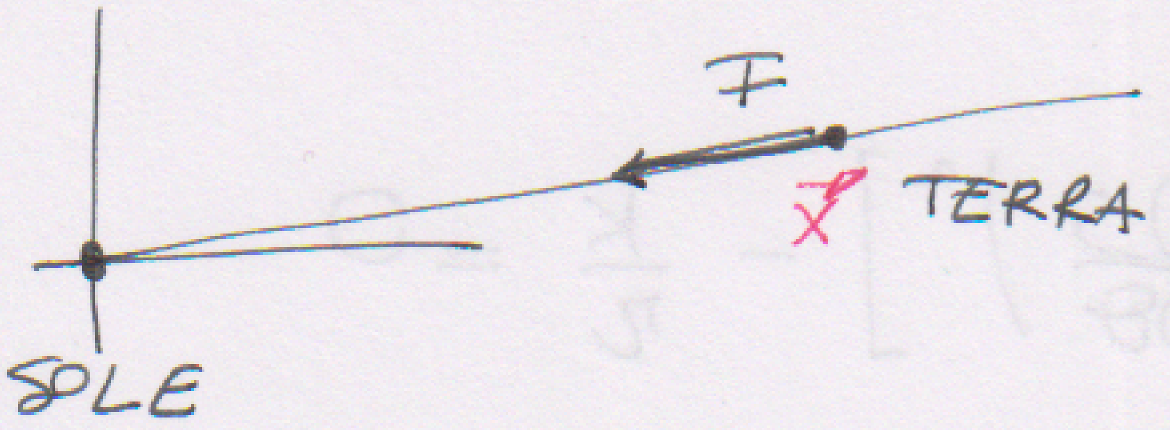
# Applicazione th H-J al problema di Keplero

Due corpi si attraggono con una forza che è inversamente prop. al quad. della loro distanza ed è diretta come la congiungente fra i corpi




$$F = -\frac{\vec{x}}{|x|^3} K = -\frac{\hat{x}}{|x|^2} K$$

$$\hat{x} = \frac{x}{|x|}$$

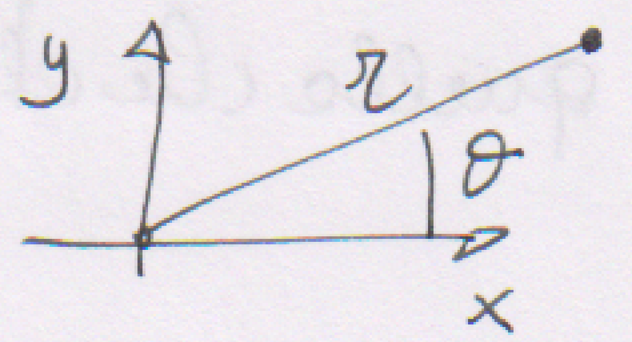
La forza F è conservativa $F = -\nabla U$ con $U = -\frac{K}{|x|}$

Verifichiamo $-\nabla U = -\sum_i \vec{e}_i \frac{\partial U}{\partial x_i} =$

$$= +K \sum \vec{e}_i \frac{\partial}{\partial x_i} \frac{1}{\sqrt{\sum x_i^2}} = -K \sum_i \frac{\vec{e}_i \, x_i}{(\sum_j x_j)^{3/2}} = -K \frac{\vec{x}}{|x|^3}$$

$$\vec{x} = \sum \vec{e}_i \, x_i$$

Descriviamo il moto del corpo in coordinate polari piane




$$U = -\frac{K}{r}$$

$$H = T + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{K}{r}$$

Notiamo subito che $\theta$ è variab. ciclica $\Rightarrow p_\theta = \text{cost.}$

Dalle relazioni costitutive $p_i = \frac{\partial S}{\partial q_i}$

$$p_r = \frac{\partial S}{\partial r} \qquad p_\theta = \frac{\partial S}{\partial \theta}$$

Eq. H-J

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial S}{\partial \theta}\right)^2\right] - \frac{k}{r} = 0$$

Tecnica di separazione delle variabili

$$S = S_1(r) + S_2(\theta) + S_3(t)$$

$$+ \frac{1}{2m}\left[\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{r}\left(\frac{dS_2}{d\theta}\right)^2\right] - \frac{k}{r} = \underbrace{-\frac{dS_3}{dt}}_{\text{costante}}$$

$$-\frac{dS_3}{dt} = E \text{ cost.} = S_3 = -Et$$

$$\frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 - \frac{k}{r} = E$$

Isolo la comp. che dip. da $\theta$ ~~dalla~~ da quella che dip da $E$

$$\left(\frac{dS_2}{d\theta}\right)^2 = r^2\left[E\,2m + K\frac{2m}{r} - \left(\frac{dS_1}{dr}\right)^2\right]$$

$$f(\theta) = g(r)$$

come prima $\left(\frac{dS_2}{d\theta}\right)^2 = \beta^2 = \text{cost.} \Rightarrow S_2 = \beta\theta$

$$\beta^2 = r^2\left[2mE + \frac{2Km}{r} - \left(\frac{dS_1}{dr}\right)^2\right]$$

Esplicito $\frac{dS_1}{dr}$

$$\frac{dS_1}{dr} = \sqrt{-\frac{\beta^2}{r^2} + 2mE + \frac{2Km}{r}}$$

$$S_1 = \int^r \sqrt{2mE + \frac{2Km}{r} - \frac{\beta^2}{r^2}}\, dr$$

Funzione H-J

$$S = \int^r \sqrt{2mE + \frac{2Km}{r} - \frac{\beta^2}{r^2}} - Et + \beta\theta$$

Quindi sapendo che i nuovi momenti congiugati $P_r$ e $P_\theta$ sono costanti, posso scegliere $P_\theta = \beta$ $P_r = E$

Formula costitutiva

$$Q_i = \frac{\partial S}{\partial P_i} \Rightarrow Q_r = \frac{\partial S}{\partial P_r} \quad Q_\theta = \frac{\partial S}{\partial P_\theta}$$

$$Q_r = \frac{\partial S}{\partial E} = \frac{\partial}{\partial E}\int^r \sqrt{2mE + \frac{2Km}{r} - \frac{\beta^2}{r^2}} - t$$

$$Q_\theta = \frac{\partial S}{\partial \beta} = \frac{\partial}{\partial \beta}\int \sqrt{2mE + \frac{2Km}{r} - \frac{\beta^2}{r^2}} + \theta$$

$$Q_\theta = \theta + \frac{\partial}{\partial\beta}\int \sqrt{2mE + \frac{2Km}{r} - \frac{\beta^2}{r^2}}\, dr$$

$$= \theta + \int^{r} \frac{(-)\beta}{r^2\sqrt{2mE + \frac{2Km}{r} - \frac{\beta^2}{r^2}}}\, dr \overset{r = 1/u}{=}$$

L'integrale è della forma $\left[\; = \theta + \frac{\beta}{\beta}\int^{u} \frac{1}{\sqrt{\frac{2mE}{\beta^2} + \frac{2Kmu}{\beta^2} - u^2}}\, du\right.$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx \Rightarrow a < 0 \quad 4ac - b^2 < 0$$

$$= -\frac{1}{\sqrt{-a}} \arcsin\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right)$$

$$Q_\theta = \theta - \arcsin\left(\frac{-2u + \frac{2Km}{\beta^2}}{\sqrt{\frac{4K^2m^2}{\beta^4} + 4\frac{2mE}{\beta^2}}}\right) =$$

$$= \theta + \arcsin\left(\frac{-\frac{2Km}{\beta^2} + 2u}{\sqrt{\frac{4K^2m^2}{\beta^4}}\sqrt{\left(1 + \frac{2E\beta^2}{mK^2}\right)}}\right) =$$

$$= \theta + \arcsin\left(\frac{-1 + u\frac{\beta^2}{km}}{\sqrt{1 + \frac{2E\beta^2}{mK^2}}}\right)$$

Sostituiamo $r = 1/u$ e invertiamo la formula. Dopo un po' di calcoli si ottiene

$$r = \frac{\beta^2}{mK} \frac{1}{\left(1+\sqrt{1+\frac{2E\beta^2}{mk^2}}\cdot\cos\left(\theta-\underbrace{\theta_0+\pi/2}_{\text{costante!}}\right)\right)}$$

Descrive la traiettoria nel piano: $r = r(\theta)$

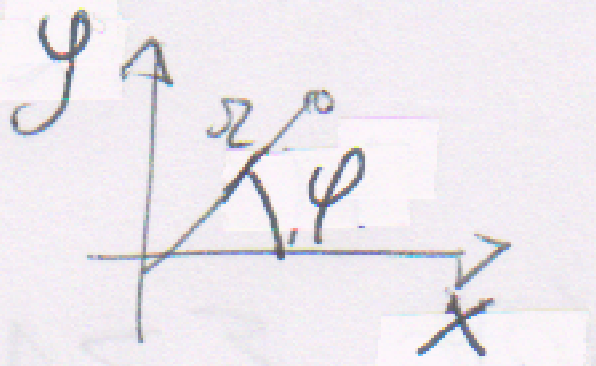
la distanza dall'origine è espressa in funzione dell'angolo

Mettendola nella forma

$$r = \frac{p}{1+\varepsilon\cos\varphi} \qquad \text{con } p = \frac{\beta^2}{mK} \qquad \varphi = \underbrace{\theta-\theta_0+\pi/2}_{\text{costanti}}$$

$$\varepsilon = \sqrt{1+\frac{2E\beta^2}{K^2 m}} \Rightarrow \text{costante}$$

$$r + \varepsilon \overbrace{r\cos\varphi}^{x} = p \qquad r\cos\varphi = x$$




$$r = p - \varepsilon x \Rightarrow r^2 = p^2 + \varepsilon^2 x^2 - 2\varepsilon p x$$

$$x^2 + y^2 = p^2 + \varepsilon^2 x^2 - 2\varepsilon p x$$

$$x^2(1-\varepsilon^2)+y^2+2\varepsilon px=p^2 \Rightarrow$$ traiettoria del pt in coord. cartesiane

caso $\varepsilon=1$

$$y^2+2px=p^2 \Rightarrow x=\frac{p^2-y^2}{2p} \Rightarrow \text{PARABOLA}$$

caso $\varepsilon<1$ ; poniamo $x'=x+\frac{\varepsilon p}{1-\varepsilon^2}$ trasl. om

$$x=x'-\frac{\varepsilon p}{1-\varepsilon^2}$$

$$x'^2(1-\varepsilon^2)-\cancel{2x'\varepsilon p}+\frac{\varepsilon^2 p^2}{1-\varepsilon^2}+y^2+\cancel{2\varepsilon px'}-2\frac{\varepsilon^2 p^2}{1-\varepsilon^2}=p^2$$

$$(x')^2\underbrace{(1-\varepsilon^2)}_{>0}+y^2=p^2+\frac{\varepsilon^2p^2}{1-\varepsilon^2}=k \Rightarrow \text{ELLISSE}$$

caso $\varepsilon>1$ l'eq. precedente ha la forma

$$-(x')^2\alpha+y^2=k \Rightarrow \text{IPERBOLE}$$

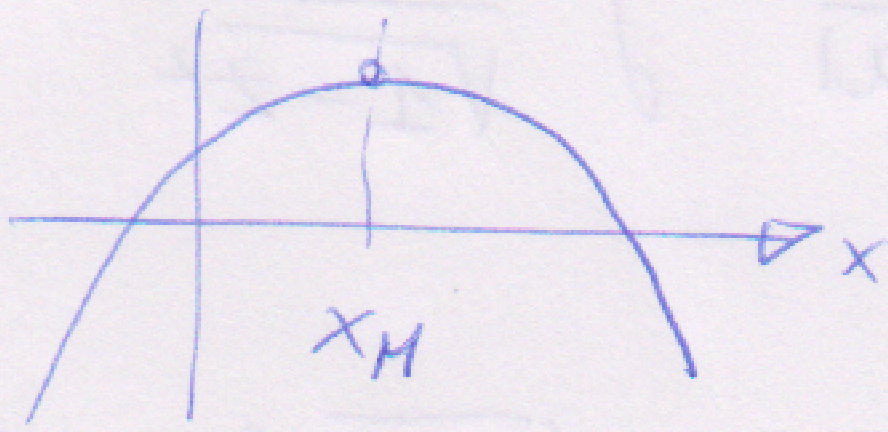
$\downarrow$

$\alpha>0$

Calcoliamo $\displaystyle\int^x \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

Cerchiamo una traslazione che elimini il termine lineare

$ax^2+bx+c \qquad a<0$




$b^2-4ac>0$

Cerco il max

$$\frac{d}{dx}(ax^2+bx+c) = 2ax_M+b=0 \Rightarrow x_M = -\frac{b}{2a}$$

Poniamo $y = x - x_M = x + \frac{b}{2a} \qquad x = y - \frac{b}{2a}$

$$ax^2+bx+c = a\left(y-\frac{b}{2a}\right)^2 + b\left(y-\frac{b}{2a}\right)+c =$$

$$= a\left(y^2+\frac{b^2}{4a^2}-\cancel{\frac{by}{a}}\right)+\cancel{by}-\frac{b^2}{2a}+c =$$

$$= ay^2 - \frac{b^2}{4a} + c = ay^2+c' \quad \text{con } c' = c-\frac{b^2}{4a}$$

NOTA $c' = \dfrac{4ac-b^2 \;{<0}}{4a \;{<0}} > 0$

$$\int^x \frac{1}{\sqrt{ax^2+bx+c}}\,dx \overset{y=x+\frac{b}{2a}}{=} \int^{x+b/2a} \frac{1}{\sqrt{ay^2+c'}}\,dy$$

$a = -|a|$ $\qquad z = y\sqrt{\frac{|a|}{c'}} \qquad dy = dz\sqrt{\frac{c'}{|a|}}$

$$= \frac{1}{\sqrt{c'}} \int^{x+b/2a} \frac{1}{\sqrt{1 - \frac{|a|}{c'} y^2}} =$$

$$= \frac{1}{\sqrt{|a|}} \int^{\left(x+\frac{b}{2a}\right)\sqrt{\frac{|a|}{c'}}} \frac{1}{\sqrt{1-z^2}}\, dz = \frac{1}{\sqrt{|a|}} \arcsin\left(\left(x+\frac{b}{2a}\right)\sqrt{\frac{|a|}{c'}}\right)$$

Riscriviamo $\sqrt{\frac{|a|}{c'}}\left(x + \frac{b}{2a}\right) = \sqrt{\frac{|a|}{c'}}\left(\frac{2ax+b}{2a}\right)$

$$= \sqrt{\frac{|a|}{c'}}\left(\frac{-(2ax+b)}{2|a|}\right) = \frac{-(2ax+b)}{\sqrt{|a| 4 c'}} =$$

$$= \frac{-(2ax+b)}{\sqrt{4|a|c - \frac{b^2|a|}{a}}} = \frac{-(2xa+b)}{\sqrt{b^2-4ac}}$$

Otteniamo $\displaystyle\int^x \frac{1}{\sqrt{ax^2+bx+c}}\, dx = -\frac{1}{\sqrt{|a|}} \arcsin\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)$