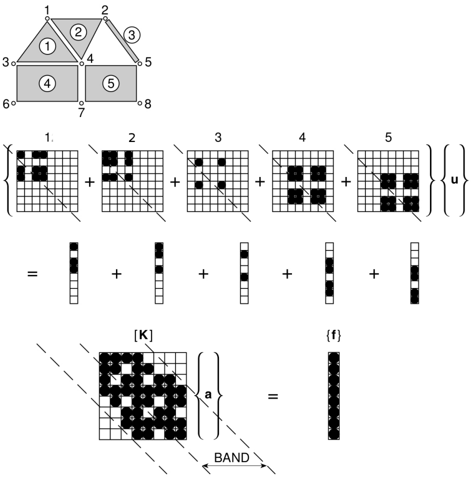


Figure 16 Credits: [6]

The Final Remarks

The real FE basic computation

As the real models are usually extremely large, as in Fig. 17, the direct method of solving Eq. 90 isn't applicable. Instead, numerical recursive procedures, typically based on Gaussian elimination and Galerkin approach, are adopted.

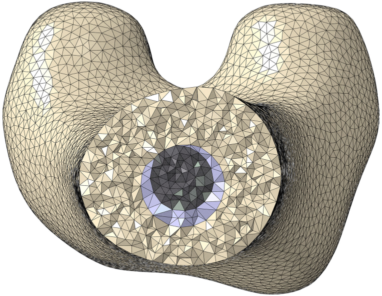


Figure 17 Credits: Materialize Inc.

An extension to dynamics

Defining, similarly to $\boldsymbol{K}$, a mass matrix $\boldsymbol{M}$ and a damping matrix $\boldsymbol{D}$, the *free* (natural) and *forced* frequencies/modes of a system can be computed by means of $\left(-\omega^2\boldsymbol{M}+\boldsymbol{K}\right)\boldsymbol{U}=[0]$ and $\boldsymbol{M}\ddot{\boldsymbol{U}}+\boldsymbol{C}\dot{\boldsymbol{U}}+\boldsymbol{K}\boldsymbol{U}=\boldsymbol{F}$, respectively.

Demonstration of the Virtual Work Theorem

Due to the symmetry of $\varepsilon^\star_{ij}$ we can write $W_i = \int_V \sum_{i,j} \sigma_{ij}\varepsilon^\star_{ij}\, dV = \int_V \sum_{i,j} \sigma_{ij} \frac{\partial u^\star_i}{\partial X_j}\, dV$.
As the product $\sigma_{ij}\frac{\partial u^\star_i}{\partial X_j}$ is equal to the sum $\frac{\partial}{\partial X_j}(\sigma_{ij}u^\star_i) - \frac{\partial \sigma_{ij}}{\partial X_j}u^\star_i$, we have $W_i = \int_V \sum_{ij} \frac{\partial}{\partial X_j}(\sigma_{ij}u^\star_i)\, dV - \int_V \sum_{ij} \frac{\partial \sigma_{ij}}{\partial X_j}u^\star_i\, dV$, where $\int_V \sum_{ij} \frac{\partial}{\partial X_j}(\sigma_{ij}u^\star_i)\, dV = \int_S \sum_{ij} \sigma_{ij} n_j u^\star_i\, dS$ is equal to $F_i u^\star_i$ and $\int_V \sum_{ij} \frac{\partial \sigma_{ij}}{\partial X_j} u^\star_i\, dV$ is equal to $-f_j u^\star_i$ (Eq. 17).
Thus, we have $W_e = W_i$.

Return

The Shear Area

Many textbooks refer to the *shear areas* $A_y^\star$ and $A_x^\star$ when treating the shear effects, defined as $A_y^\star = \dfrac{J_x^2}{\int_A \left(\dfrac{S_x(y)}{b(y)}\right)^2 dA}$ and $A_x^\star = \dfrac{J_y^2}{\int_A \left(\dfrac{S_y(x)}{h(x)}\right)^2 dA}$ where $b(y)$ is the width of the cross section at position y from the principal x axis, $h(x)$ is the width of the cross section at position x from the principal y axis, $S_x(y)$ is equal to m_r in Eq. 45, and $S_y(x)$ is the counterpart of $S_x(y)$. The ratios $\dfrac{A}{A_y^\star} \approx \chi$ and $\dfrac{A}{A_x^\star}$ are tabulated for widely used cross sections[14].

[14] For instance, computing the integrals for the rectangular cross section $b \times h$ in the example 42 gives $\dfrac{A}{A_y^\star} = \chi = \dfrac{6}{5}$

Return

Solving the Timoshenko Beam Equation I

Assuming p, E, A, and J as constants, Eq. 51c, because of Eq. 55 and recalling that $\gamma = v' + \varphi$, can be rewritten as

$$EJ\varphi'' - \frac{GA}{\chi}\left(v' + \varphi\right) = 0 \tag{98}$$

The solution of the second equation in Eq. 57, rewritten as $EJ\varphi''' = -p$, is

$$\varphi(z) = -\frac{p}{6EJ}z^3 + \frac{C_1}{2}z^2 + C_2 z + C_3 \tag{99}$$

Substituting Eq. 99 in Eq. 98 and solving for φ gives

$$v(z) = \frac{p}{24EJ}z^4 - \frac{C_1}{6}z^3 + \left(-\frac{C_2}{2} - \frac{\chi p}{2GA}\right)z^2 + \left(\frac{C_1 EJ\chi}{GA} - C_3\right)z + C_4 \tag{100}$$

so that

$$T(z) = -pz + C_1 EJ \tag{101}$$

$$M(z) = -\frac{p}{2}z^2 + C_1 EJz + C_2 EJ \tag{102}$$

Solving the Timoshenko Beam Equation II

Assuming again p, E, A, and J as constants, according to the Bernoulli simplification $\gamma = 0$, integrating the second equation in Eq. 55 gives

$$v(z) = \frac{p}{24EJ}z^4 + \frac{C_1}{6}z^3 + \frac{C_2}{2}z^2 + C_3 z + C_4 \tag{103}$$

so that

$$\varphi(z) = -\frac{p}{6EJ}z^3 - \frac{C_1}{2}z^2 - C_2 z - C_3 \tag{104}$$

$$T(z) = -pz - C_1 EJ \tag{105}$$

$$M(z) = -\frac{p}{2}z^2 - C_1 EJz - C_2 EJ \tag{106}$$

Solving the Timoshenko Beam Equation III

The constants C_1, C_2, C_3, and C_4 in Eqs. 99 to 106 are determined by the boundary conditions. As an example, we take a beam fixed at both ends, i.e. $v(0) = v(l) = \varphi(0) = \varphi(l) = 0$. In such a case, the equations of φ (Eqs. 99 and 104), M (Eqs. 102 and 106), and T (Eqs. 101 and 105) are equal for both the Timoshenko and the Bernoulli models: (Eqs. 107 to 109).

$$\varphi(z) = -\frac{p}{6EJ}z^3 + \frac{lp}{4EJ}z^2 - \frac{l^2p}{12EJ}z \tag{107}$$

$$M(z) = -\frac{p}{2}z^2 + \frac{lp}{2}z - \frac{l^2p}{12} \tag{108}$$

$$T(z) = -pz + \frac{lp}{2} \tag{109}$$

Solving the Timoshenko Beam Equation IV

Yet, the governing equations $v(z)$ for the Timoshenko and the Bernoulli beams match only if $\dfrac{J\chi(1+\nu)}{Al^2} \ll 1$, as shown in Eqs. 110 and 111, respectively.

$$v(z) = \frac{pl^4}{24EJ}\left[\left(\frac{z}{l}\right)^4 - 2\left(\frac{z}{l}\right)^3 + \left(1 - \frac{24J\chi + 24J\chi\nu}{Al^2}\right)\left(\frac{z}{l}\right)^2 + \left(\frac{24J\chi + 24J\chi\nu}{Al^2}\right)\left(\frac{z}{l}\right)\right] \quad (110)$$

$$v(z) = \frac{pl^4}{24EJ}\left[\left(\frac{z}{l}\right)^4 - 2\left(\frac{z}{l}\right)^3 + \left(\frac{z}{l}\right)^2\right] \quad (111)$$

Fact

The ratio of the maximum displacements, which occurs at $l/2$ for a beam fixed at both ends, between the Timoshenko and the Bernoulli formulations is equal to $1 + \dfrac{96J\chi}{Al^2}(1+\nu)$.
The Bernoulli beam is a very good approximation of the Timoshenko one if it is slender *enough, so that the shear contribution may be neglected.*

Example ($b \times h \times l$ beam)

For a rectangular cross-section beam with $b = .1$, $h = .3$, $\chi = 6/5$, $\nu = .3$, the error on the maximum displacement when approximating the Timoshenko formulation with the Bernoulli one is $> 3\%$ if $L < 7$.

Finding the Beam Curvature I

The center of curvature $C = [z_c; v_c]$ of $v(z)$ between z_1 and $z_2 = z_1 + dz$, obtained by solving the linear system Eq. 112

$$\begin{cases} v - v_1 = k_1(z - z_1) \\ v - v_2 = k_2(z - z_2) \end{cases} \tag{112}$$

where $k_1 = \tan(\alpha_1 + \pi/2)$ and $k_2 = \tan(\alpha_2 + \pi/2) = \tan(\alpha_1 + d\alpha + \pi/2)$, being $\alpha_1 = \arctan\left(\frac{dv}{dz}\Big|_{z=z_1}\right)$ and $\alpha_2 = \arctan\left(\frac{dv}{dz}\Big|_{z=z_2}\right)$, is given by Eq. 113

$$C = \left[z - \frac{\tan(\alpha)}{\frac{d\alpha}{dz}}, v + \frac{1}{\frac{d\alpha}{dz}}\right] \tag{113}$$

Thus the radius of curvature, $R = \sqrt{(z - z_c)^2 + (v - v_c)^2}$, is given by Eq. 114.

$$R = \frac{1}{\frac{d\alpha}{dz}\cos(\alpha)} \tag{114}$$

Finding the Beam Curvature II

Under the hypothesis of small deflections, i.e. if $\alpha \ll 1 \rightsquigarrow \alpha = \sin\alpha = \tan\alpha$, Eqs. 113 and 114 are simplified into their linear forms as in Eqs. 115 and 116

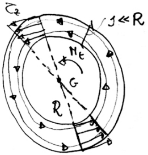
$$C = \left[z - \frac{\alpha}{\frac{d\alpha}{dz}}, v + \frac{1}{\frac{d\alpha}{dz}} \right] \tag{115}$$

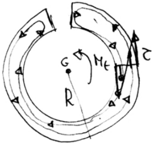
$$R = \frac{1}{\frac{d\alpha}{dz}} \tag{116}$$

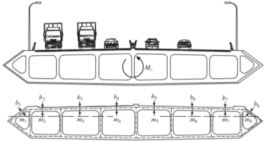
and the curvature variation, $\kappa = \frac{d\alpha}{ds} = \frac{d\alpha}{Rd\alpha} = \frac{1}{R} = \frac{d\alpha}{dz}\cos\alpha$, becomes $\kappa = -\frac{d\alpha}{dz}$, where the minus sign has been introduced in order to match the moment direction.

Return

Closed vs Non-Closed Sections ◂ Return I







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