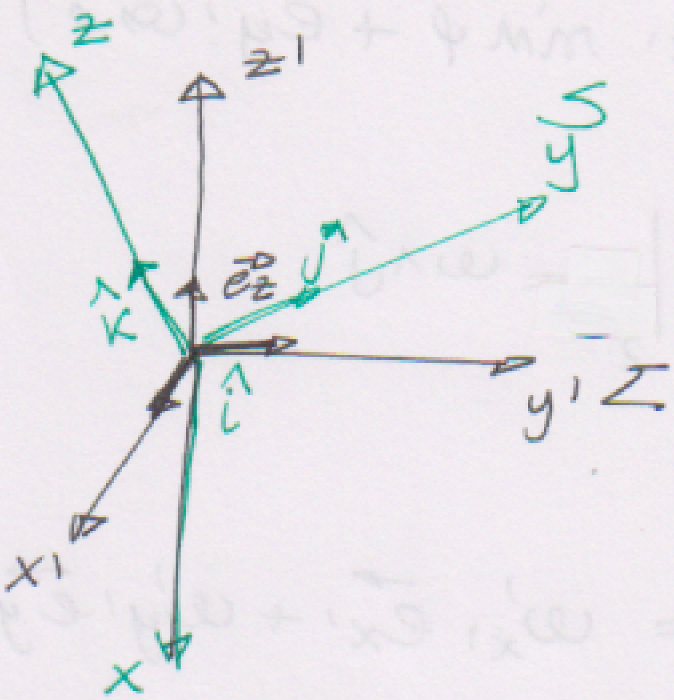
# Richiami

## Velocità angolare




$$\left.\frac{d}{dt}\vec{v}\right|_{\Sigma} = \left.\frac{d\vec{v}}{dt}\right|_{S} + \vec{\omega} \wedge \vec{v}$$

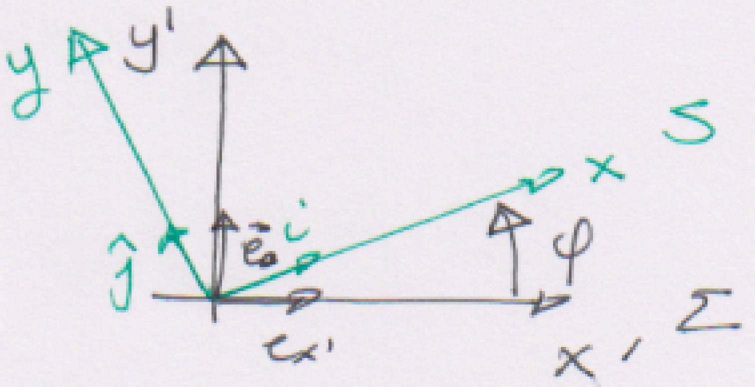
↳ veloc. angolare

$\omega$ è un vettore che può essere espresso in coord S o Σ

a seconda della convenienza

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} = \omega'_{x'} \vec{e}_{x'} + \omega_{y'} \vec{e}_{y'} + \omega_{z'} \vec{e}_{z'}$$

Caso particolare: rotazione piana




prendo $v = \hat{i}$ ⇒ $\left.\frac{d\hat{i}}{dt}\right|_{\Sigma} = \cancel{\left.\frac{d\hat{i}}{dt}\right|_{S}}^{\,0} + \vec{\omega} \wedge \hat{i}$

$\hat{i} = \vec{e}_{x'} \cos\varphi + \vec{e}_{y'} \sin\varphi$ $\quad \left.\frac{d\hat{i}}{dt}\right|_{\Sigma} = (-\vec{e}_{x'} \sin\varphi + \vec{e}_{y'} \cos\varphi)\dot{\varphi}$

$$\left(-\vec{e_{x'}} \sin\varphi + \vec{e_{y'}} \cos\varphi\right)\dot{\varphi} = \vec{\omega} \wedge \hat{i} \quad *$$

$$\left.\frac{d\hat{j}}{dt}\right|_{\Sigma} = \omega \wedge \hat{j} \qquad \vec{e_{k'}} = \hat{k} \Rightarrow \omega \wedge k = 0$$

$$\omega \wedge \vec{e_k} = 0$$

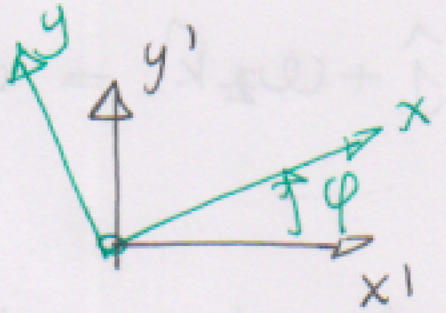
$$\vec{\omega} = \omega'_{x'} \vec{e_{x'}} + \omega'_{y'} \vec{e_{y'}} + \omega'_{z'} \vec{e_{z'}}$$

$$\vec{\omega} \wedge \vec{e_k} = 0 \Rightarrow \quad -\omega'_{x'} \vec{e_{y'}} + \omega'_{y'} \vec{e_{x'}} = 0$$

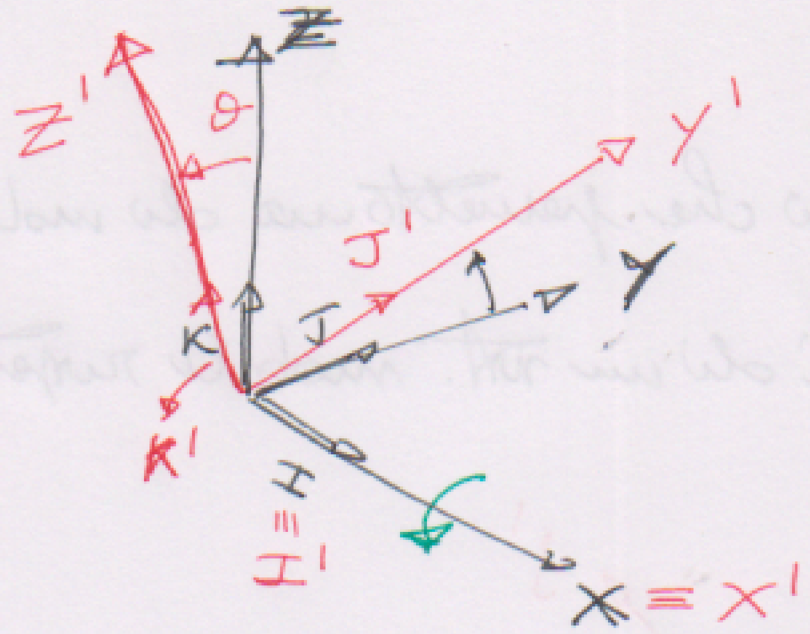
$$\Rightarrow \omega'_{x'} = \omega'_{y'} = 0$$

$$\vec{\omega} = \omega' \hat{k} = \omega' \vec{e_{k'}}$$

$$* \Rightarrow \quad \omega' \hat{k} \wedge \hat{i} = \omega'(+\hat{j}) = \dot{\varphi}\ \underbrace{\left(-\vec{e_x}\sin\varphi + \vec{e_{y'}}\cos\varphi\right)}_{+\hat{j}}$$

$\omega' = \dot{\varphi} \quad \Rightarrow$  $\Rightarrow \vec{\omega} = \dot{\varphi}\hat{k}$

2)




Ricaviamo

$$\hat{J}' = \hat{J}\cos\theta + \hat{K}\sin\theta$$

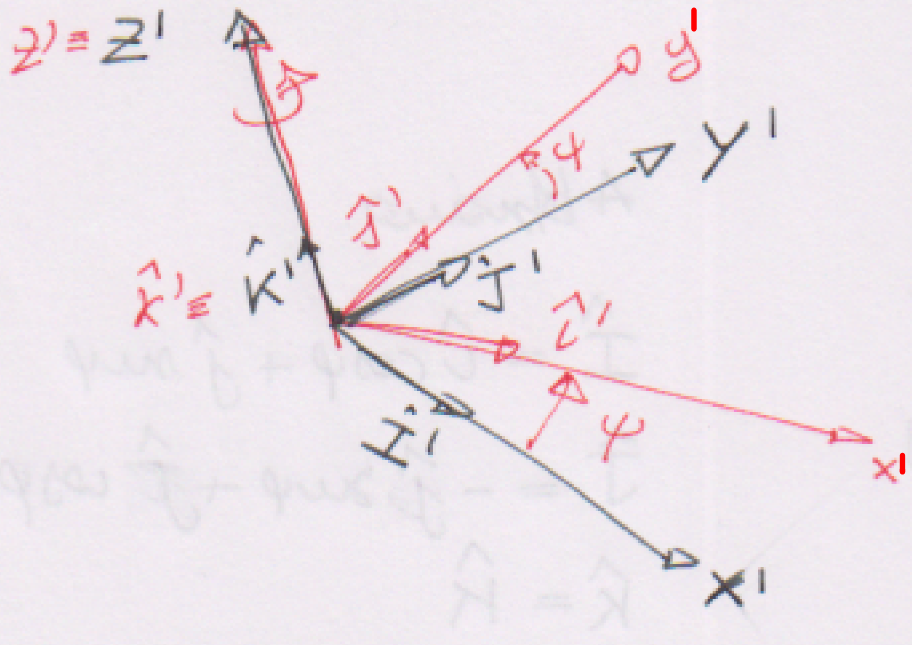
$$\hat{K}' = \hat{K}\cos\theta - \hat{J}\sin\theta$$

$$\hat{I} = \hat{I}'$$

$$\hat{J} = \hat{J}'\cos\theta - \hat{K}'\sin\theta$$

$$\hat{K} = \hat{K}'\cos\theta + \hat{J}'\sin\theta$$

3)




$$\hat{\imath}' = \hat{I}'\cos\psi + \hat{J}'\sin\psi$$

$$\hat{\jmath}' = \hat{J}'\cos\psi - \hat{I}'\sin\psi$$

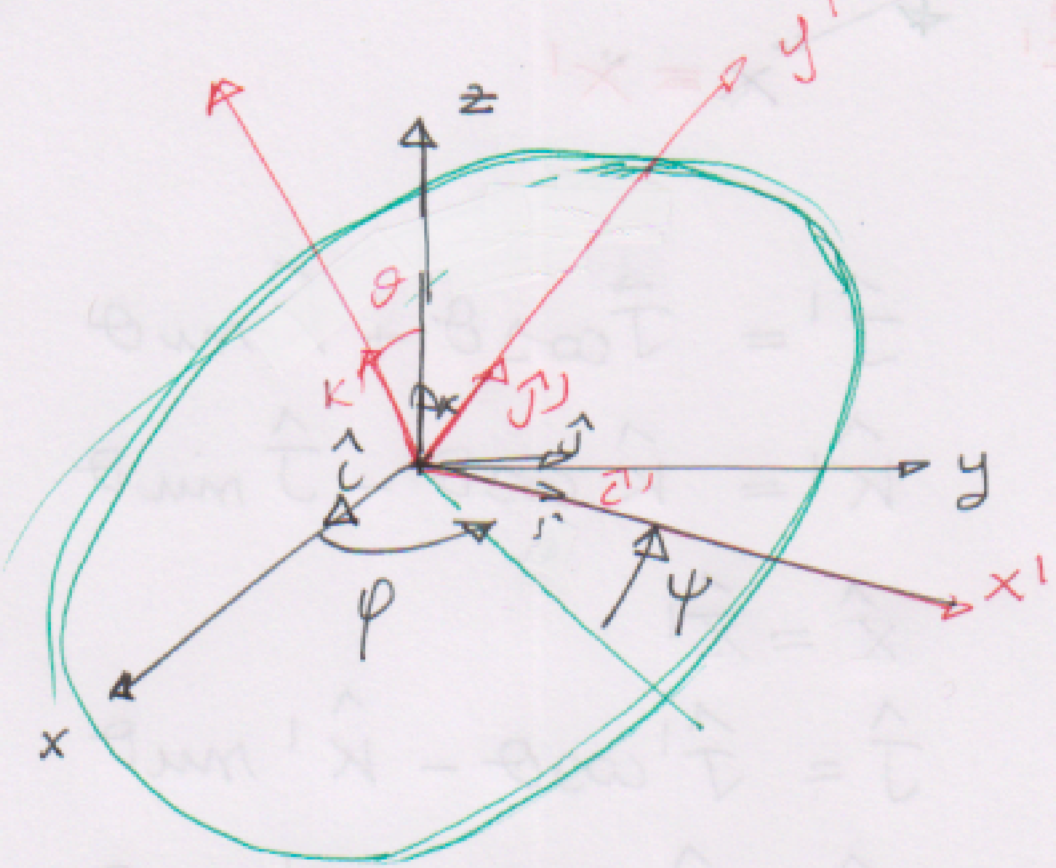
$$\kappa' = K'$$

$$\hat{I}' = \hat{\imath}'\cos\psi - \jmath'\sin\psi$$

$$\hat{J}' = \hat{\jmath}'\cos\psi + \hat{\imath}'\sin\psi$$

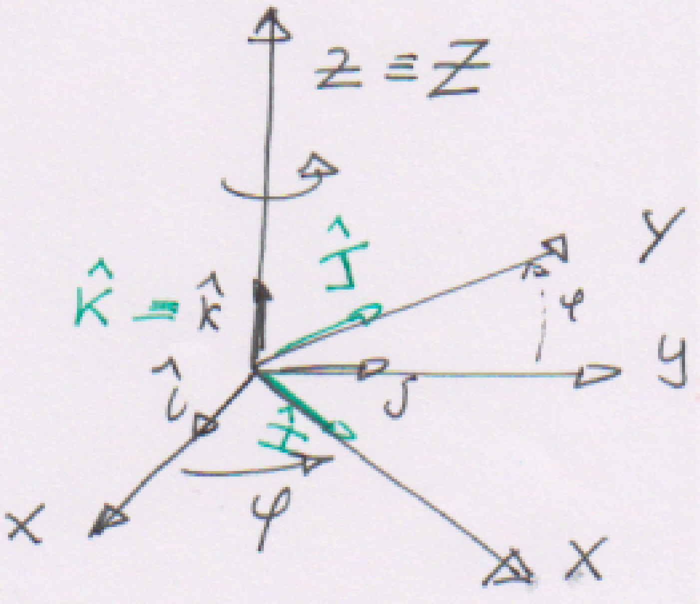
# Angoli di Eulero

Successione di rotazioni che permettono di individuare l'orientamento degli assi di un rif. mobile rispetto a uno fisso




Passiamo da $(x, y, z) \rightarrow (x', y', z')$ tramite 3 rotazioni

1)




Abbiamo

$$\hat I = \hat\imath \cos\varphi + \hat\jmath \sin\varphi$$

$$\hat J = -\hat\imath \sin\varphi + \hat\jmath \cos\varphi$$

$$\hat K = \hat k$$

$$\hat\imath = \hat I \cos\varphi - \hat J \sin\varphi$$

$$\hat\jmath = \hat J \cos\varphi + \hat I \sin\varphi$$

Utilizzo le relazioni precedenti per scrivere i versori di una terna sull'altra

$$\hat{\imath} = \hat{I}\cos\varphi - \hat{J}\sin\varphi = \hat{I}'\cos\varphi - (\hat{J}'\cos\theta - \hat{K}'\sin\theta)\sin\varphi$$

$$= \hat{I}'\cos\varphi - \hat{J}'\cos\theta\sin\varphi + \hat{K}'\sin\theta\sin\varphi$$

$$= \cos\varphi(\hat{\imath}'\cos\psi - \hat{\jmath}'\sin\psi) - \cos\theta\sin\varphi(\hat{\jmath}'\cos\psi + \hat{\imath}'\sin\psi)$$

$$+ \hat{k}\sin\theta\sin\varphi$$

$$\hat{\imath} = \hat{\imath}'(\cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi) +$$
$$\hat{\jmath}'(-\cos\varphi\sin\psi - \cos\theta\sin\varphi\cos\psi) + \hat{k}'\sin\theta\sin\varphi$$

analogamente

$$\hat{\jmath} = \hat{\imath}'(\sin\varphi\cos\psi + \cos\varphi\cos\theta\sin\psi) +$$
$$\hat{\jmath}'(-\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi) - \cos\varphi\sin\theta\, k'$$

$$\hat{k} = \sin\theta\sin\psi\,\hat{\imath}' + \sin\theta\cos\psi\,\hat{\jmath}' + \cos\theta\,\hat{k}'$$

Immaginiamo che $(x', y', z')$ sia la terna solidale ad un rigido: quanto vale la velocità angolare del rigido?

Gli angoli di Eulero descrivono 3 succ. rotazioni, a cui corrispondono 3 contributi alla velocità angolare

$$\vec{\omega} = \underbrace{\dot{\varphi}\,\hat{k} + \dot{\vartheta}\,\hat{I}' + \dot{\psi}\,\hat{k}'}_{\text{versori di terne differenti}}$$

Utilizzando le formule precedenti possiamo scrivere $\vec{\omega}$ in $(x, y, z)$ o $(x', y', z')$

Scrivo rispetto a $(x, y, z)$

$$\vec{\omega} = \dot{\varphi}\,\hat{k} + \dot{\vartheta}\,(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi) +$$
$$\dot{\psi}\,(\hat{k}\cos\vartheta - \hat{J}\sin\vartheta)$$
$$= \dot{\varphi}\,\hat{k} + \dot{\vartheta}\cos\varphi\,\hat{\imath} + \dot{\vartheta}\sin\varphi\,\hat{\jmath} + \dot{\psi}\cos\vartheta\,\hat{k}$$
$$- \dot{\psi}\sin\vartheta\,(-\hat{\imath}\sin\varphi + \hat{\jmath}\cos\varphi)$$
$$= \hat{\imath}\,(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi) +$$
$$\hat{\jmath}\,(\dot{\vartheta}\sin\varphi - \dot{\psi}\sin\vartheta\cos\varphi)$$
$$\hat{k}\,(\dot{\varphi} + \dot{\psi}\cos\vartheta)$$

analog. $\vec{\omega} = \hat{\imath}'(\dot{\varphi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi) + \hat{\jmath}'(\dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi)$

$+ k'(\dot{\varphi}\cos\vartheta + \dot{\psi})$

Energia cinetica di rotazione

$$T = \frac{1}{2}\, \vec{\omega}' \cdot (\sigma\, \vec{\omega}')$$

$\to$ mat. inerzia

Se gli assi solidali sono anche principali di inerzia $\Rightarrow$ $\sigma = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

quindi T nel solidale diventa

$$T = \frac{1}{2}\left(\omega_{x'}'^{2} I_{xx} + \omega_{y'}'^{2} I_{yy} + \omega_{z'}'^{2} I_{zz}\right)$$

$$\vec{\omega}' = \omega_{x'}'\, \hat{i}' + \omega_{y'}'\, \hat{j}' + \omega_{z'}'\, \hat{k}'$$

otteniamo

$$T = \frac{1}{2} I_{xx} \left(\dot{\varphi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi\right)^2 +$$

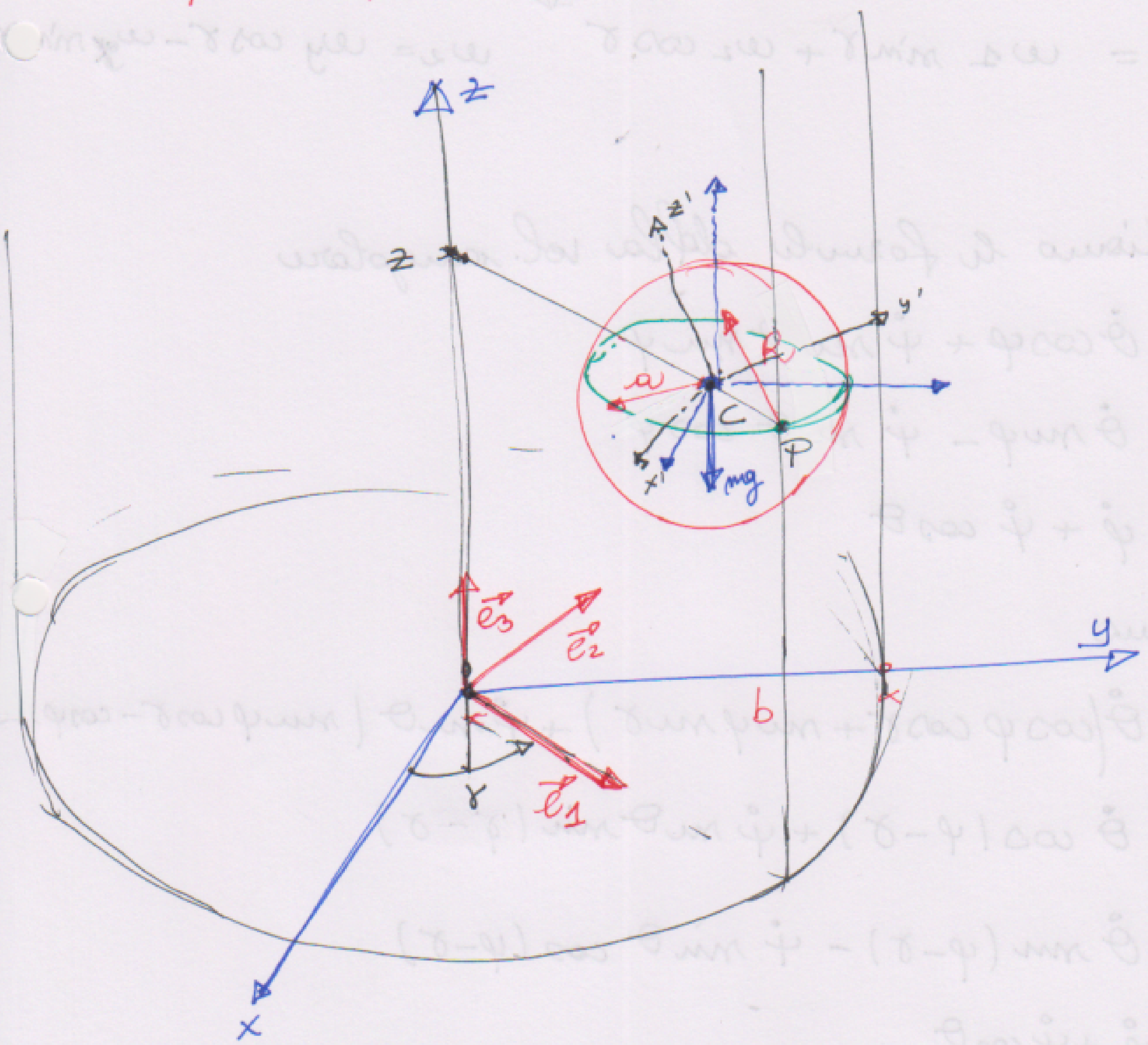
$$+ \frac{1}{2} I_{yy} \left(\dot{\varphi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi\right)^2$$

$$+ \frac{1}{2} I_{zz} \left(\dot{\varphi} \cos\vartheta + \dot{\psi}\right)^2$$

caso particolare $I_{xx} = I_{yy} = I_{zz}$

$$\boxed{T = \frac{1}{2} I \left(\dot{\varphi}^2 + \dot{\psi}^2 + \dot{\vartheta}^2 + 2 \dot{\varphi} \dot{\psi} \cos\vartheta\right)}$$

Esempio: Sfera che rotola lungo un cilindro




Sfera di raggio $a$, cilindro raggio $b$

Coord. lag. : $\gamma, z, \underbrace{\varphi, \vartheta, \psi}$

angoli di Eulero del sist. sol. alla sfera rispetto al sistema fisso $(x, y, z)$

Possiamo scrivere $\vec{\omega}$ in $(x, y, z)$ o in $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

$$\vec{\omega} = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$$

$$\omega_x = \omega_1 \cos\gamma - \omega_2 \sin\gamma \quad \Rightarrow \quad \omega_1 = \omega_x \cos\gamma + \omega_y \sin\gamma$$

$$\omega_y = \omega_1 \sin\gamma + \omega_2 \cos\gamma \qquad \omega_2 = \omega_y \cos\gamma - \omega_x \sin\gamma$$

Riprendiamo le formule della vel. angolare

$$\omega_x = \dot\theta \cos\varphi + \dot\psi \sin\theta \sin\varphi$$

$$\omega_y = \dot\theta \sin\varphi - \dot\psi \sin\theta \cos\varphi$$

$$\omega_z = \dot\varphi + \dot\psi \cos\theta$$

Otteniamo

$$\omega_1 = \dot\theta(\cos\varphi\cos\gamma + \sin\varphi\sin\gamma) + \dot\psi \sin\theta(\sin\varphi\cos\gamma - \cos\varphi\sin\gamma)$$

$$= \dot\theta \cos(\varphi-\gamma) + \dot\psi \sin\theta \sin(\varphi-\gamma)$$

$$\omega_2 = \dot\theta \sin(\varphi-\gamma) - \dot\psi \sin\theta \cos(\varphi-\gamma)$$

$$\omega_3 = \dot\varphi + \dot\psi \cos\theta$$

Condizione di non strisciamento della sfera lungo il cilindro

$$\vec{v}_P = \vec{v}_C + \vec{\omega} \wedge (\vec{e}_1 a) = 0 \qquad \rho = b - a$$

$$\vec{v}_C = \frac{d}{dt}(C - O) = \frac{d}{dt}(z\vec{e}_3 + \rho\vec{e}_1) = \dot z \vec{e}_3 + \rho \dot{\vec{e}}_1$$

$$(\omega \wedge \vec{e}_1) = (\omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3) \wedge \vec{e}_1 =$$

$$-\omega_2 \vec{e}_3 + \omega_3 \vec{e}_2$$

$$a\left(-\omega_2\,\vec{e_3}+\omega_3\,\vec{e_2}\right)=-\mathring{z}\,\vec{e_3}-\rho\,\dot{\vec{e_1}}$$

Molt. per. $\vec{e_1} \Rightarrow 0=0$

per $\vec{e_2} \Rightarrow a\omega_3=-\rho\,\vec{e_2}\,\dot{\vec{e_1}}=-\rho\mathring{\gamma} \Rightarrow \boxed{a\omega_3+\rho\mathring{\gamma}=0}$ *

$\vec{e_1}=\hat{\imath}\cos\gamma+\hat{\jmath}\sin\gamma \overset{\dot{\vec{e_1}}}{\Rightarrow} (-\hat{\imath}\sin\gamma+\hat{\jmath}\cos\gamma)\mathring{\gamma}=\vec{e_2}\mathring{\gamma}$

$\vec{e_2}=\hat{\jmath}\cos\gamma-\hat{\imath}\sin\gamma$

Molt. per $\vec{e_3} \Rightarrow -\omega_2 a=-\mathring{z} \Rightarrow \boxed{\mathring{z}-a\omega_2=0}$

Vincolo di rotol. puro 1) $a\omega_3+\rho\mathring{\gamma}=0$

2) $\mathring{z}-a\omega_2=0$

$$\boxed{\begin{aligned}&1)\Rightarrow a\mathring{\varphi}+a\mathring{\psi}\cos\vartheta+\rho\mathring{\gamma}=0\\&2)\Rightarrow \mathring{z}-a\mathring{\vartheta}\sin(\varphi-\gamma)-a\mathring{\psi}\sin\vartheta\cos(\varphi-\gamma)=0\end{aligned}}$$

1): $\sum A_{1,i}\mathring{q}_i=0 \Rightarrow A_{1,\varphi}=a \quad A_{1\psi}=a\cos\vartheta \quad A_{1\gamma}=\rho$

$A_{1\vartheta}=0 \quad A_{1,z}=0$

2): $\sum A_{2i}\mathring{q}_i=0 \Rightarrow A_{1,\varphi}=0 \quad A_{1\psi}=-a\sin\vartheta\cos(\varphi-\gamma)$

$A_{1,\vartheta}=-a\sin(\varphi-\gamma) \; A_{1z}=1 \quad A_{1,\gamma}=0$

Lagrangiana $(m=1)$

$$T = \frac{I}{2}\left(\dot\theta^2+\dot\varphi^2+\dot\psi^2+2\cos\theta\,\dot\varphi\dot\psi\right)+\frac{1}{2}\underbrace{\left(\dot z^2+\rho^2\dot\gamma^2\right)}_{v_c^2}$$

$$U = gz$$

Eq. E-L.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}-\frac{\partial L}{\partial q_i}=\lambda_1 A_{1,q_i}+\lambda_2 A_{2,q_i} \qquad q_i=\gamma,z,\varphi,\theta,\psi$$

$$\gamma \Rightarrow \frac{d}{dt}\left(\frac{1}{2}\rho^2 2\dot\gamma\right)=\lambda_1\rho \Rightarrow \rho\ddot\gamma=\lambda_1$$

$$z \Rightarrow \ddot z + g = \lambda_2$$

$$\varphi \Rightarrow I\ddot\varphi + I\frac{d}{dt}\left(\cos\theta\,\dot\psi\right)=\lambda_1 a$$

$$\theta \Rightarrow I\ddot\theta = -\lambda_2 a\sin(\varphi-\gamma) - I\sin\theta\,\dot\varphi\dot\psi$$

$$\psi \Rightarrow I\ddot\psi + I\frac{d}{dt}\left(\cos\theta\,\dot\varphi\right)=\lambda_1 a\cos\theta - \lambda_2 a\sin\theta\cos(\varphi-\gamma)$$

Sostituiamo $\lambda_1$

A $\quad I\left(\ddot\varphi+\frac{d}{dt}(\cos\theta\,\dot\psi)\right)=a\rho\ddot\gamma$

B $\quad I\ddot\theta + a(\ddot z+g)\sin(\varphi-\gamma)+I\sin\theta\,\dot\varphi\dot\psi=0$

C $\quad I\left(\ddot\psi+\frac{d}{dt}(\cos\theta\,\dot\varphi)\right)=a\cos\theta\,\rho\ddot\gamma - a(\ddot z+g)\sin\theta\cos(\varphi-\gamma)$

a cui vanno aggiunte 1)+2)

Consideriamo A e 1)

$I\left(\frac{d}{dt}\left(\dot\varphi+\cos\theta\,\dot\psi\right)\right)=a\,f\,\ddot\gamma$

$a\dot\varphi+a\dot\psi\cos\theta+f\dot\gamma=0$

$\Rightarrow \frac{f}{a} I \frac{d}{dt}\dot\gamma = a f \ddot\gamma \qquad I,a\neq 0 \Rightarrow \ddot\gamma=0$

$\dot\gamma=\Omega=\text{cost}$

$* \Rightarrow \boxed{\omega_3=-\frac{f}{a}\Omega} \qquad \dot\varphi+\dot\psi\cos\theta=-\frac{f}{a}\Omega$

Confrontiamo C e B. eliminiamo il termine $\ddot z+g$.

$B\sin\theta\cdot\cos(\varphi-\gamma)-C\sin(\varphi-\gamma)=$

$I\left(\frac{d}{dt}\left(\dot\psi+\cos\theta\,\dot\varphi\right)\right)\sin(\varphi-\gamma)+I\left(\ddot\theta+\sin\theta\,\dot\varphi\dot\psi\right)\sin\theta\cos(\varphi-\gamma)=0$ $*_2$

Notiamo

vincolo $\dot\varphi+\dot\psi\cos\theta=-\frac{f}{a}\Omega$

$\ddot\varphi+\ddot\psi\cos\theta-\dot\psi\dot\theta\sin\theta=0$

$\ddot\varphi\cos\theta=-\ddot\psi\cos^2\theta+\dot\psi\dot\theta\sin\theta\cos\theta$ $*$

Consideriamo il termine

$$\frac{d}{dt}\left(\dot{\psi}+\dot{\varphi}\cos\theta\right)=\ddot{\psi}+\ddot{\varphi}\cos\theta-\dot{\varphi}\dot{\theta}\sin\theta=$$

$$\ddot{\psi}\left(1-\cos^2\theta\right)+\dot{\psi}\dot{\theta}\sin\theta\cos\theta-\dot{\varphi}\dot{\theta}\sin\theta=$$

$$\sin\theta\left[\ddot{\psi}\sin\theta+\dot{\theta}\left(\dot{\psi}\cos\theta-\dot{\varphi}\right)\right]$$

$$*_2\Rightarrow\left(\ddot{\psi}\sin\theta+\dot{\theta}\left(\dot{\psi}\cos\theta-\dot{\varphi}\right)\right)\sin(\varphi-\gamma)+$$

$$+\left(\ddot{\theta}+\sin\theta\,\dot{\varphi}\,\dot{\psi}\right)\cos(\varphi-\gamma)=0$$

Valutiamo adesso $\dot{\omega}_1$

$$\dot{\omega}_1=\ddot{\theta}\cos(\varphi-\gamma)+\dot{\theta}\left(-\sin(\varphi-\gamma)\,(\dot{\varphi}-\dot{\gamma})\right)+$$

$$\ddot{\psi}\sin\theta\sin(\varphi-\gamma)+\dot{\psi}\cos\theta\,\dot{\theta}\sin(\varphi-\gamma)+\dot{\psi}\sin\theta\cos(\varphi-\gamma)\,(\dot{\varphi}-\dot{\gamma})$$

$$=\cos(\varphi-\gamma)\left[\ddot{\theta}+\dot{\psi}\sin\theta\,(\dot{\varphi}-\dot{\gamma})\right]+$$

$$+\sin(\varphi-\gamma)\left[-\dot{\theta}\,(\dot{\varphi}-\dot{\gamma})+\ddot{\psi}\sin\theta+\dot{\psi}\dot{\theta}\cos\theta\right]$$

Uso $*_2$

$$\cos(\varphi-\gamma)\left(-\dot{\gamma}\dot{\psi}\sin\theta\right)+$$

$$\sin(\varphi-\gamma)\left[\cancel{-\ddot{\psi}\sin\theta}-\dot{\theta}\left(\cancel{\dot{\psi}\cos\theta}-\dot{\varphi}\right)-\dot{\theta}(\dot{\varphi}-\dot{\gamma})+\right.$$

$$\left.+\cancel{\ddot{\psi}\sin\theta}+\dot{\psi}\cancel{\dot{\theta}\cos\theta}\right]$$

$$= \cos(\varphi-\gamma)\,(-\dot\gamma\dot\varphi\sin\theta) +$$

$$+\sin(\varphi-\gamma)\,[\,\underbrace{\dot\theta\dot\gamma}_{\ddot e}\,]$$

$$\dot\omega_1 = \ell[\cos(\varphi-\gamma)\,(-\dot\varphi\sin\theta) + \dot\theta\sin(\varphi-\gamma)]$$

$$= \Omega\,\omega_2$$

Analogamente si può trovare l'espressione

$$I(\dot\omega_2 + \Omega\omega_1) = -a(\ddot z + g)$$

utilizzando $\dot z = a\omega_2 \Rightarrow \ddot z = a\dot\omega_2$

$$I(\dot\omega_2 + \Omega\omega_1) = -a(a\dot\omega_2 + g)$$

$$\dot\omega_2\,(I+a^2) + I\Omega\omega_1 = -ag$$

derivando

$$\ddot\omega_2\,(I+a^2) + I\Omega^2\omega_2 = 0$$

$$\ddot\omega_2 + \frac{I\Omega^2}{I+a^2}\,\omega_2 = 0 \Rightarrow \omega_2 = A\sin(Kt-\eta)$$

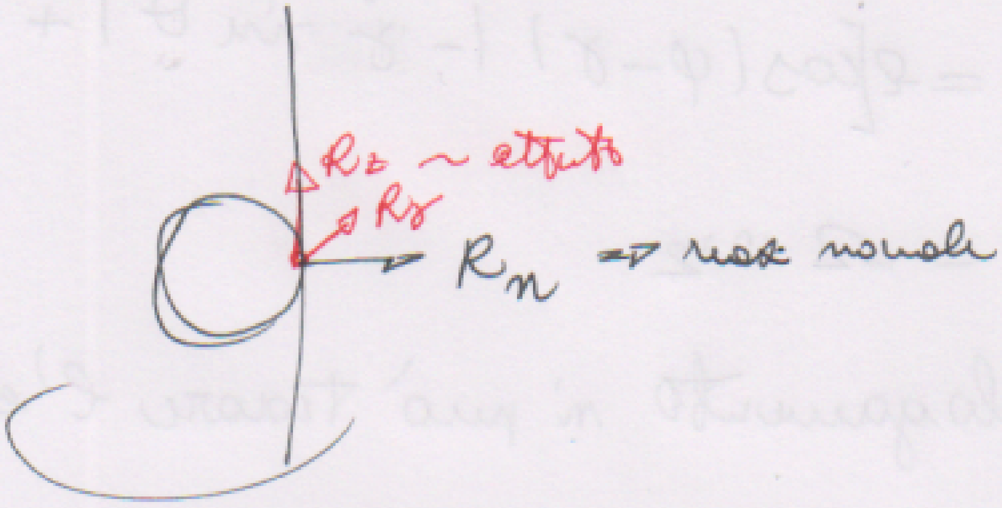
$$K = \sqrt{\frac{I\Omega^2}{I+a^2}}$$

$$z = a\int^t \omega_2 = -\frac{A}{K}a\cos(Kt-\eta) + z_0$$

Moto periodico lungo $\cancel{z}$!

Perché avviene in realtà il distaccamento della sfera!

Reazione di contatto




Si trova $R_n = -(b-a)\Omega^2 = \text{cost}$

$$R_\gamma = 0$$

$$R_z = \ddot{z} + g = a\,K^2 A\cos(Kt+\eta) + g$$

Condiz. di non slittamento

$$a\,K A + g \le \mu\,(b-a)\,\Omega^2$$

↳ coeff. attrito radente