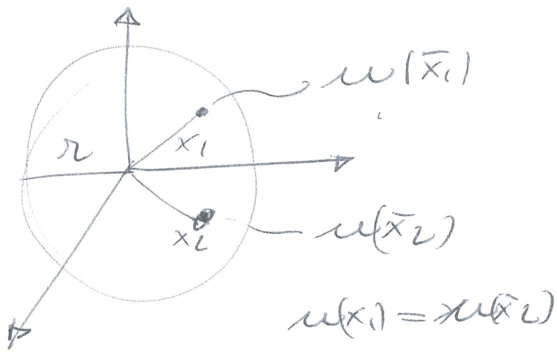
Soluzioni fondamentale dell'equazione di Laplace

Definiamo la coordinata radiale, ovvero la norma Euclidea del vettore $\vec{x} = (x_1, x_2 \dots x_n)$

$$r = |\vec{x}| = \sqrt[2]{\sum_{i=1}^{n} x_i^2}$$

Cerco una soluz. particolare dell'eq. di Laplace che dipende solo da r (a simmetria sferica)

$$u(\bar{x}) \equiv u(r)$$



$$\partial_{x_i} u = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i} = \frac{x_i}{r}\frac{\partial u}{\partial r}$$

$$\boxed{\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{\sum_j x_j^2} = \frac{1}{\cancel{2}\sqrt{\sum_j x_j^2}} \cdot \cancel{2}\sum_j \big(\overset{\delta_{ij}}{\overset{\shortparallel}{\partial_{x_i} x_j}}\big) x_j = \frac{1}{r} x_i}$$

per comodità di notazione indichiamo

$$\frac{\partial u}{\partial r} = u'$$

$$\partial^2_{x_i^2} u = \frac{\partial}{\partial x_i}\left[\frac{x_i}{r} u'\right] =$$

$$= \frac{u'}{r} + x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) u' + \frac{x_i}{r}\frac{\partial}{\partial x_i} u'$$

calcoliamo $\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_i}\left(\sum_j x_j^2\right)^{-1/2} =$

$$= -\frac{1}{2}\underbrace{\left(\sum_j x_j\right)^{-3/2}}_{= r^{-3}} \cdot 2x_i = -\frac{x_i}{r^3}$$

$$\partial^2_{x_i} u = \frac{u'}{r} - \frac{x_i^2}{r^3} u' + \frac{x_i}{r} u'' \underbrace{\frac{\partial r}{\partial x_i}}_{= \frac{x_i}{r}}$$

$$\partial^2_{x_i} u = \frac{u'}{r} - \frac{x_i^2}{r^3} u' + \frac{x_i^2}{r^2} u''$$

$$\Big| \sum_{i=1}^{n}$$

$$\Delta u = \sum_{i=1}^{n}\left(\frac{u'}{r} - \frac{x_i^2}{r^3} u' + \frac{x_i^2}{r^2} u''\right)$$

$$\sum_{i=1}^{n} x_i^2 = r^2$$

$$\Delta u = \frac{n}{r} u' - \frac{1}{r} u' + u''$$

$$\Delta u = 0 \Rightarrow \quad u'' + \frac{n-1}{r} u' = 0 \quad \text{ODE}$$

Risolviamo riscrivendo

$$\frac{u''}{u'} = \frac{(1-n)}{r}$$

$$\frac{u''}{u'} = \frac{d}{dr}\left(\ln(u')\right)$$

$$\frac{d}{dr}\left(\ln(u')\right) = \frac{(1-n)}{r}$$

integro in r $\Rightarrow$

$$\ln(u') = (1-n)\ln(r) + c$$

$$\ln\left|\frac{u'}{r^{1-n}}\right| = c$$

$$u' = r^{1-n} \underbrace{e^{c}}_{=a}$$

$$u' = r^{1-n} a$$

Distinguo 2 casi (1) $n>2$

$$u' = r^{1-n} a \Rightarrow u = \frac{r^{2-n}}{2-n} a + b = \frac{d}{r^{n-2}} + b$$

(2) $n=2 \Rightarrow u' = \frac{a}{r} \Rightarrow u = a\ln(r) + b'$

In sintesi

$$\Delta u = 0 \overset{u=u(r)}{\Longrightarrow} \qquad u = \begin{cases} a \ln(r) + b' & n=2 \\ \dfrac{d}{r^{n-2}} + b & n \geq 3 \end{cases}$$

Fissando opportunamente le costanti si ottiene la soluz. fond. dell'eq. di Laplace $\phi(r)$

$$\phi(r) = \begin{cases} -\dfrac{1}{2\pi} \ln(r) & n=2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{r^{n-2}} & n \geq 3 \end{cases}$$

Nota: per costruzione $\Delta\phi = 0$ in $\mathbb{R}^n - \{0\}$

ϕ diverge nell'origine