Appendix C

LEMMAS ON DIAGRAMS

We give here proofs of Lemmas 1, 2 and 4 of Section 3.7; the proofs of Lemmas 3 and 5 are similar to those of 2 and 4, respectively, and are therefore omitted.

PROOF OF LEMMA 1. Let $\{g_1, \dots, g_N\}$ be the set of diagrams in G (N may be infinite). A typical diagram, Γ, in the set H is the star product of n_1 diagrams g_1, n_2 diagrams $g_2, \dots,$ and n_N diagrams g_N, where some of the numbers n_i may be zero; we express this result symbolically by writing

$$\Gamma = (g_1 ** n_1) * (g_2 ** n_2) * \cdots * (g_N ** n_N) \tag{C.1}$$

The value of g_i is by definition $[g_i] = I_i / S_i$, where S_i is the symmetry number, I_i is the integral associated with g_i, and we temporarily adopt the notation $[\cdots]$ to denote the value of a diagram. Then the value of Γ is

$$[\Gamma] = I/S = (1/S) \prod_{i=1}^{N} I_i^{n_i} \tag{C.2}$$

where the symmetry number is

$$S = \prod_{i=1}^{N} n_i! \times \prod_{i=1}^{N} S_i^{n_i} \tag{C.3}$$

The factors $n_i!$ take care of the permutations of the n_i identical diagrams g_i; note that (C.3) is true only for diagrams that are star irreducible. Equation (C.2) can be rewritten as

$$[\Gamma] = \prod_{i=1}^{N} I_i^{n_i} S_i^{-n_i} \bigg/ \prod_{i=1}^{N} n_i! = \prod_{i=1}^{N} [g_i]^{n_i} \bigg/ \prod_{i=1}^{N} n_i! \tag{C.4}$$

We now sum over all diagrams in H and find that

$$\sum_{\Gamma} [\Gamma] = -1 + \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \prod_{i=1}^{N} \frac{[g_i]^{n_i}}{n_i!} = -1 + \prod_{i=1}^{N} \sum_{n_i=0}^{\infty} \frac{[g_i]^{n_i}}{n_i!}$$

$$= \prod_{i=1}^{N} \exp\bigl([g_i]\bigr) - 1 = \exp\left(\sum_{i=1}^{N}[g_i]\right) - 1 \tag{C.5}$$

We subtract unity in the first line of (C.5) to exclude the case when all $n_i = 0$.

PROOF OF LEMMA 2. If S is the symmetry number and m is the number of black circles of Γ, the number of topologically inequivalent diagrams that are generated by attaching labels $1, \ldots, m$ to the black circles in all possible ways is $\nu = m!/S$. These diagrams we denote by Γ_i. It follows from the definition of a value of a diagram given by (3.7.3) that

$$\Gamma = \frac{1}{m!}\sum_{i=1}^{\nu}\Gamma_i \tag{C.6}$$

We now take the functional derivative of Γ with respect to $\gamma(\mathbf{r})$. Since

$$\frac{\delta\gamma(\mathbf{r}_i)}{\delta\gamma(\mathbf{r})} = \delta(\mathbf{r} - \mathbf{r}_i) \tag{C.7}$$

the differentiation corresponds diagrammatically to replacing successively each black γ-circle in (C.6) by a white 1-circle. In this way, νm diagrams are generated, each containing one white circle and $m - 1$ black circles. These we denote by $\Gamma_i^{(j)}$, where j is the label carried by the whitened circle. Thus

$$\frac{\delta\Gamma}{\delta\gamma(\mathbf{r})} = \frac{1}{m!}\sum_{i=1}^{\nu}\sum_{j=1}^{m}\Gamma_i^{(j)} = \frac{1}{(m-1)!}\sum_{i=1}^{\nu}\Gamma_i^{(1)} \tag{C.8}$$

In the second step we have replaced the sum over j by m times the contribution for $j = 1$; this is permissible, since the value of any $\Gamma_i^{(j)}$ is independent of j for given i.

The ν diagrams $\Gamma_i^{(1)}$ can now be divided into μ groups, chosen according to the topologically distinct diagrams into which each reduces when the labels of the $m - 1$ black circles are removed. If these diagrams are denoted by $\Gamma_1', \ldots, \Gamma_\mu'$, definition (3.7.3) implies that

$$\frac{\delta\Gamma}{\delta\gamma(\mathbf{r})} = \Gamma_1' + \cdots + \Gamma_\mu' \tag{C.9}$$

which is the required result.

PROOF OF LEMMA 4. Let m be the number of black circles in Γ. Any diagram in the set H can be expressed as $h(\Gamma; \{g_i\})$, where $\{g_i\} \equiv \{g_1, \ldots, g_m\}$ is a set of diagrams drawn from G that are attached to the black circles of Γ; some of the g_i may be identical. Two diagrams h obtained from two distinct sets $\{g_i\}$ are not necessarily different. Lemma 4 can then be written in more compact form as

$$\sum_{\{g_i\}}{}' h(\Gamma;\{g_i\}) = \big[\text{the diagram obtained from } \Gamma \text{ by associating the function } \mathcal{G}(\mathbf{r}) \text{ with each of the black circles}\big] \tag{C.10}$$

The sum in (C.10) is taken over all sets $\{g_i\}$, with the restriction (denoted by the prime) that the diagrams $h(\Gamma;\{g_i\})$ must be topologically distinct.

Let $S(\Gamma)$ be the symmetry number of Γ, and let $S(g_i)$ and $S(\Gamma;\{g_i\})$ be, respectively, the symmetry numbers of the diagrams in G and H; $S(\Gamma)$ is obviously also the symmetry number of the right-hand side of (C.10). According to the definition (3.7.4):

$$h\big(\Gamma;\{g_i\}\big) = h\big(\Gamma';\{g_i'\}\big)\big/S\big(\Gamma;\{g_i\}\big) \tag{C.11}$$

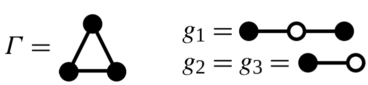
where $h(\Gamma';\{g_i'\})$ is a diagram derived from $h(\Gamma;\{g_i\})$ by labelling its black circles in an arbitrary way. Let $h(\Gamma';\{g_i\})$ be the diagram obtained from $h(\Gamma';\{g_i'\})$ by removing the labels of the black circles of the g_i', but retaining the labels of the black circles of Γ', and let $S^*(\Gamma;\{g_i\})$ be the number of permutations of the m labels of $h(\Gamma';\{g_i\})$ that give rise to topologically equivalent diagrams. For each of the S^* permutations there are $\prod_{i=1}^m S(g_i)$ permutations of the black circles of the g_i that yield diagrams equivalent to $h(\Gamma';\{g_i'\})$. We can therefore write

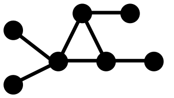
$$S\big(\Gamma;\{g_i\}\big) = S^*\big(\Gamma;\{g_i\}\big)\prod_{i=1}^m S(g_i) \tag{C.12}$$

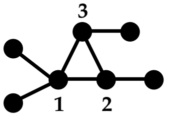
We now require a relation between $S(\Gamma)$ and $S^*(\Gamma;\{g_i\})$. Note that $S(\Gamma) \geqslant S^*(\Gamma;\{g_i\})$, since the process of decorating the black circles of Γ can never lead to an increase in symmetry number. Let $n(\Gamma;\{g_i\})$ be the number of labellings that give rise to diagrams $h(\Gamma';\{g_i\})$ that are topologically inequivalent, but yield diagrams Γ' (i.e. labelled versions of Γ on its own) that are equivalent. Consider now the set of $S(\Gamma)$ diagrams that are obtained from $h(\Gamma';\{g_i\})$ by making the $S(\Gamma)$ permutations that leave Γ' topologically unaltered. This set can be divided into precisely $n(\Gamma;\{g_i\})$ groups, such that the diagrams in different groups are topologically inequivalent to each other. Each of the $n(\Gamma,\{g_i\})$ groups consists of $S^*(\Gamma;\{g_i\})$ topologically inequivalent diagrams. Thus

$$S(\Gamma) = n\big(\Gamma;\{g_i\}\big)S^*\big(\Gamma;\{g_i\}\big) \tag{C.13}$$

Illustration.



$$h\big(\Gamma; \{g_i\}\big) =$$ 

$$h\big(\Gamma; \{g_i\}\big) =$$ 

In this example, $S(\Gamma) = 6$; $S^*(\Gamma; g_1, g_2, g_3) = 2$, because labels 2 and 3 can be permuted in $h(\Gamma'; g_1, g_2, g_3)$; and $n(\Gamma; g_1, g_2, g_3) = 3$, because permutation of labels 1 and 2 or 1 and 3 in $h(\Gamma'; g_1, g_2, g_3)$ generates diagrams that are topologically inequivalent.

By combining (C.12) and (C.13) we find that

$$S\big(\Gamma; \{g_i\}\big) = S(\Gamma) \prod_{i=1}^{m} S(g_i)/n\big(\Gamma; \{g_i\}\big) \tag{C.14}$$

If use is made of (C.11) and (C.14), the left-hand side of (C.10) can be rewritten as

$$\sum_{\{g_i\}}{}' \frac{n(\Gamma; \{g_i\})}{S(\Gamma)\prod_{i=1}^{m} S(g_i)} h\big(\Gamma'; \{g_i'\}\big) \tag{C.15}$$

or, from (3.7.4):

$$\sum_{\{g_i\}}{}' \frac{n(\Gamma; \{g_i\})}{S(\Gamma)} h\big(\Gamma'; \{g_i\}\big) \tag{C.16}$$

Remembering the significance of $n(\Gamma; \{g_i\})$, we see that (C.16) can also be expressed as

$$\sum_{g_i} \cdots \sum_{g_m} h(\Gamma'; g_1, \ldots, g_m)/S(\Gamma) \tag{C.17}$$

where the m summations are now unrestricted. But (C.17) is just a labelled diagram obtained from Γ' by associating the function $\mathcal{G}(\mathbf{r})$ with each black circle and dividing by the symmetry number $S(\Gamma)$. It follows from (3.7.4) that (C.17) is equal to the right-hand side of (C.10).