Esercizi limiti 1

PAG 1

ES 4 $\lim_{x\to 0} (\cos x)^{\frac{1}{\sin^2 x}}$

OSS $\left.\begin{array}{l}\lim_{x\to 0} \cos x = 1 \\ \lim_{x\to 0} \frac{1}{\sin^2 x} = +\infty\end{array}\right\} \Rightarrow$ si tratta di una forma indet. del tipo "$1^{+\infty}$" che si risolve riconducendosi al limite notevole

$$\lim_{\xi\to 0} (1+\xi)^{\frac{1}{\xi}} = e$$

quindi $(\cos x)^{\frac{1}{\sin^2 x}} = \left(1-(1-\cos x)\right)^{\frac{1}{\sin^2 x}} = \left(1-(1-\cos x)\right)^{\frac{1}{1-\cos x}\cdot\frac{1-\cos x}{\sin^2 x}}$

ora: $\lim_{x\to 0} \left(1-(1-\cos x)\right)^{\frac{1}{1-\cos x}} = e^{-1}$

$$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} = \lim_{x\to 0} \frac{1-\cos x}{x^2}\cdot\frac{x^2}{\sin^2 x} = \lim_{x\to 0} \frac{1-\cos x}{x^2}\cdot \lim_{x\to 0}\frac{x^2}{\sin^2 x} = \frac{1}{2}$$

ne segue $\lim_{x\to 0} (\cos x)^{1/\sin^2 x} = e^{-1/2}$

ES 8 $\lim_{x\to +\infty} = e^x - x^{\ln x}$

OSS $\lim_{x\to +\infty} e^x = +\infty$, $\lim_{x\to +\infty} x^{\ln x} = +\infty \Rightarrow$ il limite è una forma indeterminata del tipo "$\infty - \infty$" occorre capire quale dei due infiniti è di ordine superiore.

ricordiamo $x = e^{\ln x} \Rightarrow x^{\ln x} = e^{\ln^2 x}$.

quindi $e^x - x^{\ln x} = e^x - e^{\ln^2 x} = e^x\left(1-e^{\ln^2 x - x}\right)$

OSS $\lim_{x\to +\infty} \ln^2 x - x = -\infty$ per ordine di infiniti

quindi $\lim_{x\to +\infty} 1-e^{\ln^2 x - x} = 1$

Poiché $\lim_{x\to +\infty} e^x = +\infty$ segue

$$\lim_{x\to +\infty} e^x - x^{\ln x} = +\infty$$

PAG 2

ES (2) $\lim_{x\to+\infty} \ln(\sqrt{2^x+x^2})-x$

OSS: È una forma indet "$\infty-\infty$"

OSS. $\ln\sqrt{2^x+x^2} = \ln\left(\sqrt{2^x}\sqrt{1+x^2/2^x}\right) = \ln\sqrt{2^x} + \ln\sqrt{1+x^2/2^x} =$
$= \frac{x}{2}\ln 2 + \ln\sqrt{1+x^2/2^x}$

quindi $\ln\sqrt{2^x+x^2} - x = \frac{x}{2}\ln 2 - x + \ln\sqrt{1+x^2/2^x} =$
$= x\left(\frac{\ln 2}{2}-1\right) + \ln\sqrt{1+x^2/2^x}$

OSS: • $\lim_{x\to+\infty} \ln\sqrt{1+x^2/2^x} = 0$

• $\frac{\ln 2}{2}-1<0 \Rightarrow \lim_{x\to+\infty} x\left(\frac{\ln 2}{2}-1\right) = -\infty$

si ottiene: $\lim_{x\to+\infty} \ln\sqrt{2^x+x^2} - x = -\infty$.

ES (10) $\lim_{x\to+\infty} x^2\left(e^{\sin(1/(x^2+\cos x))} - \cos(1/x)\right)$

OSS. $\lim_{x\to+\infty} \frac{1}{x^2+\cos x} = 0 \Rightarrow \lim_{x\to+\infty} e^{\sin(1/(x^2+\cos x))} = 1$

$\lim_{x\to+\infty} \cos\frac{1}{x} = 1$

$\lim_{x\to+\infty} x^2 = +\infty$ $\Big\}$ $\Rightarrow$ IL limite è una forma indeterminata del tipo "$+\infty\cdot 0$"

occorre capire quale è l'ordine di infinitesimo della funzione $e^{\sin(1/(x^2+\cos x))} - \cos(1/x)$.

Denotiamo con ξ la quantità $\sin\left(1/(x^2+\cos x)\right)$. Quindi $\xi \xrightarrow[x\to+\infty]{} 0$

$$e^{\sin\xi} - \cos\left(\frac{1}{x}\right) = e^{\sin(\xi)}-1 + 1-\cos(1/x) =$$
$$= \frac{e^{\sin\xi}-1}{\sin\xi}\cdot\frac{\sin(\xi)}{\xi}\cdot\frac{1}{x^2+\cos x} + \frac{1-\cos(1/x)}{(1/x)^2}\cdot\frac{1}{x^2} =$$
$$= \frac{1}{x^2}\left(\frac{e^{\sin\xi}-1}{\sin\xi}\cdot\frac{\sin\xi}{\xi}\cdot\frac{x^2}{x^2+\cos x} + \frac{1-\cos(1/x)}{(1/x)^2}\right)$$

ne segue $x^2\left(e^{\sin\xi}-\cos\left(\frac{1}{x}\right)\right) = \frac{e^{\sin\xi}-1}{\sin\xi}\cdot\frac{\sin\xi}{\xi}\cdot\frac{x^2}{x^2+\cos x} + \frac{1-\cos(1/x)}{(1/x)^2}$

da cui $\lim_{x\to+\infty} x^2\left(e^{\sin\xi} - \cos\left(\frac{1}{x}\right)\right) =$

$$\underset{\substack{\lim_{x\to+\infty}\xi(x)=0}}{\overset{\uparrow}{=}} \underbrace{\lim_{\xi\to 0} \frac{e^{\sin\xi}-1}{\sin\xi}}_{=1} \cdot \underbrace{\lim_{\xi\to 0}\frac{\sin\xi}{\xi}}_{=1} \cdot \underbrace{\lim_{x\to+\infty}\frac{x^2}{x^2+\cos x}}_{=1} + \underbrace{\lim_{x\to+\infty}\frac{1-\cos(1/x)}{(1/x)^2}}_{\lim_{y\to 0}\frac{1-\cos y}{y^2}=\frac{1}{2}} =$$

$$= 1 + \frac{1}{2} = \frac{3}{2}.$$

PAG 3 - limSUP e limINF

ES 1 $\lim_{x\to+\infty} \ln(3^x - x)\left(\ln(x^4+1) - 2\ln(x^2-x)\right)$

. $\ln(3^x - x) = \ln 3^x + \ln(1 - x/3^x) = x\cdot\ln 3 + \ln(1-x/3^x)$

. $\ln(x^4+1) - 2\ln(x^2-x) = \ln\left(\frac{x^4+1}{(x(x-1))^2}\right) = \ln\left(\cancel{\frac{x^4}{x^4}}\,\frac{1+1/x^4}{(1-1/x)^2}\right)$

da cui $\ln(3^x - x)\left(\ln(x^4+1) - 2\ln(x^2-x)\right) =$

$$= \left(x\ln 3 + \ln(1-x/3^x)\right)\cdot \ln\left(\frac{1+1/x^4}{(1-1/x)^2}\right) =$$

$$= \underbrace{\underbrace{x}_{\to+\infty}\ln 3 \cdot \underbrace{\ln\left(\frac{1+1/x^4}{(1-1/x)^2}\right)}_{\to 0}} + \underbrace{\ln\left(1-\frac{x}{3^x}\right)\cdot \ln\left(\frac{1+\frac{1}{x^4}}{(1-\frac{1}{x})^2}\right)}_{=o(1)}$$

$$x\ln\left(\frac{1+1/x^4}{(1-1/x)^2}\right) = \ln\left(1 - \underbrace{\left(1 - \frac{(1+\frac{1}{x^4})}{(1-\frac{1}{x})^2}\right)}_{\doteq\,\xi(x)\to 0}\right)^x =$$

$$= \underbrace{\ln\left(1-\xi(x)\right)^{\frac{1}{\xi(x)}}}_{\to e^{-1}} \cdot \underbrace{\xi(x)\cdot x}_{?}$$

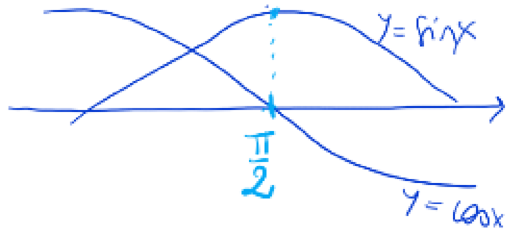
$$\xi(x)\cdot x = \left(1 - \frac{(1+\frac{1}{x^4})}{(1-\frac{1}{x})^2}\right)\cdot x = \left(1 - \frac{x^4+1}{x^4\frac{(x-1)^2}{x^2}}\right)\cdot x = \left(1 - \frac{x^4+1}{x^2(x-1)^2}\right)\cdot x =$$

$$= \left(\frac{x^2(x-1)^2 - x^4 - 1}{x^2(x-1)^2}\right)\cdot x = \left(\frac{x^2(x^2-2x+1) - x^4 - 1}{x^2(x-1)^2}\right)x =$$

$$= \frac{\cancel{x^4} - 2x^3 + x^2 - \cancel{x^4} - 1}{x^2(x-1)^2}\cdot x = \cancel{\frac{x^4}{x^4}}\,\frac{-2+1/x-1/x^3}{(1-\frac{1}{x})^2}$$

$$\xrightarrow[x\to+\infty]{} -2$$

da cui $\lim_{x\to+\infty} \ln(3^x-x)\left(\ln(x^4+1)-2\ln(x^2-x)\right) = e^2$.



ES ③. $\lim_{x\to 2} \dfrac{\cos\pi/x\,(\sin\pi/x-1)}{(x-2)^2\ln(3-x)}$

$t = \frac{\pi}{2}-\frac{\pi}{x}$ allora $t\to 0$.

$\frac{\pi}{x} = \frac{\pi}{2}-t$. $\cos\pi/x = \cos\left(\frac{\pi}{2}-t\right) = \sin t$

. $\sin\pi/x = \sin\left(\frac{\pi}{2}-t\right) = \cos t$

da cui: $\cos\frac{\pi}{x}\left(\sin\frac{\pi}{x}-1\right) = \sin t\cdot(\cos t-1) = \frac{\sin t}{t}\cdot\frac{\cos t-1}{t^2}\cdot t^3$

considero il denominatore $(x-2)^2\ln(3-x) = (x-2)^2\ln\left(1+(2-x)\right) =$

$\xi = 2-x$ $\qquad = \xi^2\ln(1+\xi) = \xi^3\cdot\frac{\ln(1+\xi)}{\xi}$

segue $\dfrac{\cos\pi/x\,(\sin\frac{\pi}{x}-1)}{(x-2)^2\ln(3-x)} = \dfrac{\sin t}{t}\,\dfrac{\cos t-1}{t^2}\cdot\dfrac{\xi}{\ln(1+\xi)}\cdot\dfrac{t^3}{\xi^3}$

$t = \frac{\pi}{2}-\frac{\pi}{x} = \frac{\pi}{2}\frac{1}{x}(x-2)$

$\xi = 2-x$

considero $\frac{t}{\xi} = \frac{\pi}{2x}\frac{(x-2)}{2-x} = -\frac{\pi}{2x}$

ora $\lim_{x\to 2} t = 0$ $\qquad\Rightarrow\qquad \lim_{x\to 2}\frac{\sin t}{t}\cdot\frac{\cos t-1}{t^2} = -\frac{1}{2}$

$\lim_{x\to 2}\xi = 0$ $\qquad\qquad \lim_{x\to 2}\frac{\xi}{\ln(1+\xi)} = 1$

inoltre $\lim_{x\to 2}\frac{t}{\xi} = -\frac{\pi}{4}$

segue $\lim(\dots) = -\frac{1}{2}\cdot 1\cdot\left(-\frac{\pi}{4}\right)^3 = \frac{\pi^3}{27}$.

ES ⑦ $\lim_{x\to+\infty}\left(3^{(x^2-\cos x)\sin\frac{1}{\sqrt{x}}} - x^x\right)$

$$3^{(x^2-\cos x)\sin\frac{1}{\sqrt{x}}} - x^x = e^{\ln 3\,(x^2-\cos x)\sin\frac{1}{\sqrt{x}}} - e^{x\ln x}$$

<u>OSS</u> $\ln 3\,(x^2-\cos x)\sin\frac{1}{\sqrt{x}} = \ln 3\,\overbrace{\left(1-\frac{\cos x}{x^2}\right)}^{\to 1}\overbrace{\left(\sin\frac{1}{\sqrt{x}}\cdot\sqrt{x}\right)}^{\to 1}\cdot x^{3/2} \xrightarrow[x\to+\infty]{} +\infty$

da cui

$$3^{(x^2-\cos x)\sin\frac{1}{\sqrt{x}}} - x^x = e^{\ln 3\left(1-\frac{\cos x}{x^2}\right)\left(\sin\frac{1}{\sqrt{x}}\cdot\sqrt{x}\right)\cdot x^{3/2}}\left(1 - e^{x\ln x - \ln 3(\dots)x^{3/2}}\right)$$

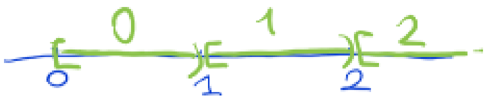
considero $x\ln x - \ln 3\left(1-\frac{\cos x}{x^2}\right)\left(\sin\frac{1}{\sqrt{x}}\cdot\sqrt{x}\right)x^{3/2} =$

$$= \underbrace{x^{3/2}}_{\to+\infty}\Bigg(\underbrace{\underbrace{\frac{\ln x}{\sqrt{x}}}_{\to 0} - \ln 3\underbrace{\left(1-\frac{\cos x}{x^2}\right)\left(\sin\frac{1}{\sqrt{x}}\cdot\sqrt{x}\right)}_{\to 1}}_{\to -1}\Bigg) \xrightarrow[x\to+\infty]{} -\infty$$

da cui $\lim_{x\to+\infty}(\dots) = 0$.

<u>ES</u> 8 a) $\lim_{x\to 0} \frac{1}{x} - \lfloor\frac{1}{x}\rfloor$

$\lfloor t\rfloor = \{$ il più grande $n\in\mathbb{N}$ t.c. $n\le t$



<u>oss</u> $t-1 < \lfloor t\rfloor \le t$

- sia $x_n = \frac{1}{n}$. Allora $x_n\to 0$ se $n\to+\infty$

considero

$\frac{1}{x_n} - \lfloor\frac{1}{x_n}\rfloor = n - \lfloor n\rfloor = 0$

$\Rightarrow \lim_{x_n\to 0} \frac{1}{x_n} - \lfloor\frac{1}{x_n}\rfloor = 0$

- sia $y_n = \frac{1}{n-\frac{1}{n}}$ Allora $\lim_{n\to+\infty} y_n = 0$

$\frac{1}{y_n} - \lfloor\frac{1}{y_n}\rfloor = n-\frac{1}{n} - \lfloor n-\frac{1}{n}\rfloor \underset{\uparrow}{=} n-\frac{1}{n} - (n-1) = 1-\frac{1}{n} \xrightarrow[n\to+\infty]{} 1$

$\lfloor n-\varepsilon\rfloor = n-1 \quad \forall\, 0<\varepsilon\le 1$

$\Rightarrow \lim_{y_n\to 0} \frac{1}{y_n} - \lfloor\frac{1}{y_n}\rfloor = 1$

segue $\nexists \lim_{x\to 0} x - \lfloor\frac{1}{x}\rfloor$

<u>OSS</u>: $\frac{1}{x}-1 < \lfloor\frac{1}{x}\rfloor \le \frac{1}{x}$ quindi $0 \le \frac{1}{x} - \lfloor\frac{1}{x}\rfloor < 1 \Rightarrow$ $\underline{\lim}\, \frac{1}{x} - \lfloor\frac{1}{x}\rfloor \ge 0$, $\overline{\lim}\, \frac{1}{x} - \lfloor\frac{1}{x}\rfloor \le 1$

poiché $\exists \{x_n\}, \{y_n\}: x_n, y_n \to 0$ e $\frac{1}{x} - \lfloor\frac{1}{x}\rfloor \to 0, \to 1$ rispettivamente

segue $0 = \underline{\lim}_{x\to 0} \frac{1}{x} - \lfloor\frac{1}{x}\rfloor$

$1 = \overline{\lim}_{x\to 0} \frac{1}{x} - \lfloor\frac{1}{x}\rfloor$

ES ⑨ $f(x) = e^{-x^2 + \frac{1}{|x|}} + \sin \frac{1}{|x|+1}$ $\qquad D = \mathbb{R} \setminus \{0\}$

OSS: $f \in C^0((0, +\infty))$

Ora: $\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to +\infty} f(x) = 0$ $\Big\} \Rightarrow$ per il teo valori intermedi $(0, +\infty) \subseteq \operatorname{Im} f(D)$

OSS: $|x|+1 > 1 \Rightarrow 0 < \frac{1}{|x|+1} < 1 \Rightarrow \sin\left(\frac{1}{|x|+1}\right) \in (0; \sin 1)$

$\Rightarrow f(x) > 0 \ \forall x \in D$ essendo somma di funzioni positive.

Ne segue $\operatorname{Im} f = (0, +\infty)$ e $\nexists\, x \in D$ t.c. $f(x) = 0$.

(IN ALTERNATIVA: studiate la derivata $f'(x)$ per $x > 0$, $x < 0$)

Ultima modifica: 23 Mar 2020