$$q_n(t) = A e^{i\omega_n t} \Rightarrow a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

Formulazione Lagrangiana per sistemi continui

Un sistema elastico di equazione

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x^2} = 0$$

è associato ad una Lagrangiana integrale

$$\mathcal{L} = \frac{1}{2} \int \left[\left(\frac{du}{dt}\right)^2 - \left(\frac{du}{dx}\right)^2 \right] dx$$

Vediamo come costruire un metodo variazionale che leghi direttamente $\mathcal{L}$ con l'eq. evoluzione.

Consideriamo $\mathcal{L}$, la lagrangiana di un sistema continuo come un funzionale

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, x, t\right)$$

~~Passando~~ ~~a~~ ~~densità~~ Lagrangiana totale di un sistema continuo

$$L = \iint \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, x, t\right) dx\, dt$$

$\downarrow$

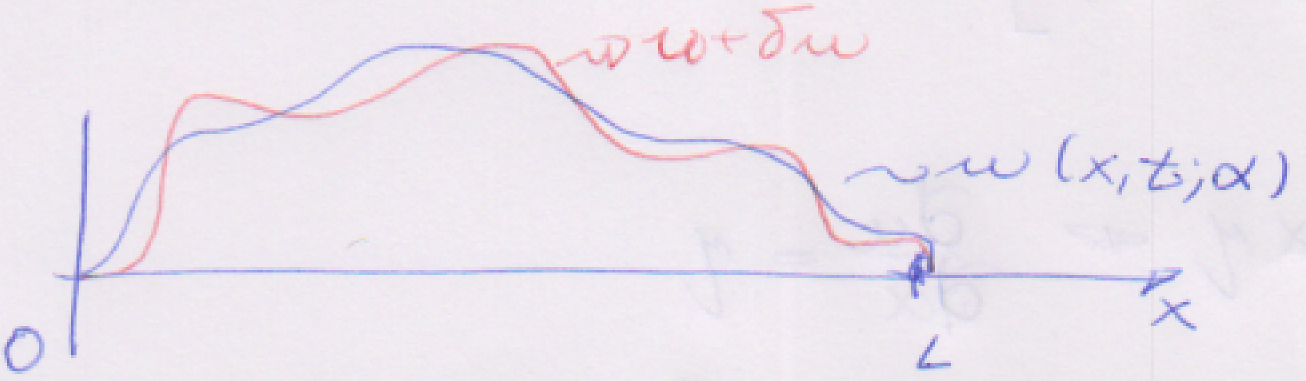
densità lagrangiana

In analogia col princ. variat. di Ham., consideriamo le variazioni di $\mathcal{L}$ al variare dell'incognita u (e delle sue derivate)

Nota: adesso le variabili spazio e tempo non vengono variate

Sist. discreto $\Rightarrow$ Variaz. di traiettorie del moto

Sist continuo $\Rightarrow$ Variaz. delle funzioni incognite



$$u(x,t;\alpha) = u(x,t,0) + \alpha \underbrace{\eta(x,t)}_{\text{variazione}}$$

in analogia al caso discreto al bordo $\eta(0) = \eta(L) = 0$: le variazioni non cambiano le c.c. del problema.

Definiamo

$$I(\alpha) = \iint dx\, dt\ \mathcal{L}(u+\alpha\eta, \partial_x(u+\alpha\eta), \partial_t(u+\alpha\eta), x, t)$$

Variazione di $I \Rightarrow \frac{dI}{d\alpha}$

Come nel caso discreto calcoliamo $\left.\frac{dI}{d\alpha}\right|_{\alpha=0}$

$$\left.\frac{d}{d\alpha}I\right|_{\alpha=0} \doteq \delta \iint \mathcal{L}\, dx\, dt$$

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2}\int_{x_1}^{x_2} dx\, dt \left[\frac{\partial \mathcal{L}}{\partial u}\frac{du}{d\alpha} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)}\frac{d}{d\alpha}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)}\frac{d}{d\alpha}\left(\frac{\partial u}{\partial t}\right)\right]$$

$$u = u_0 + \alpha \eta \Rightarrow \frac{du}{d\alpha} = \eta$$

$$\frac{d}{d\alpha}\left(\frac{\partial u}{\partial x}\right) = \frac{d}{d\alpha}\left(\frac{\partial u_0}{\partial x} + \alpha\frac{\partial \eta}{\partial x}\right) = \frac{\partial \eta}{\partial x}$$

$$\frac{d}{d\alpha}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial \eta}{\partial t}$$

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial \mathcal{L}}{\partial u}\cdot\eta + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)}\frac{\partial \eta}{\partial x} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)}\frac{\partial \eta}{\partial t}\right] dx\, dt$$

Integriamo per parti

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)}\frac{\partial \eta}{\partial x}\, dx\, dt = \int_{t_1}^{t_2}\left(\int F(x,t)\frac{\partial \eta}{\partial x}(x,t)\, dx\right) dt$$

$$F(x,t) = \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x}\right)}$$

$$\int_{x_1}^{x_2} F(x,t)\frac{\partial \eta}{\partial x}\,dx = \underbrace{\eta(x,t)F(x,t)\Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2}\frac{\partial F}{\partial x}\eta\,dx$$

$$\Rightarrow \int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x}\right)}\frac{\partial \eta}{\partial x}\,dx\,dt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial x}\frac{\partial L}{\partial\left(\frac{\partial u}{\partial x}\right)}\eta\,dx\,dt$$

Analogamente

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{\partial L}{\partial\left(\frac{\partial u}{\partial t}\right)}\frac{\partial \eta}{\partial t}\,dx\,dt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial t}\frac{\partial L}{\partial\left(\frac{\partial u}{\partial t}\right)}\eta\,dx\,dt$$

Abbiamo ottenuto

$$\delta\iint \mathcal{L}\,dx\,dt = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)}\right]\eta\,dx\,dt$$

$\delta\int \mathcal{L} = 0 \;\; \forall \eta \Rightarrow$ Eq. EULERO LAGR. per sistemi continui

$$\boxed{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} - \frac{\partial \mathcal{L}}{\partial u} = 0}$$

Analogia con E-L per sist. discreto

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0$$

Verif. che il principio Ham. fornisce l'eq. alle onde dalla Lag. ~~ottenuta~~ come limite di oscillatori armonici

$$L=\frac{1}{2}\underbrace{\iint\left[\left(\frac{\partial u}{\partial t}\right)^2-\left(\frac{\partial u}{\partial x}\right)^2\right]dx\,dt}$$

$$\mathcal{L}=\frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2-\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)}=-\frac{1}{2}\,2\,\frac{\partial u}{\partial x}\Rightarrow\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)}=-\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)}=\frac{1}{2}\,2\,\frac{\partial u}{\partial t}\xRightarrow{\frac{\partial}{\partial t}}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \mathcal{L}}{\partial u}=0$$

$$\Rightarrow-\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t^2}=0$$

Notiamo che nel pr. Ham. le variabili x e t sono trattate in maniera compl. simm. se poniamo

$x_0 = t \quad x_1 = x$

$$\Rightarrow \sum_i \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x_i}\right)} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

Il formalismo si generalizza immediatamente a sistemi di dimensione n:

$$\mathcal{L}\left[u(x_1, \dots x_n), \frac{\partial u}{\partial x_i}, x_i\right]$$

$$\Rightarrow \delta \int \mathcal{L} \, dx_1 \dots dx_n = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x_i}\right)} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

Es. equazione onde in $\mathbb{R}^3$

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 - \left(\vec{\nabla} u \cdot \vec{\nabla} u\right)\right] =$$

$$\frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 - \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i}\right)^2\right]$$

poniamo $x_0 = t$ $\quad x_1 = x,\ x_2 = y,\ x_3 = z$

$$\sum_{j=0}^{3} \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x_j}\right)} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} = 0$$

$$j=0 \Rightarrow \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} = \frac{\partial^2}{\partial t^2} u$$

$$j=1 \Rightarrow \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} = -\frac{\partial}{\partial x}\left[\frac{\partial}{\partial\left(\frac{\partial u}{\partial x}\right)} \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right)\frac{1}{2}\right]$$

$$= -\frac{\partial}{\partial x} \frac{\partial u}{\partial x} = -\frac{\partial^2 u}{\partial x^2}$$

analog. $\frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial y}\right)} = -\frac{\partial^2 u}{\partial y^2}$ $\quad \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial z}\right)} = -\frac{\partial^2 u}{\partial z^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} \underbrace{- \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2}}_{-\Delta u} = 0$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} - \Delta u = 0}$$

Eq. onde in $\mathbb{R}^3$