Formula del valor medio

Sia $u \in C^2(U)$ e $\Delta u = 0$ in U

allora

$$u(x) = \fint_{\partial B(x,r)} u\,dS = \fint_{B(x,r)} u\,dV$$

per ogni palla $B(x,r) \subset U$

Dimo

Si definisce $\varphi(r) \doteq \fint_{\partial B(x,r)} u\,dS$

$\downarrow$

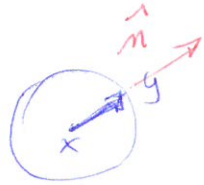
mi concentro sulla variabile r e dimostro che $\varphi(r)$ è costante

$$\varphi(r) = \underbrace{\frac{1}{n\,\alpha(n)\,r^{n-1}}}_{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\,dS(y)$$

$y = \vec{x} + r\vec{z}$ $\quad dS(y) = r^{n-1}\,dS(z)$

$$\varphi(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dV(z)$$

$$\frac{\partial \varphi}{\partial r} = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla_y u \Big|_{y=(x+zr)} \cdot \vec{z}\, dV(z)$$



$$= \fint_{\partial B(x,r)} \nabla_y u \cdot \underbrace{\left(\frac{\vec{y}-\vec{x}}{r} \right)}_{\hat{n}} dS(y)$$

$\hat{n}$ normale uscente alla sfera

$$= \frac{1}{n\alpha(n)\, r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \hat{n}} \overset{= \nabla u \cdot \hat{n}}{} dS(y)$$

teo. div.

$$\longrightarrow \frac{1}{n\alpha(n)\, r^{n-1}} \int_{B(x,r)} \Delta_y u\, dV(y) = 0$$

Quindi $\varphi(r) = \varphi(0) = \lim_{r\to 0} \varphi(r) =$

$$= \lim_{r\to 0} \fint_{\partial B(x,r)} u(y)\, dS = u(x)$$

$\Downarrow$

teo. medie integ. di Lebesgue