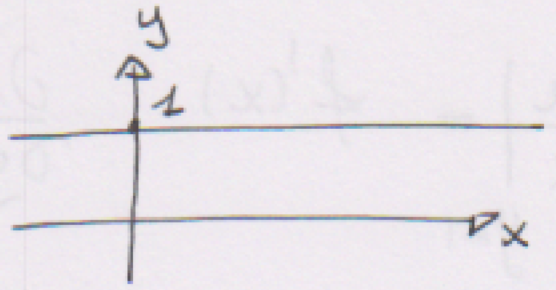
Es. Metodo Riemann

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0$$



$$u\Big|_{y=1} = f(x) \qquad \frac{\partial u}{\partial y}\Big|_{y=1} = F(x)$$

il cambio di variabili $\xi = xy$, $\eta = y/x$ porta

alla forma canonica

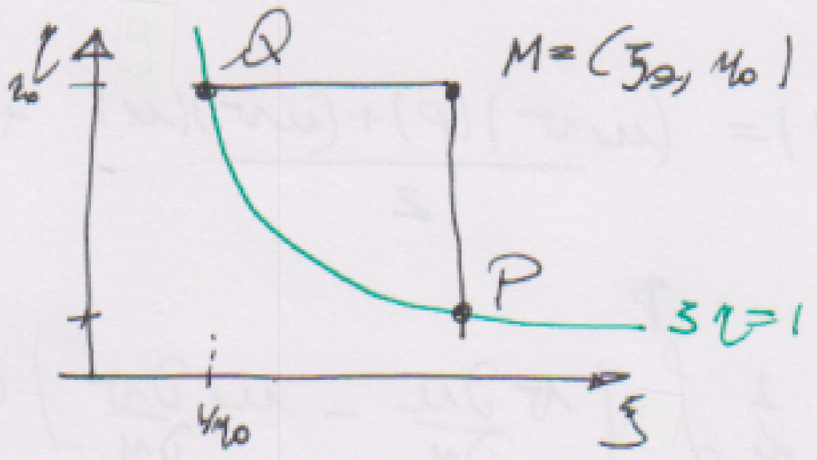
$$\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{1}{2\xi}\frac{\partial u}{\partial \eta} = 0$$

La retta $y=1$ diventa nelle nuove variabili

$\xi\eta = 1$ un' iperbole

$x = \sqrt{\xi/\eta}$

$y = \sqrt{\xi\eta}$



$$Q = \left(\frac{1}{\eta_0}, \eta_0\right) \qquad P = \left(\xi_0, \frac{1}{\xi_0}\right)$$

Calcoliamo il valore delle derivate di u nella curva $\xi\eta = 1$

$$\frac{\partial u}{\partial \xi}\Big|_{\xi\eta=1} = \left(\frac{1}{2}\frac{\partial u}{\partial x}\frac{1}{\sqrt{\xi}}\frac{1}{\sqrt{\eta}} + \frac{\partial u}{\partial y}\frac{1}{\sqrt{\eta}}\frac{1}{\xi^{3/2}}\frac{1}{2}\right)\Big|_{\xi\eta=1} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2\xi}\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \eta}\Big|_{\xi\eta=1} = \left(\frac{\partial u}{\partial x}\sqrt{\xi}\,(-)\frac{1}{2}\frac{1}{\eta^{3/2}} + \frac{\partial u}{\partial y}\sqrt{\xi}\,\frac{1}{2}\frac{1}{\sqrt{\eta}}\right)\Big|_{\xi\eta=1}$$

$$= -\frac{\xi^2}{2}\frac{\partial u}{\partial x} + \frac{1}{2}\xi\frac{\partial u}{\partial y}$$

dalle c.c.

$$\left.\frac{\partial u}{\partial x}\right|_{y=1} = f'(x) \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = F(x) \qquad \underbrace{x=\sqrt{\frac{\xi}{\eta}} \quad y=\sqrt{\xi\eta}}_{\xi\eta=1}$$

$$\Downarrow$$

$$x=\xi \qquad y=1$$

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi\eta=1} = \frac{1}{2} f'(\xi) + \frac{1}{2\xi} F(\xi)$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\xi\eta=1} = -\frac{1}{2}\xi^2 f'(\xi) + \frac{1}{2}\xi F(\xi)$$

Applichiamo la formula di Riemann con

$$e=-\frac{1}{2\xi} \qquad d=0 \quad f=0 \quad g=0$$

$$u(M) = \frac{(uv)(P)+(uv)(Q)}{2} + \frac{1}{2}\int_Q^P \left(v\frac{\partial u}{\partial \xi} - u\frac{\partial v}{\partial \xi}\right) d\xi$$

$$-\frac{1}{2}\int_Q^P \left(v\frac{\partial u}{\partial \eta} - u\frac{\partial v}{\partial \eta} - \frac{1}{\xi} vu\right) d\eta$$

Calcoliamo la funzione di Riemann

L'eq. aggiunta $\quad \dfrac{\partial^2 v}{\partial \xi \partial \eta} + \dfrac{1}{2\xi}\dfrac{\partial v}{\partial \eta} = 0$

$$v(\xi,\eta_0;\xi_0,\eta_0) = e^{-\int_{\xi_0}^{\xi}\frac{1}{2s}ds} = \sqrt{\frac{\xi_0}{\xi}} \quad \text{in } \overline{MQ}$$

$$v(\xi_0,\eta;\xi_0,\eta_0) = e^{\int_{\eta_0}^{\eta} 0\, d\eta} = 1 \quad \text{in } \overline{MP}$$

Poichè v sul bordo non dip. da η, funziona anche una soluz. dell'eq. diff.

$$v(\xi, \eta; \xi_0, \eta_0) = \sqrt{\frac{\xi_0}{\xi}} \Rightarrow \text{Funz. Riemann.}$$

Calcoliamo $u(M)$

$$u(P) = u\left(\xi_0, \frac{1}{\xi_0}\right) = f(\overset{x=\xi}{\underset{\vee}{\xi_0}})$$

$$u(Q) = u\left(\frac{1}{\eta_0}, \eta_0\right) = f(1/\eta_0)$$

$$v(P) = v\left(\xi_0, \frac{1}{\xi_0}, \xi_0, \eta_0\right) = 1$$

$$v(Q) = v\left(\frac{1}{\eta_0}, \eta_0, \xi_0, \eta_0\right) = \sqrt{\xi_0 \eta_0}$$

Otteniamo

$$u(\xi_0, \eta_0) = \frac{1}{2} f(\xi_0) + \frac{1}{2}\sqrt{\xi_0 \eta_0}\, f\left(\frac{1}{\eta_0}\right) +$$

$$\frac{1}{2}\int_{1/\eta_0}^{\xi_0} \left(\sqrt{\frac{\xi_0}{\xi}}\left(\frac{1}{2} f'(\xi) + \frac{1}{2\xi} F(\xi)\right) + f(\xi)\,\frac{1}{2}\,\frac{\sqrt{\xi_0}}{\xi^{3/2}}\right) d\xi$$

$$+\frac{1}{2}\underbrace{\int_{\eta_0}^{1/\xi_0} \left(\sqrt{\frac{\xi_0}{\xi}}\left(-\frac{1}{2} f'(\xi)\xi^2 + \frac{1}{2}\xi F(\xi)\right) - \sqrt{\frac{\xi_0}{\xi}}\,\frac{f(\xi)}{\xi}\right)\Bigg|_{\xi=\frac{1}{\eta}} d\eta}$$

cambio variabile $\xi = \frac{1}{\eta} \Rightarrow d\eta = -\frac{1}{\xi^2} d\xi$

$$\Rightarrow \frac{1}{4}\int_{1/\eta_0}^{\xi_0} \left(\sqrt{\frac{\xi_0}{\xi}} f'(\xi) + \sqrt{\xi_0}\,\frac{1}{\xi^{3/2}} F(\xi) - \sqrt{\xi_0}\,\frac{f(\xi)}{\xi^{3/2}}\right) d\xi$$

Otteniamo

$$u(\xi_0, \eta_0) = \frac{1}{2} f(\xi_0) + \frac{1}{2} \sqrt{\xi_0 \eta_0}\, f\left(\frac{1}{\eta_0}\right) +$$

$$- \frac{1}{4} \sqrt{\xi_0} \int_{\xi_0}^{1/\eta_0} f(s)\, s^{-3/2}\, ds + \frac{1}{2} \sqrt{\xi_0} \int_{\xi_0}^{1/\eta_0} F(s)\, s^{-3/2}\, ds$$

Nelle variabili originarie $\xi_0 = xy \quad \eta_0 = y/x$

$$u(x, y) = \frac{1}{2} f(xy) + \frac{1}{2} y\, f\left(\frac{x}{y}\right) +$$

$$- \frac{1}{4} \sqrt{xy} \int_{xy}^{x/y} f(s)\, s^{-3/2}\, ds + \frac{1}{2} \sqrt{xy} \int_{xy}^{x/y} F(s)\, s^{-3/2}\, ds$$