Volume palla unitaria in $\mathbb{R}^n$

$$B^n(0,1) = \left\{ \bar{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \le 1 \right\}$$

per comodità di notazione indico la compon. $x_n = t$

$$\bar{x} = (\underbrace{x_1, x_2, \dots, x_{n-1}}_{\tilde{x} \in \mathbb{R}^{n-1}}, t) \equiv (\tilde{x}, t)$$

Fisso la coordinata $x_n = t$, Indico la sezione della palla col piano x_n costante con E_t

$$B^n(0,1) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n-1} x_i^2 \le 1 - t^2 \right\} \equiv$$

$$\left\{ (\tilde{x}, t) \in \mathbb{R}^n \mid \|\tilde{x}\|^2_{\mathbb{R}^{n-1}} \le 1 - t^2 , \quad -1 < t < 1 \right\}$$

$$E_t = \left\{ \tilde{x} \in \mathbb{R}^{n-1} \mid \|\tilde{x}\|^2 \le 1 - t^2 \right\}$$

$$\equiv \begin{cases} B^{n-1}(0, \sqrt{1-t^2}) & -1 < t < 1 \\ 0 & |t| > 1 \end{cases}$$

La sezione di una palla con un piano è una palla di dimensione minore

$$|B^n(0,1)| = \int_{B^n(0,1)} dx_1\, dx_2 \dots dx_n =$$

$$\int_{B^n(0,1)} (dx_1 \dots dx_{n-1})\, dt = \int_{-1}^{1} dt \left(\int_{E_t} dx_1\, dx_2 \dots dx_{n-1} \right)$$

E_t = sezione della palla col piano

$$= \int_{-1}^{1} dt \int_{B^{n-1}(0,\sqrt{1-t^2})} dV^{n-1}(\tilde{x})$$

$$\int_{B^{n-1}(0,\sqrt{1-t^2})} dV^{n-1}(\tilde{x}) \xrightarrow{\bar{y} = \tilde{x}\sqrt{1-t^2}} = \int_{B^{n-1}(0,1)} (1-t^2)^{\frac{n-1}{2}} dV^{n-1}(y)$$

$$dV^{n-1}(\tilde{x}) = (1-t^2)^{\frac{n-1}{2}} dV(y)$$

$$= (1-t^2)^{\frac{n-1}{2}} \int_{B^{n-1}(0,1)} dV^{n-1}(y) = (1-t^2)^{\frac{n-1}{2}} |B^{n-1}(0,1)|$$

$$|B^n(0,1)| = |B^{n-1}(0,1)| \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} dt$$

$$= 2|B^{n-1}(0,1)| \int_0^1 (1-t^2)^{\frac{n-1}{2}} dt$$

calcoliamo $\displaystyle\int_0^1 (1-t^2)^{\frac{n-1}{2}}\, dt$

cambio variabile $\quad t = \operatorname{sen}\varphi$

$\qquad\qquad\qquad\qquad dt = \cos\varphi \, d\varphi$

$$= \int_0^{\pi/2} (\cos\varphi)^{(n-1)} \cos\varphi \, d\varphi$$

$$= \int_0^{\pi/2} (\cos\varphi)^{n} \, d\varphi$$

Integ. x parti

$$\int_0^{\pi/2} \cos^{(n-1)}\varphi \underset{D(\operatorname{sen}\varphi)}{\underbrace{\cos\varphi \, d\varphi}} = \cancel{\operatorname{sen}\varphi \cos^{n-1}\varphi}\Big|_0^{\pi/2} \to 0$$

$$+ \int_0^{\pi/2} (n-1) \cos^{n-2}\varphi \underset{1-\cos^2\varphi}{\underbrace{\operatorname{sen}^2\varphi}} \, d\varphi$$

$$(n-1)\int_0^{\pi/2} \cos^n\varphi + \int_0^{\pi/2} \cos^n\varphi = (n-1)\int_0^{\pi/2} \cos^{n-2}\varphi \, d\varphi$$

$$\boxed{\int_0^{\pi/2} \cos^n\varphi = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2}\varphi \, d\varphi}$$

Caso n pari $\quad n=2K$

$$\int_0^{\pi/2} \cos^{2K}\varphi = \frac{2K-1}{2K} \cdot \int_0^{\pi/2} \cos^{2K-2}(\varphi)\, d\varphi =$$

$$\frac{2K-1}{2K} \underbrace{\frac{(2K-2)-1}{2K-2}}_{\frac{2K-3}{2K-2}} \int_0^{\pi/2} \cos^{2K-4}\varphi\, d\varphi = \dots$$

$$= \frac{(2K-1)!!}{(2K)!!} \underbrace{\int_0^{\pi/2} \cos^0\varphi\, d\varphi}_{\pi/2}$$

Caso n dispari $\quad n=2K+1$

$$\int_0^{\pi/2} \cos(\varphi)^{2K+1} d\varphi = \frac{(2K)!!}{(2K+1)!!} \underbrace{\int_0^{\pi/2} \cos\varphi\, d\varphi}_{= 1}$$

Si è ottenuto

$n = 2K$

$$|B^{2K}(0,1)| = 2\,|B^{2K-1}(0,1)|\,\frac{(2K-1)!!}{(2K)!!}\,\frac{\pi}{2}$$

$n = 2K+1$

$$|B^{2K+1}(0,1)| = 2\,|B^{2K}(0,1)|\,\frac{(2K)!!}{(2K+1)!!}$$

caso dispari

$$|B^{2k+1}(0,1)| = 2^2\,|B^{2k-1}(0,1)|\,\frac{(2k)!!}{(2k+1)!!}\,\frac{(2k-1)!!}{(2k)!!}\,\frac{\pi}{2}$$

$$= \cancel{2^2}\,|B^{2k-1}(0,1)|\,\frac{\cancel{}}{2k+1}\,\frac{\pi}{\cancel{2}}$$

$$= 2\,|B^{2k-3}(0,1)|\,\frac{\pi}{2k+1}\,\frac{\pi}{2k-1}\,2 = \ldots$$

$$= \frac{2^k\,\pi^k}{(2k+1)!!}\,|B^1(0,1)| = \frac{2^{k+1}\,\pi^k}{(2k+1)!!}$$

Nota

$B^1(0,1) = [-1,1]$

analogamente per n pari $\quad |B^{2k}(0,1)| = \frac{\pi^k}{k!}$

Integrali di superficie in $\mathbb{R}^n$

Superf.: definizione implicita

$$V: \{x \in \mathbb{R}^n : g(x) = 0\}$$

$$\downarrow C^1$$

$$g(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$$

Esempio tipico $\quad g(\vec{x}) = \sum x_i^2 - r^2$

$$g(x) = 0 \Rightarrow \underbrace{\sum x_i^2 = r^2}_{\text{sup. sferica}} = \partial B(0, r)$$

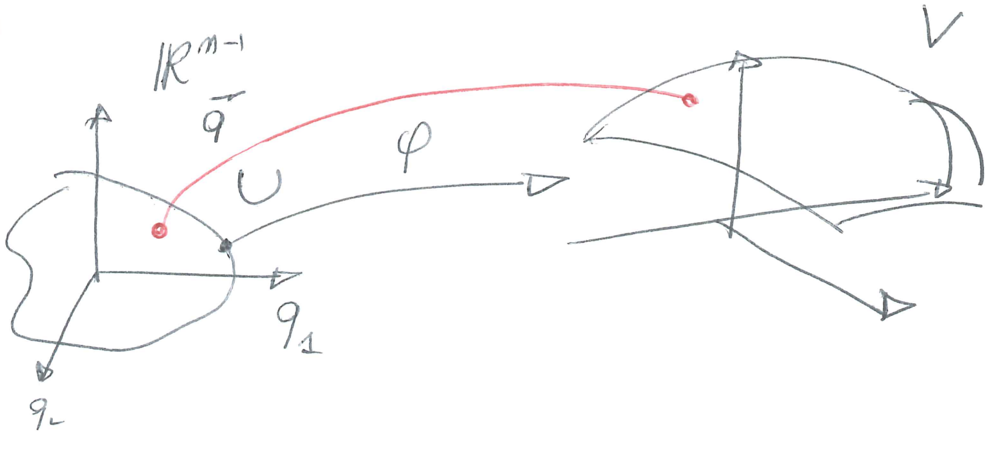
$$\int_V f(x)\, dS(x) \qquad f \in C^1(\mathbb{R}^n)$$

$$f : \mathbb{R}^n \to \mathbb{R}$$

$V \subset \mathbb{R}^{n-1}$

$dS(x)$: superficie $\downarrow$ misura di Hausdorff

Per valutare l'integrale immag. di disporre di una mappa $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ $\quad \varphi(U) = V$ iniettiva



Tratto V come una varietà ed utilizzando la parametrizz. indotta da φ si ottiene che l'integrale di superficie si scrive

$$\int_V f(\vec{x})\, d\overset{n}{S}(\vec{x}) = \int_U f(\varphi(\vec{q}))\, |J|\, \underbrace{d\overset{n-1}{V}(\vec{q})}_{\text{Lebesgue}}$$

$$|J| \doteq \sqrt{\det(J J^T)}$$

J: matrice Jacobiana

$$J = \frac{\partial \vec{\varphi}}{\partial \vec{q}}$$

$$J = {\scriptstyle n-1}\left\{ \overbrace{\begin{pmatrix} \frac{\partial \varphi_1}{\partial q_1} & \frac{\partial \varphi_2}{\partial q_1} & \frac{\partial \varphi_3}{\partial q_1} & \cdots & \frac{\partial \varphi_n}{\partial q_1} \\ \frac{\partial \varphi_1}{\partial q_2} & & & & \\ \vdots & & & & \\ \frac{\partial \varphi_1}{\partial q_{n-1}} & & & & \frac{\partial \varphi_n}{\partial q_{n-1}} \end{pmatrix}}^{n} \right.$$

$$\boxed{J_{ij} = \frac{\partial \varphi_j}{\partial q_i}} \qquad \boxed{J^T_{ij} = \frac{\partial \varphi_i}{\partial q_j}}$$

$J: \underset{\text{righe}}{(n-1)} \times \underset{\text{col.}}{n}$ $\qquad$ $J^T = n \times (n-1)$

$J \cdot J^T$: $(n-1) \times (n-1)$ quadrata

$$(J J^T)_{ij} = \sum_{r=1}^{m} \underbrace{\frac{\partial \varphi_r}{\partial q_i}}_{J_{ir}} \underbrace{\frac{\partial \varphi_r}{\partial q_j}}_{J^T_{rj}}$$

Scaling dell'Integ. di superficie

$$\int_{\partial B(0,r)} f(x)\, dS^n(x) \overset{\vec{x}=r\vec{y}}{=} \int_{\partial B(0,1)} f(r\vec{y})\, \boxed{r^{n-1}}\, dS^n(y)$$

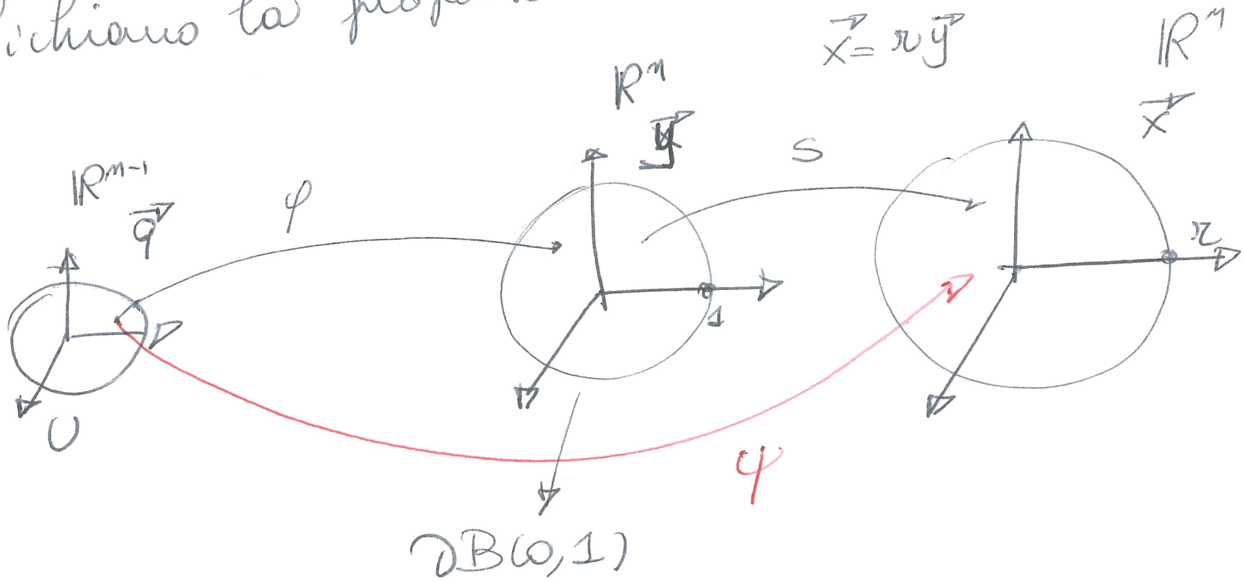
$$\bar{x} = r\bar{y} \Rightarrow dS^n(x) = r^{n-1} dS^n(y)$$

La relazione segue immediatamente dalla ~~regola~~ proprietà di scaling della misura di Hausdorff.

$$\mathcal{H}^{n-1}(\alpha\omega) = \alpha^{n-1}\mathcal{H}^{n-1}(\omega)$$

$\Downarrow$ scalare

Verifichiamo la proprietà direttamente



$$\psi = S \circ \varphi$$

$$\psi(q) = r\varphi(q)$$

Dalla definizione

$$\int_{\partial B(0,r)} f(\vec{x})\, dS^{n}(x) = \int_{U} f(\varphi(q))\, |J^{\varphi}|\, dV^{n-1}(q)$$

$$|J^{\varphi}| = \sqrt{\underbrace{\det(J^{\varphi} J^{\varphi t})}}$$

$$(J^{\varphi}J^{\varphi t})_{ij} = \sum_{n} \frac{\partial \varphi_r}{\partial q_i}\frac{\partial \varphi_r}{\partial q_j} = r^{2} \sum_{n} \frac{\partial \varphi_r}{\partial q_i}\frac{\partial \varphi_r}{\partial q_j}$$

$$= r^{2} \left(J^{\varphi} J^{\varphi t}\right)_{ij}$$

Ricordando che $\det(\lambda A) = \lambda^{n} \det(A)$
↳ mat $n \times n$

$$\det(J^{\varphi}J^{\varphi t}) = (r^{n-1})^{2} \det(J^{\varphi}J^{\varphi t})$$

$$\Rightarrow \quad |J^{\varphi}| = |J^{\varphi}|\, r^{n-1}$$

$$\int_{\partial B(0,r)} f(\vec{x})\, dS^{n}(x) = \int_{U} \underbrace{f(r\varphi(q))\, r^{n-1}}_{h(\varphi(q))} |J^{\varphi}|\, dV_q^{n-1}$$

$$= \int_{\partial B(0,1)} h(\vec{y})\, dS^{n-1}(y) = \int_{\partial B(0,1)} f(r\vec{y})\, r^{n-1}\, dS^{n-1}(y)$$

↓
dalla def. di integ. di sup