Trasformata di Fourier

Data una funzione $f \in L^1(\mathbb{R}^n)$ $\left(\text{ovvero} \int_{\mathbb{R}^n} |f(x)|\, dV(x) < \infty \right)$

Si definisce TRASFORMATA di FOURIER di f: $\hat{f} = F(f)$

$$\hat{f}(\vec{\eta}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\bar{x})\, e^{-i\vec{\eta}\cdot\bar{x}}\, dV(\bar{x})$$

Passo dalla variabile $\vec{x} \in \mathbb{R}^n$ a $\vec{\eta} \in \mathbb{R}^n$

Nota: in generale $\hat{f}(\eta) \in \mathbb{C}$

Si dimostra subito che $\hat{f}(\eta)$ è limitata

$$|\hat{f}(\eta)| \le \frac{1}{(2\pi)^{n/2}} \int |f(x)|\, dV(x) = \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1}$$

Quindi F possiamo vederla come una mappa $L^1(\mathbb{R}^n) \xrightarrow{F} L^\infty(\mathbb{R}^n)$

La Trasf. di Fourier è invertibile: si ha

Sia $f \in L^1(\mathbb{R}^n)$ t.c. $\hat{f} \in L^1(\mathbb{R}^n) \Rightarrow$

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\eta)\, e^{i\vec{\eta}\cdot\bar{x}}\, d\bar{\eta} = F^{-1}(\hat{f})$$

In forma compatta $\quad F^{-1}(F(f)) = f \qquad F^{-1} F = I$

Alcune proprietà fondamentali della Tras. Fourier

1) $F(x f)(\eta) = i \vec{\nabla}_\eta \hat{f}(\eta) = i \vec{\nabla}_\eta F(f)(\eta)$

2) $F\left(\vec{\nabla}_x f\right)(\eta) = i \vec{\eta} \hat{f}(\eta) = i \eta\, F(f)(\eta)$

3) $F^{-1}(\eta \hat{f})(x) = -i \vec{\nabla}_x f(x)$

4) $F^{-1}(\vec{\nabla}_\eta \hat{f})(x) = -i \bar{x} f(x)$

5) $$\int_{\mathbb{R}^n} |f(\bar{x})|^2 dV(\bar{x}) = \int_{\mathbb{R}^n} |\hat{f}(\bar{\eta})|^2 dV(\bar{\eta})$$

Verifica

1) $$F(x f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \vec{x} f(\bar{x}) e^{-i x \eta} dV(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\, i \nabla_\eta e^{-i x \eta} dV(x) =$$

$$i \vec{\nabla}_\eta \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i x \eta} = i \vec{\nabla}_\eta \hat{f}(\eta)$$

Ricordiamo

$$\vec{\nabla}_\eta e^{-i x \eta} = \sum \vec{e}_i \frac{\partial}{\partial \eta_i} e^{-i \sum_j x_j \eta_j} =$$

$$= \sum_i \vec{e}_i \, e^{-i\bar{x}\cdot\bar{\eta}} (-i) \sum_j \underbrace{\frac{\partial}{\partial \eta_i} x_j \eta_j}_{x_j \delta_{ij}} = \sum_i (-i) \, e^{-i\bar{x}\cdot\bar{\eta}} x_i \vec{e}_i$$

$$= -i \, \vec{x} \, e^{-i\bar{x}\bar{\eta}} \qquad \Rightarrow \quad \vec{x} = i \, \bar{\nabla}_\eta e^{-i\bar{x}\cdot\bar{\eta}}$$

Formula (2)

$$\mathcal{F}(\vec{\nabla}_x f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (\vec{\nabla}_x f) \, e^{-i\bar{x}\bar{\eta}} \, dV(x) =$$

Int per parti

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-) f(x) \underbrace{\nabla_x e^{-ix\eta}}_{-i\bar{\eta}} \, dV(x)$$

$$= i\bar{\eta} \frac{1}{(2\pi)^{n/2}} \int f(\bar{x}) \, e^{-i\bar{x}\bar{\eta}} \, dV(x) = i\bar{\eta} \, \hat{f}(\bar{\eta})$$

Trasformata di Fourier della Funzione Gaussiana

Calcoliamo $\mathcal{F}\left(e^{-\frac{|x|^2}{2}\alpha}\right)$ con $\alpha \in \mathbb{R}^+$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}\alpha - ix\eta} \, dV(x)$$

Consideriamo prima il caso $n=1$

$$F(e^{-\frac{\alpha x^2}{2}}) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{x^2\alpha}{2} - ix\eta}\, dx$$

$$y = x + \frac{i\eta}{\alpha} \quad : \quad -\frac{x^2\alpha}{2} - i\eta x = -\frac{\alpha}{2}y^2 - \frac{\eta^2}{2\alpha}$$

$$F(e^{-\frac{\alpha x^2}{2}}) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{\eta^2}{2\alpha}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}y^2}\, dy \xrightarrow{y = s\sqrt{\frac{2}{\alpha}}}$$

$$= \frac{1}{(2\pi)^{1/2}} e^{-\frac{\eta^2}{2\alpha}} \sqrt{\frac{2}{\alpha}} \underbrace{\int_{-\infty}^{\infty} e^{-s^2}\, ds}_{\sqrt{\pi}}$$

$$= \frac{1}{\alpha^{1/2}} e^{-\frac{\eta^2}{2\alpha}} \qquad \boxed{F(e^{-\frac{\alpha x^2}{2}}) = \frac{1}{\sqrt{\alpha}} e^{-\frac{\eta^2}{2\alpha}}}$$

Caso generico

$$F(e^{-\frac{\alpha}{2}|x|^2}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}\alpha - i\bar{x}\bar{\eta}}\, dV(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{n} (\frac{\alpha}{2}x_i^2 + x_i\eta_i)}\, dx_1 \dots dx_n$$

$$\mathcal{F}(\Delta u) = \mathcal{F}(\vec{\nabla} \cdot \vec{\nabla} u)$$

definito $\vec{v} = \vec{\nabla} u$

$$\mathcal{F}(\Delta u) = \mathcal{F}(\vec{\nabla} \cdot \vec{v}) \overset{\text{Prop ②}}{=} i\vec{\eta}\, \mathcal{F}(\vec{v}) =$$

$$= i\eta\, \mathcal{F}(\nabla u) \overset{②}{=} (i\eta)^2 \mathcal{F}(u) = -|\eta|^2 \hat{u}(\eta)$$

Otteniamo $\mathcal{F}(\partial_t u - \Delta u) =$

$$\mathcal{F}(\partial_t u) - \mathcal{F}(\Delta u) = \partial_t \hat{u} + |\eta|^2 \hat{u} = 0$$

$$\partial_t \hat{u} = -|\eta|^2 \hat{u}(\eta) \Rightarrow \quad \hat{u}(\eta, t) = \hat{u}(\eta, 0)\, e^{-|\eta|^2 t}$$

Cond. iniziale $\hat{u}(\eta, 0) = \mathcal{F}(u(x, 0)) = \mathcal{F}(g) = \hat{g}(\eta)$

$$\hat{u}(\eta, t) = \hat{g}(\eta)\, e^{-|\eta|^2 t}$$

$\Downarrow$

Trasf. di Fou. della sol. eq. calore

Calcoliamo $u(x, t)$

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\eta, t)) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{g}(\eta)\, e^{-|\eta|^2 t + i\eta x}\, dV(\eta)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x')\, e^{-ix'\eta}\, dV(x') \right) e^{-|\eta|^2 t + i\eta x}\, dV(\eta)$$

$$= \prod_{i=1}^{n} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2} x_i^2 + x_i \eta_i} \, dx_i}_{F\left(e^{-\frac{\alpha}{2} x^2}\right)} =$$

$$\prod_{i=1}^{n} \frac{1}{\sqrt{\cancel{2\pi}\,\alpha}} e^{-\frac{\eta_i^2}{2\alpha}} = \frac{1}{(\cancel{2\pi}\,\alpha)^{n/2}} e^{-\frac{|\eta|^2}{2\alpha}}$$

$$\boxed{F\left(e^{-\frac{\alpha}{2}|x|^2} \right) = \frac{1}{\alpha^{n/2}} e^{-\frac{|\eta|^2}{2\alpha}}}$$

Applicazione della T. Fou. all'equazione del calore

Ricaviamo la formula della soluzione di Eq. calore in $\mathbb{R}^n$ con C. I. assegnate

$$\begin{cases} \partial_t u - \Delta_x u = 0 & x \in \mathbb{R}^n \quad t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

Applichiamo la T. di Four. a equazione

Abbiamo $\quad F(\partial_t u) = \partial_t \hat{u}(\eta)$

infatti $\quad F(\partial_t u) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_t u \, e^{-i\eta x} \, dx$

derivo risp. t ma integro risp. a $\bar{x}$

Riordiniamo i termini ed invertiamo l'ordine di integrazione

$$w(\bar{x}, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x') \Bigg(\underbrace{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|\eta|^2 t + i\eta(x-x')} d\eta}_{\mathcal{F}^{-1}(e^{-|\eta|^2 t})(x-x')} \Bigg) dx'$$

Abbiamo calcolato $\mathcal{F}\left(e^{-\frac{\alpha}{2}|x|^2}\right) = \frac{1}{\alpha^{n/2}} e^{-\frac{|\eta|^2}{2\alpha}}$

applico $\mathcal{F}^{-1}$ $\Rightarrow$ $e^{-\frac{\alpha}{2}|x|^2} = \frac{1}{\alpha^{n/2}} \mathcal{F}^{-1}\left(e^{-\frac{|\eta|^2}{2\alpha}}\right)$

$$\mathcal{F}^{-1}\left(e^{-\frac{|\eta|^2}{2\alpha}}\right) = \alpha^{n/2} e^{-\frac{\alpha}{2}|x|^2}$$

$\alpha = \frac{1}{2t}$ $\Rightarrow$ $\mathcal{F}^{-1}\left(e^{-|\eta|^2 t}\right) = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}$

otteniamo $$w(x,t) = \int_{\mathbb{R}^n} g(x') \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-x'|^2}{4t}} dV(x')$$

$$= \int_{\mathbb{R}^n} g(x') \, \Psi(x-x', t) \, dV(x')$$