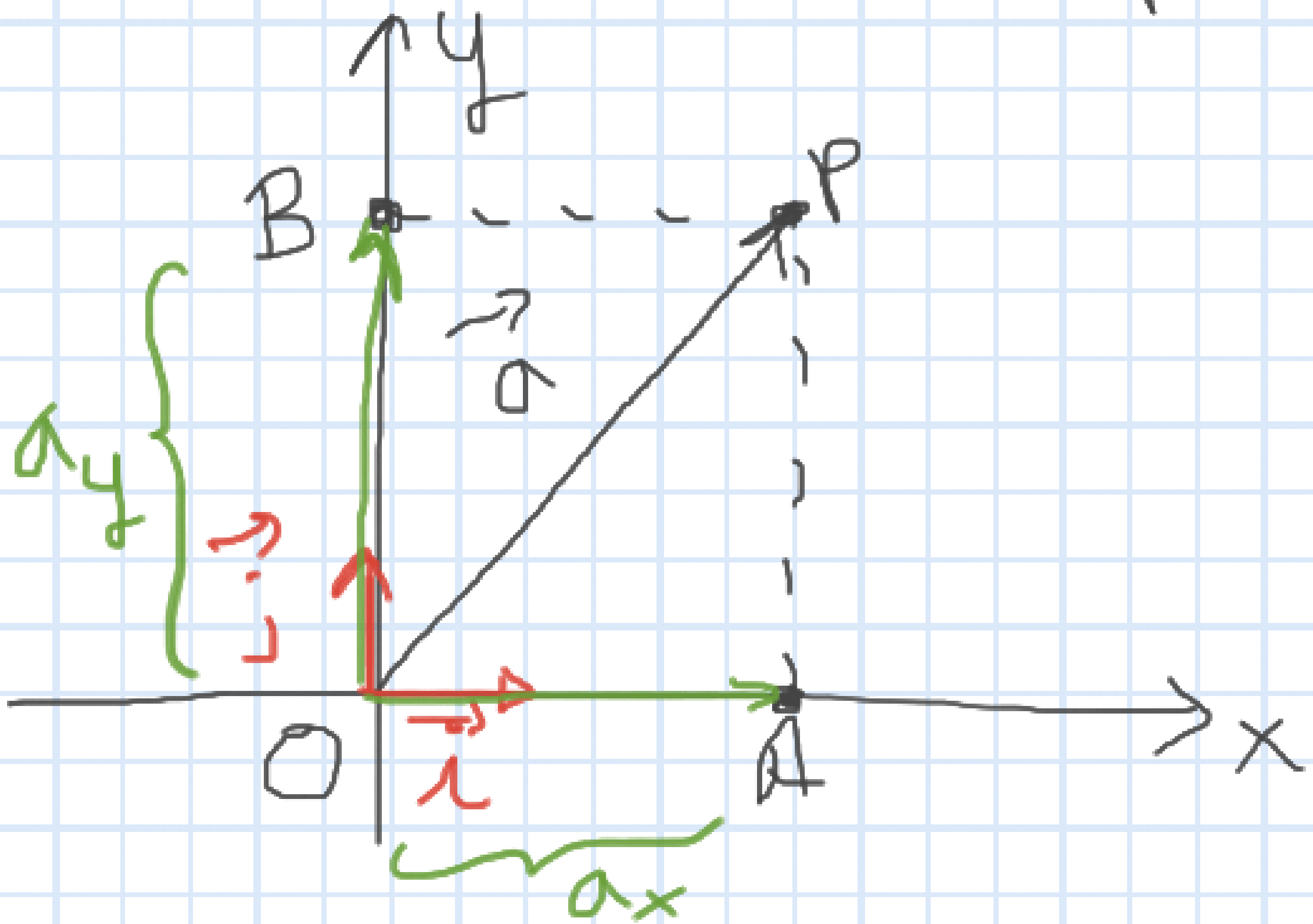
RAPPRESENTAZIONE CARTESIANA dei VETTORI

VETTORI nel PIANO

Consideriamo un piano cartesiano xOy



e disegnamo un vettore $\vec{a}$

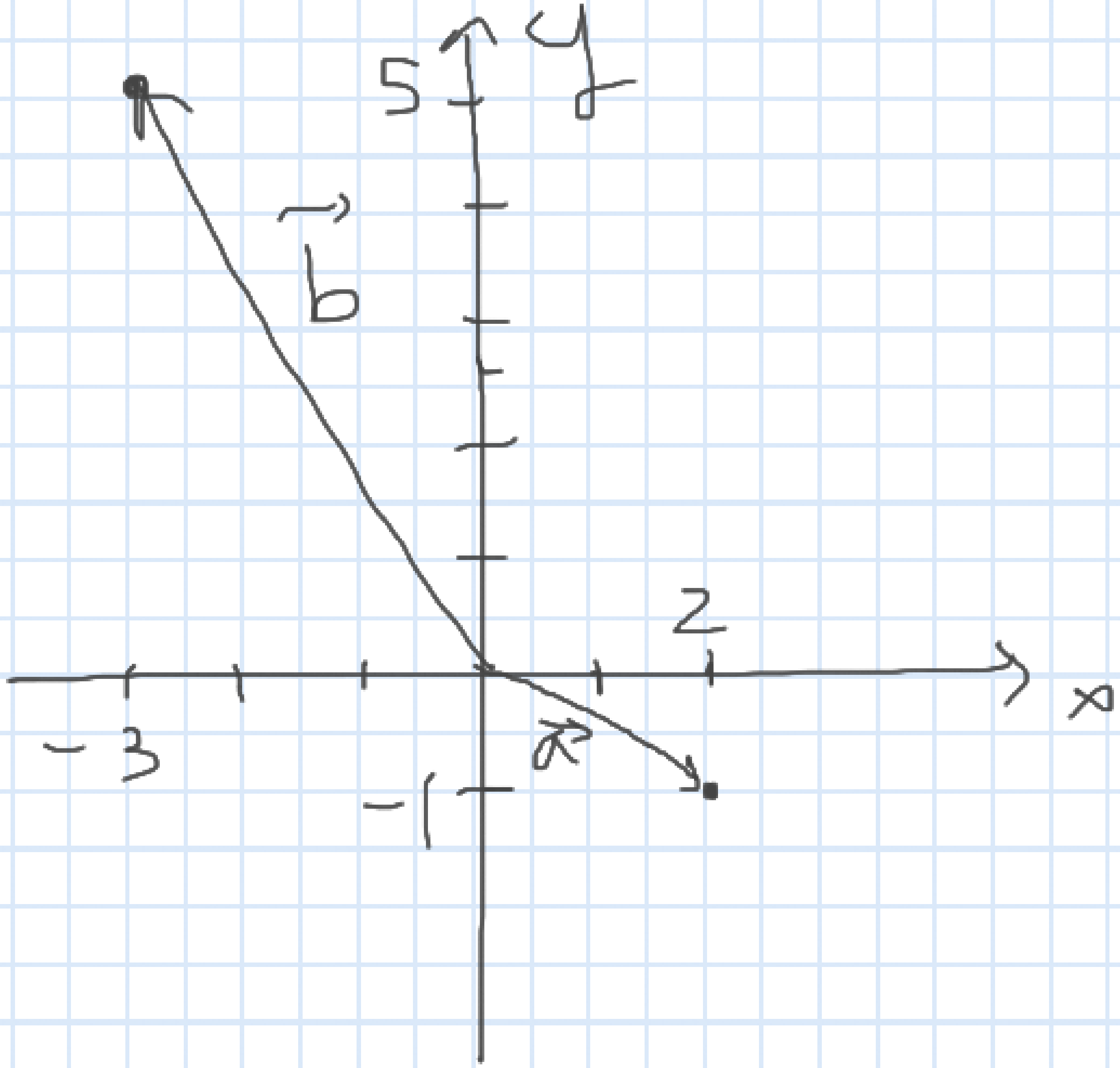
$\vec{i}, \vec{j}$ versori aventi direzioni e versi dell'asse x e y rispettivamente

$\overline{OA} = a_x \quad \overline{OB} = a_y \longrightarrow \overrightarrow{OA} = a_x \vec{i} \quad \overrightarrow{OB} = a_y \vec{j}$

$$\vec{a} = \underbrace{a_x}\vec{i} + \underbrace{a_y}\vec{j}$$

componenti cartesiane di $\vec{a}$

$$\vec{a} = (a_x ; a_y)$$



Esempio

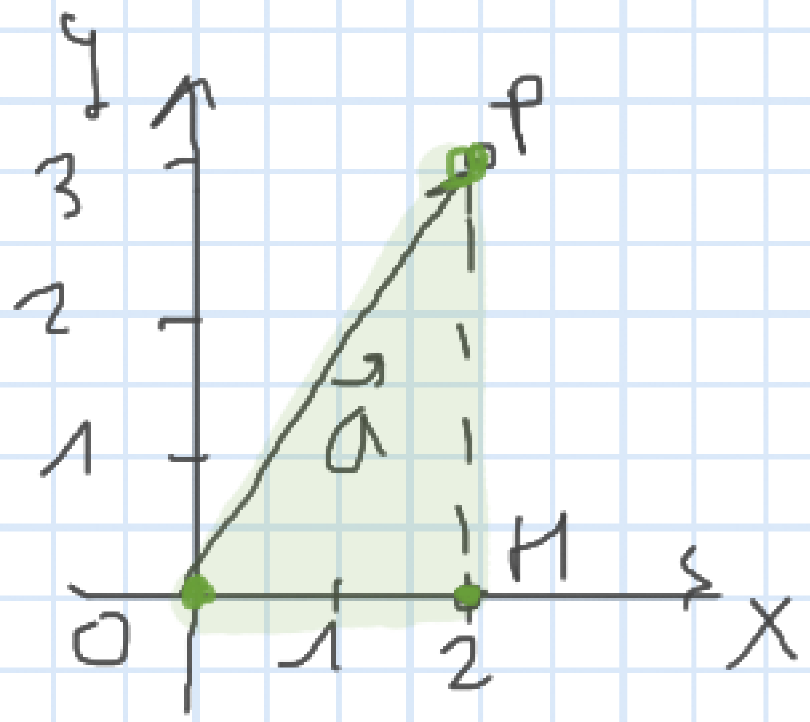
$\vec{a} = (2, -1) = 2\vec{i} - \vec{j}$

$\vec{b} = (-3, 5) = -3\vec{i} + 5\vec{j}$

$\vec{i} = (1, 0)$

$\vec{j} = (0, 1)$

Modulo e direzione



$\vec{a} = (2, 3)$

$|\vec{a}|$ = modulo = lunghezza di $\vec{a}$

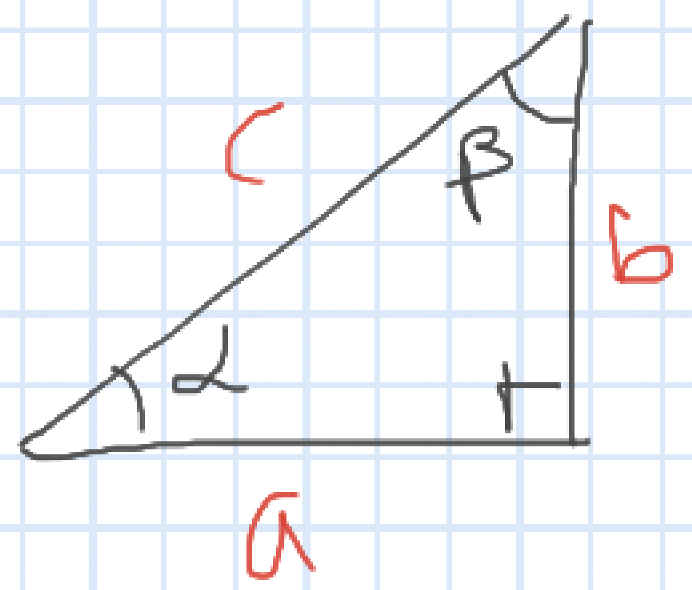
Si applica il Teo di Pitagora

$|\vec{a}| = \sqrt{\overline{OH}^2 + \overline{HP}^2}$

$$\boxed{|\vec{a}| = \sqrt{a_x^2 + a_y^2}}$$ MODULO

$|\vec{a}| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

(Parentesi) FORMULE DI TRIGONOMETRIA



Dato un triangolo rettangolo

CATETO = IPOTENUSA · COS (ANGOLO ADIACENTE)

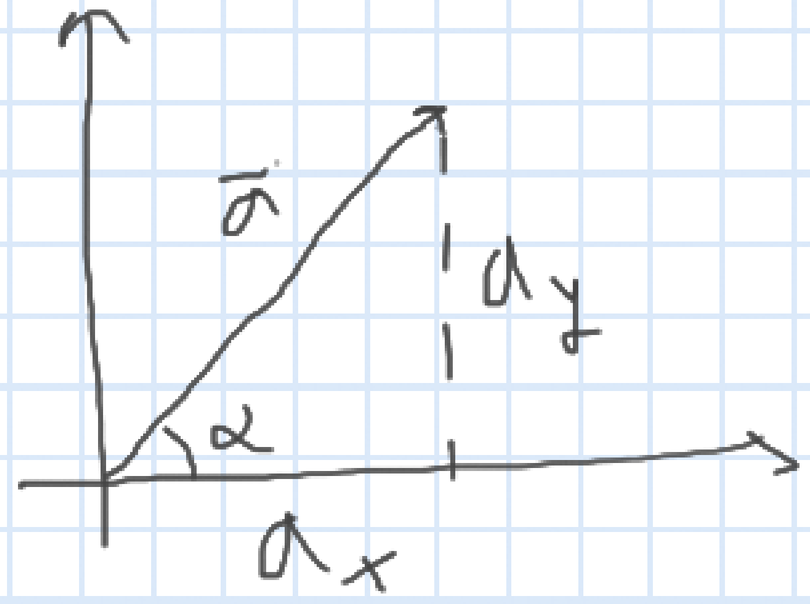
$$\boxed{a = c \cdot \cos(\alpha)}$$

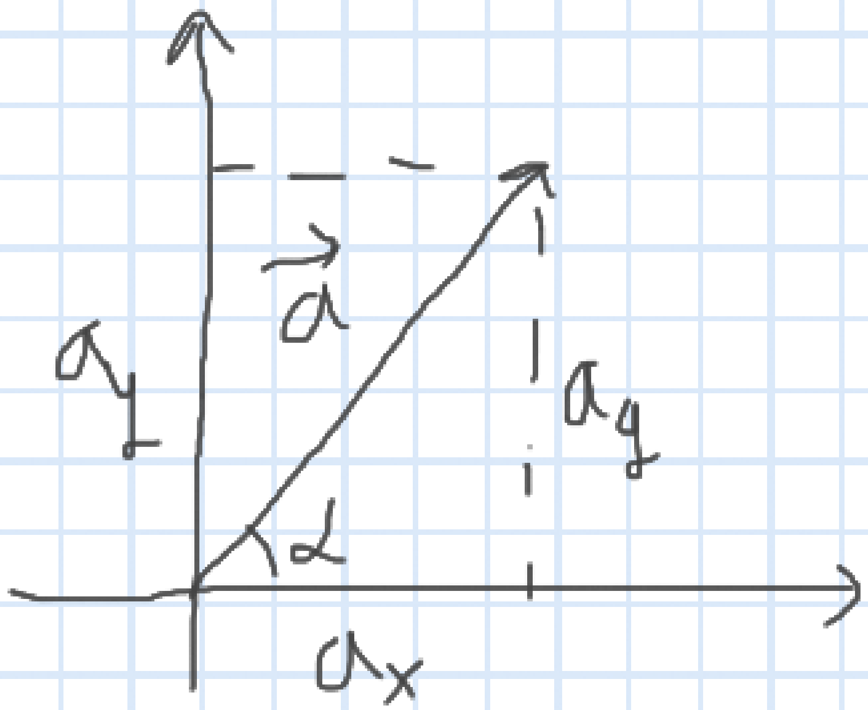
$$b = c \cdot \cos(\beta)$$

CATETO = IPOTENUSA · sen (ANGOLO OPPOSTO)

$$a = c \cdot \text{sen}(\beta)$$

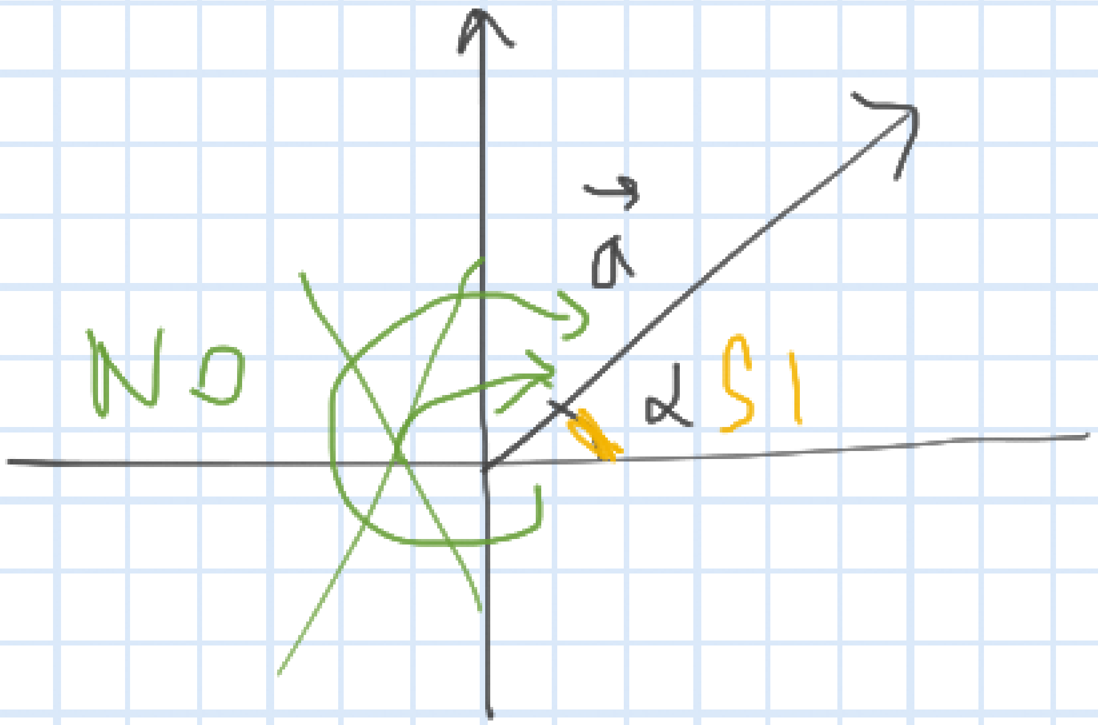
$$\boxed{b = c \cdot \text{sen}(\alpha)}$$





$$\boxed{\begin{aligned} a_x &= a \cdot \cos\alpha \\ a_y &= a \cdot \operatorname{sen}\alpha \end{aligned}}$$

Trovare la DIREZIONE del vettore $\vec{a}$ significa individuare l'ampiezza dell'angolo α che $\vec{a}$ forma con la direzione positiva dell'asse x



$$\cos(\alpha) = \frac{a_x}{a} \rightarrow \alpha = \cos^{-1}\left(\frac{a_x}{a}\right)$$

$$\operatorname{sen}(\alpha) = \frac{a_y}{a} \rightarrow \alpha = \operatorname{sen}^{-1}\left(\frac{a_y}{a}\right)$$

Operazioni tra vettori in componenti cartesiane

Consideriamo 2 vettori $\vec{a} = (a_x ; a_y) = a_x \vec{i} + a_y \vec{j}$, $\vec{b} = (b_x ; b_y) = b_x \vec{i} + b_y \vec{j}$

SOMMA

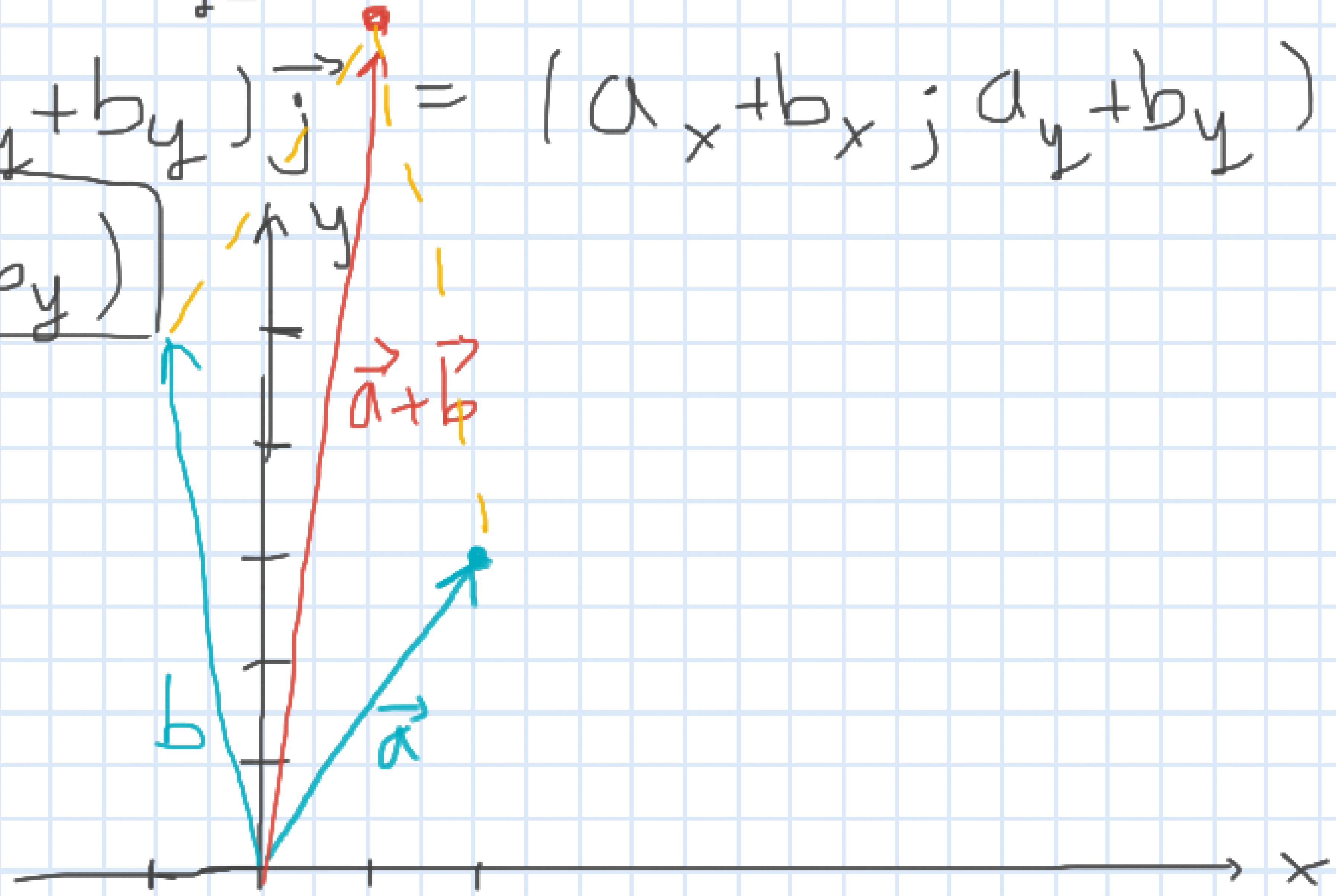
$$\vec{a} + \vec{b} = a_x \vec{i} + a_y \vec{j} + b_x \vec{i} + b_y \vec{j} =$$

$$= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} = (a_x + b_x ; a_y + b_y)$$

$$\boxed{\vec{a} + \vec{b} = (a_x + b_x ; a_y + b_y)}$$

ESEMPIO

$$\left.\begin{aligned} \vec{a} &= (2, 3) \\ \vec{b} &= (-1, 5) \end{aligned}\right\} \vec{a} + \vec{b} = (1, 8)$$

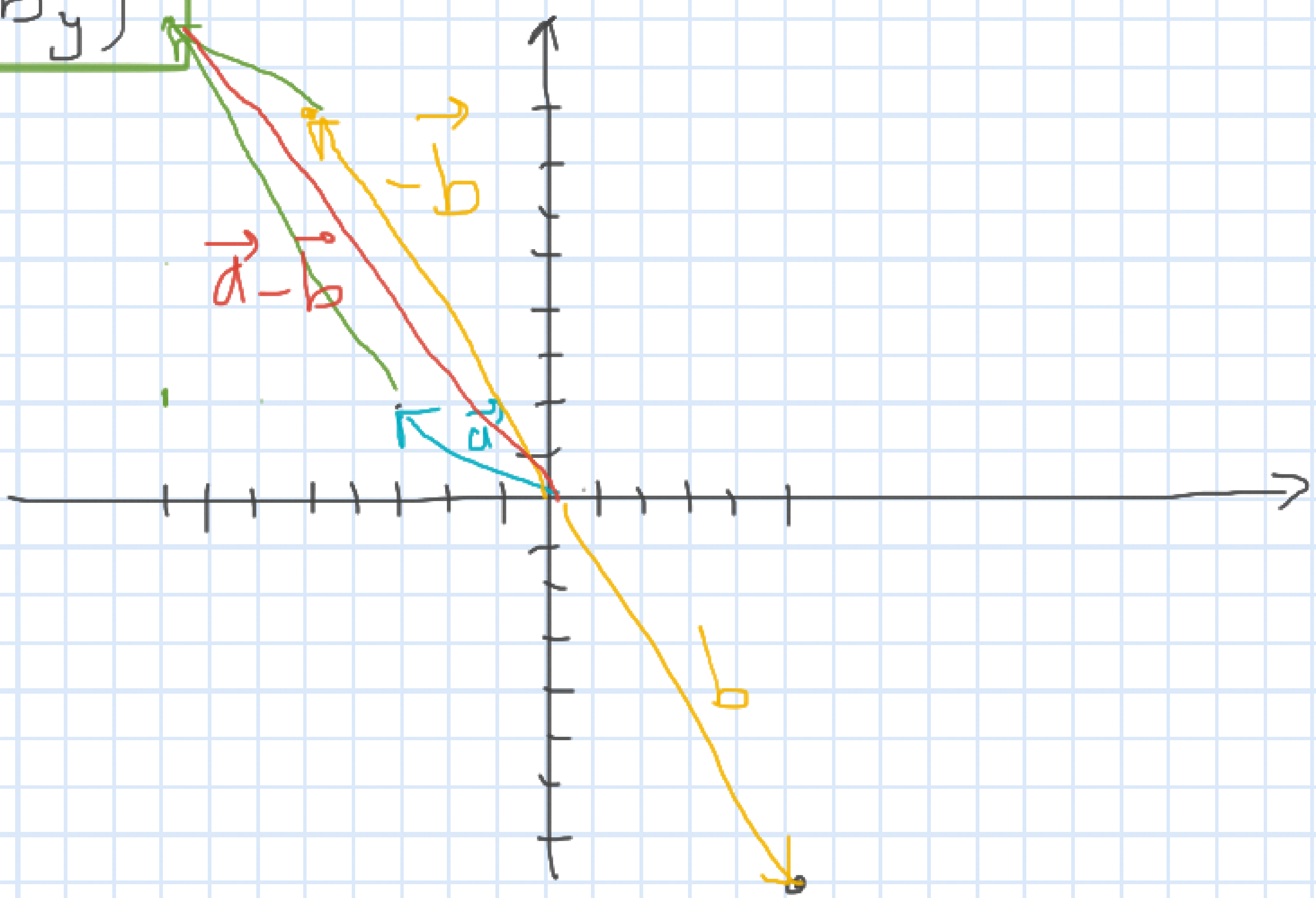


DIFFERENZA

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = a_x\vec{i} + a_y\vec{j} + (-b_x\vec{i} - b_y\vec{j}) =$$

$$= (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} = (a_x - b_x, a_y - b_y)$$

$$\boxed{\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y)}$$



ESEMPIO

$\vec{a} = (-3, 2)$

$\vec{b} = (5, -8)$

$\vec{a} - \vec{b} = (-3-5, 2-(-8))$

$= (-8, +10)$

PRODOTTO PER SCALARE

$k \in \mathbb{R}$ numero $\rightarrow$ $k \cdot \vec{a} = k(a_x \vec{i} + a_y \vec{j}) =$

$= (k a_x)\vec{i} + (k a_y)\vec{j}$

$$k\vec{a} = (k a_x ; k a_y)$$

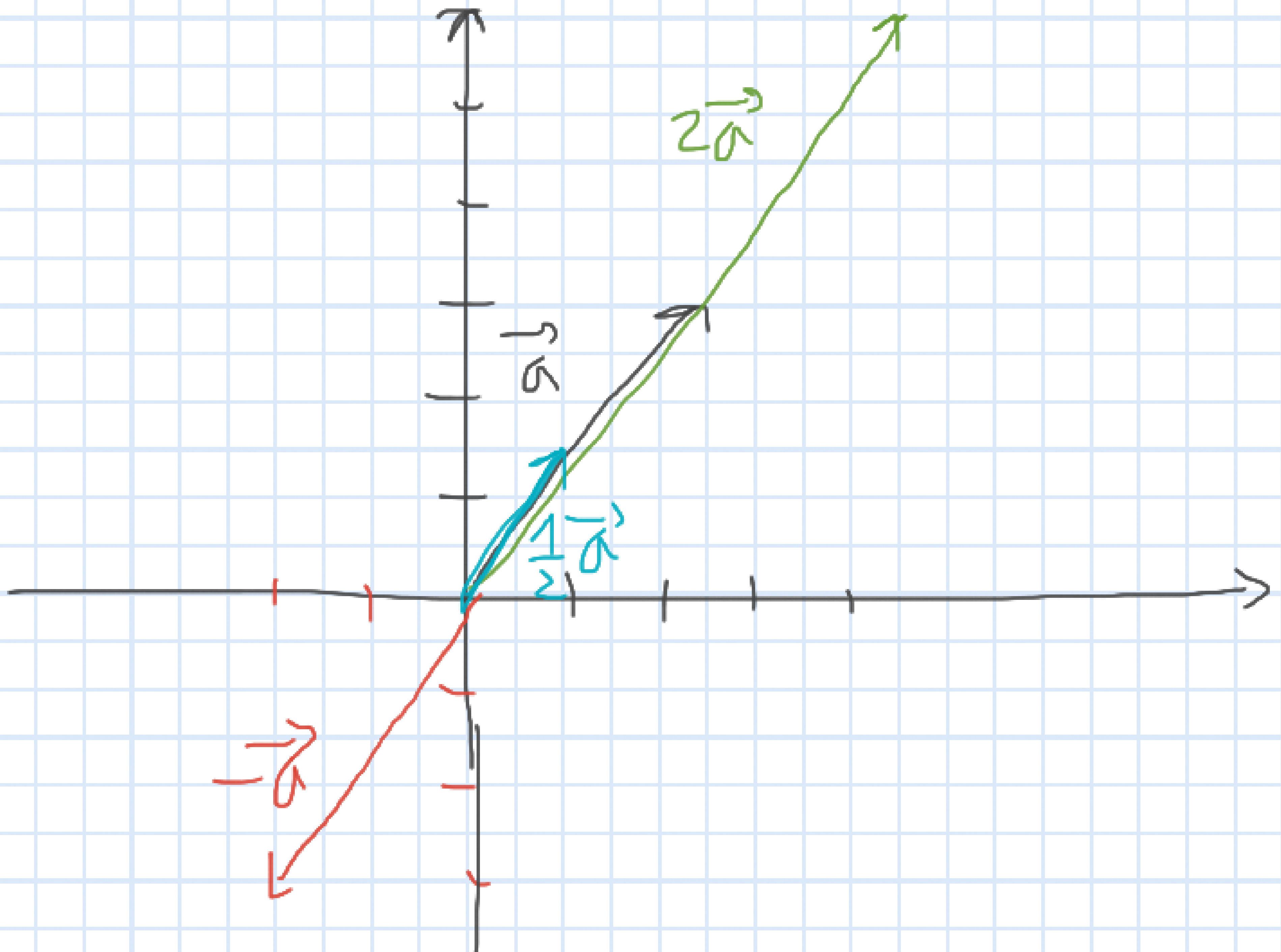
ESEMPIO

$\vec{a} = (2, 3)$

$k = 2 \rightarrow \quad 2 \cdot \vec{a} = (4, 6)$

$k = \frac{1}{2} \quad\quad \frac{1}{2}\vec{a} = \left(1, \frac{3}{2}\right)$

$k = -1 \quad\quad -1\vec{a} = (-2, 3)$



PRODOTTO SCALARE

$$\vec{a} \cdot \vec{b} = (a_x \vec{i} + a_y \vec{j}) \cdot (b_x \vec{i} + b_y \vec{j}) =$$

$$= a_x b_x \underbrace{\vec{i} \cdot \vec{i}}_{1} + \cancel{a_x b_y \underbrace{\vec{i} \cdot \vec{j}}_{0}} + \cancel{a_y b_x \underbrace{\vec{j} \cdot \vec{i}}_{0}} + a_y b_y \underbrace{\vec{j} \cdot \vec{j}}_{1}$$

$$\vec{i} \cdot \vec{i} = 1 \cdot 1 \cdot \cos(0) = 1$$

$$\vec{j} \cdot \vec{j} = 1 \cdot 1 \cdot \cos(0) = 1$$

$$\vec{i} \cdot \vec{j} = 1 \cdot 1 \cdot \cos(90) = 0$$

$$\vec{j} \cdot \vec{i} = 0$$

$$\boxed{\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y}$$

ESEMPIO

$\vec{a} = (2, -3)$

$\vec{b} = (5, -1)$

$$\vec{a} \cdot \vec{b} = (2 \cdot 5) + (-3)(-1)$$

$$= 10 + 3 = 13$$

$$\vec{a} \cdot \vec{b} = a \cdot b \cdot \underline{\cos\alpha}$$

$$\| $$

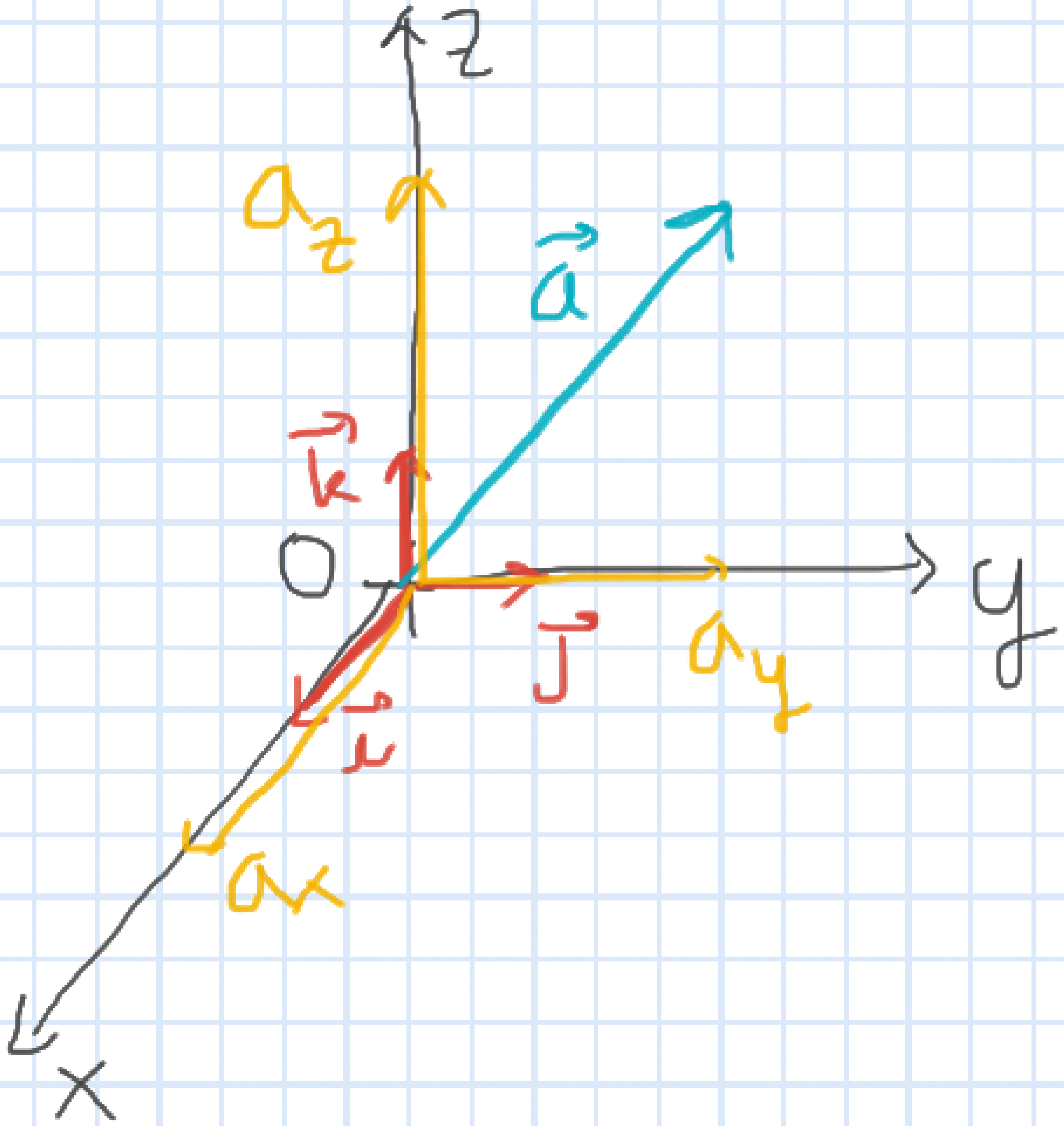
$$a_x b_x + a_y b_y$$

$$\boxed{\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{a_x b_x + a_y b_y}{a \cdot b}}$$

coseno di α angolo compreso tra $\vec{a}$ e $\vec{b}$

Vettori nello spazio

Consideriamo un sistema cartesiano $Oxyz$
e indichiamo con i $\vec{i}, \vec{j}, \vec{k}$ i versori degli assi x, y, z



$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$
$$= (a_x, a_y, a_k)$$

OPERAZIONI TRA VETTORI NELLO SPAZIO

Consideriamo $\vec{a} = (a_x, a_y, a_z)$ e $\vec{b} = (b_x, b_y, b_z)$ vettori

SOMMA $\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$

ES $\underbrace{(2,3,5)}_{\vec{a}} + \underbrace{(-1,4,2)}_{\vec{b}} = (1,7,7)$

PRODOTTO PER SCALARE $k\vec{a} = (ka_x, ka_y, ka_z)$

ES $k=2$ $2 \cdot \underbrace{(2,3,5)}_{\vec{a}} = (4,6,10)$

$k=-2$ $-2(2,3,5) = (-4,-6,-10)$

PRODOTTO SCALARE $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z \cdot b_z$

ES $(2,3,5) \cdot (-1,4,2) = -2 + 12 + 10 = 20$

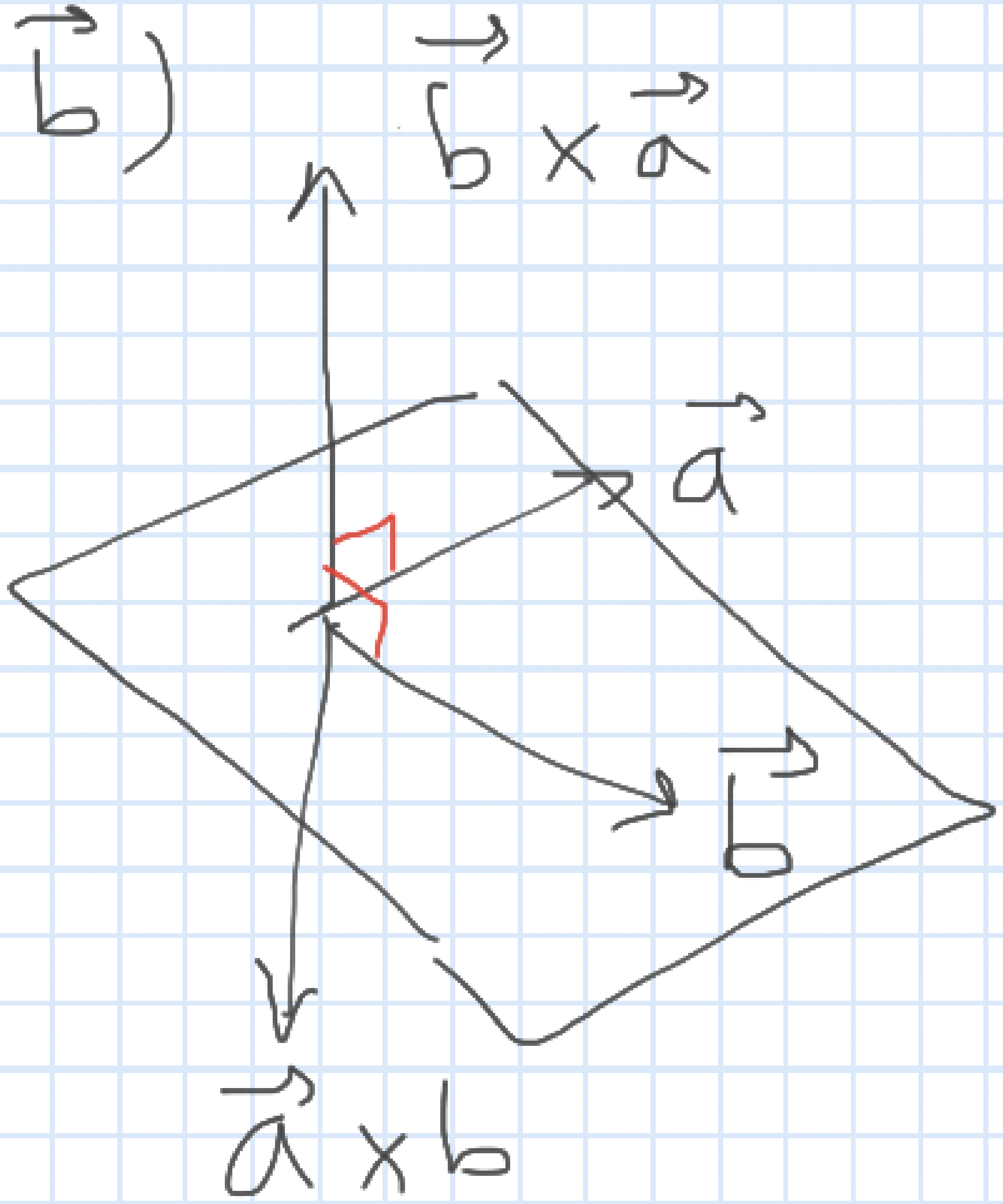
PRODOTTO VETTORIALE

$\vec{a}, \vec{b}$ 2 vettori $\longrightarrow$ $\vec{a} \times \vec{b}$ $(\vec{a} \wedge \vec{b})$

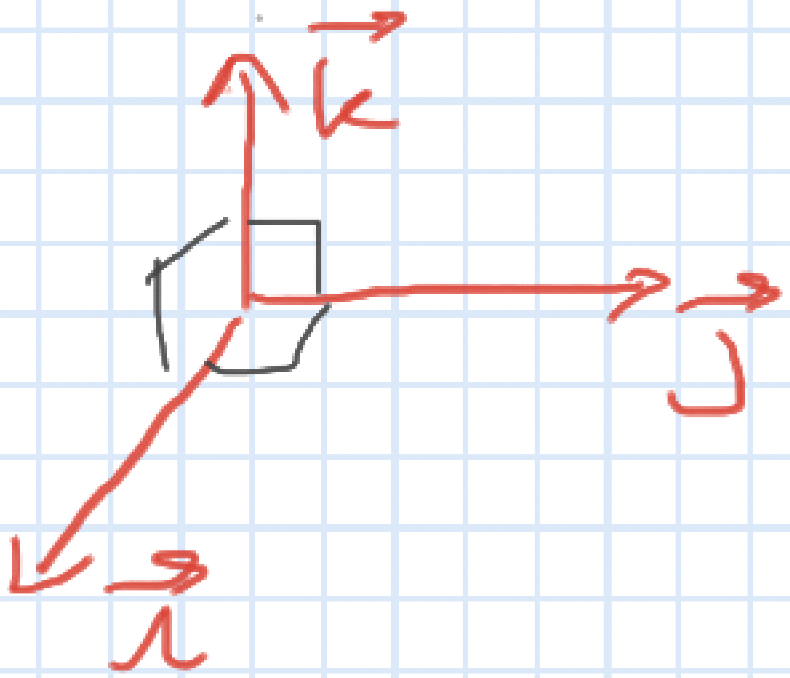
$= \vec{c}$

- modulo $|\vec{a} \times \vec{b}| = a \cdot b \cdot \text{sen}\,\alpha$
- direzione $\perp$ piano di ab
- verso $\rightarrow$ mano destra

$\vec{a} \times \vec{b}$ esce dal palmo



Consideriamo i versori $\vec{i}$ $\vec{j}$ $\vec{k}$



$$\vec{i} \times \vec{i} = 0$$

$$\left(1 \cdot 1 \cdot \underbrace{\text{sen}(0)}_{=0}\right)$$

$$\vec{j} \times \vec{j} = 0$$

$$\vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$

$$\vec{j} \times \vec{i} = -\vec{k} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

$$1 \cdot 1 \cdot \text{sen}(90) = 1$$

$$\begin{tikzcd} \vec{i} \arrow[r] & \vec{j} \arrow[r] & \vec{k} \arrow[ll, bend left] \end{tikzcd}$$

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \qquad \vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

$$\vec{a}\times\vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \times (b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) =$$

$$= \cancel{a_x b_x \underbrace{\vec{i}\times\vec{i}}_{=0}} + a_x b_y \underbrace{\vec{i}\times\vec{j}}_{\vec{k}} + a_x b_z \underbrace{\vec{i}\times\vec{k}}_{-\vec{j}} +$$

$$a_y b_x \underbrace{\vec{j}\times\vec{i}}_{-\vec{k}} + \cancel{a_y b_y \underbrace{\vec{j}\times\vec{j}}_{=0}} + a_y b_z \underbrace{\vec{j}\times\vec{k}}_{\vec{i}} +$$

$$a_z b_x \underbrace{\vec{k}\times\vec{i}}_{\vec{j}} + a_z b_y \underbrace{\vec{k}\times\vec{j}}_{-\vec{i}} + \cancel{a_z b_z \underbrace{\vec{k}\times\vec{k}}_{=0}} =$$

$$= (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}$$

Tecnica del determinante simbolico

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$$

$$= +\vec{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \vec{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \vec{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} =$$

$$= (a_y b_z - b_y a_z)\vec{i} \underbrace{- (a_x b_z - a_z b_x)\vec{j}}_{+(a_z b_x - a_x b_z)\vec{j}} + (a_x b_y - a_y b_x)\vec{k}$$

Esempio

$\vec{a} = (1, -1, 0) \qquad \vec{b} = (-2, 0, 1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{vmatrix} =$$

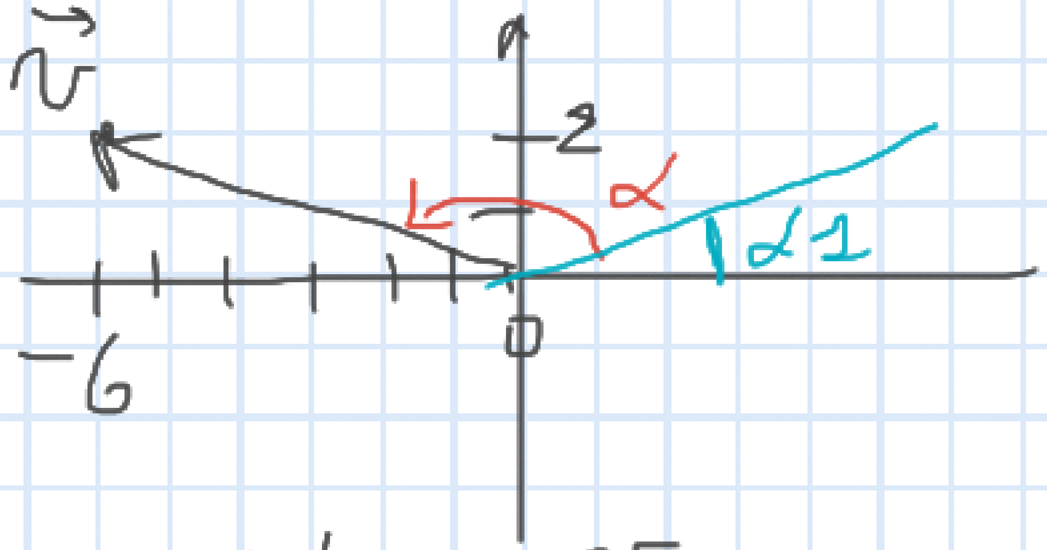
$$= \vec{i}(-1-0) - \vec{j}(1-0) + \vec{k}(0-2) =$$

$$= -\vec{i} - \vec{j} - 2\vec{k} = (-1, -1, -2)$$

Es Dato il vettore $\vec{v} = (-6, 2)$ determinare modulo e direzione

$v_x = -6, \; v_y = 2 \longrightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{36+4} = \sqrt{40}$

$= \sqrt{4 \cdot 10} = 2\sqrt{10}$



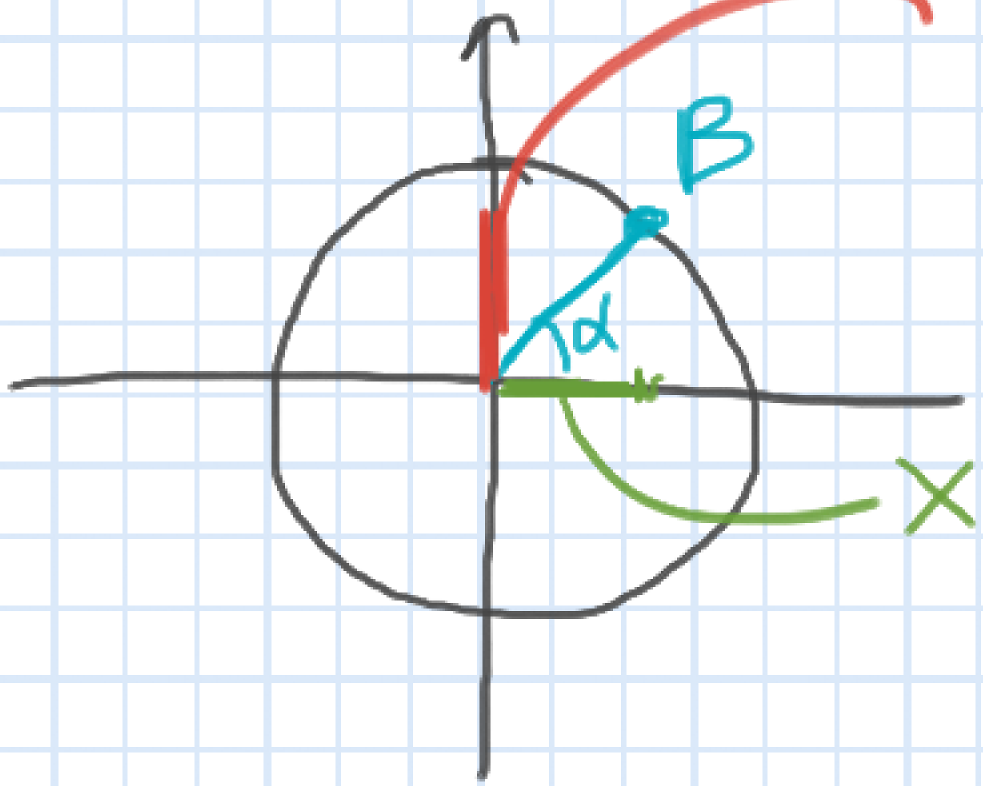
$v_x = v \cdot \cos\alpha$

$v_y = v \operatorname{sen}\alpha$

$\cos\alpha = \dfrac{v_x}{v} = \dfrac{-6}{2\sqrt{10}} = -\dfrac{3}{\sqrt{10}} \longrightarrow \alpha = \cos^{-1}\left(-\dfrac{3}{\sqrt{10}}\right) \simeq 161^\circ$

$\operatorname{sen}\alpha = \dfrac{v_y}{v} = \dfrac{2}{2\sqrt{10}} = \dfrac{1}{\sqrt{10}} \longrightarrow \alpha_1 = \operatorname{sen}^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$

$\operatorname{sen}(\alpha) = \operatorname{sen}(\alpha_1) \rightarrow \alpha = 180 - \alpha_1$

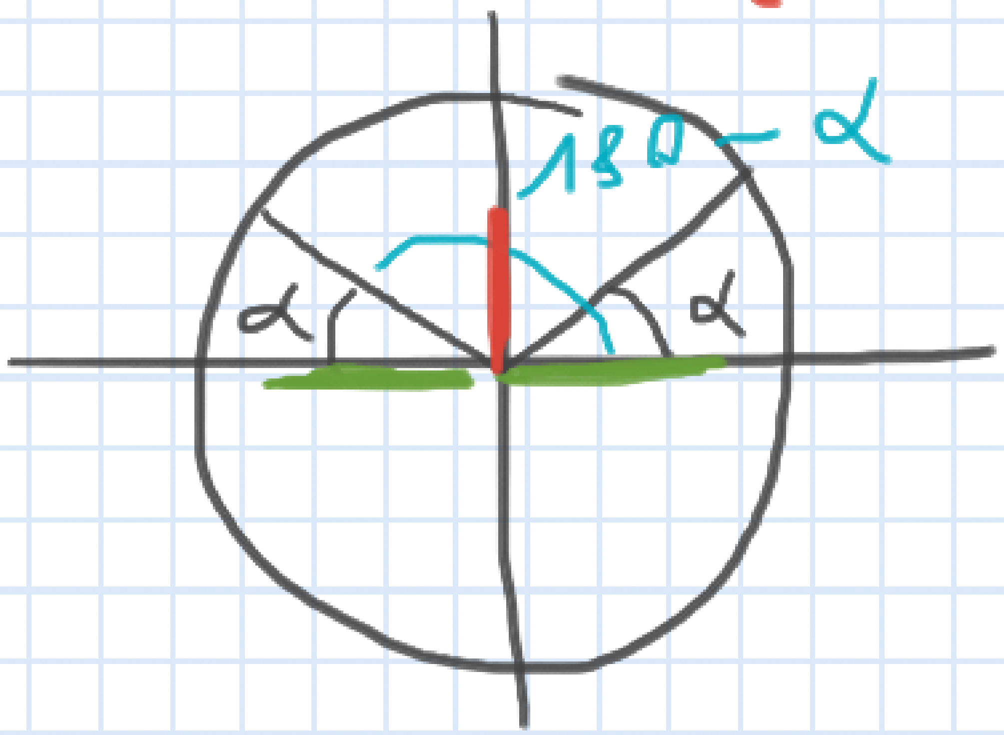


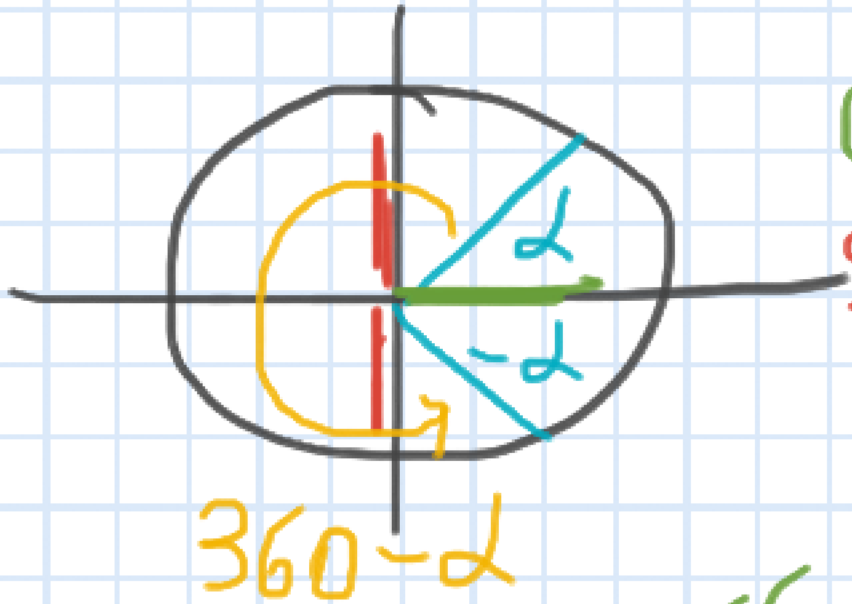
$y_B = sen(\alpha)$

$x_B = \cos(\alpha)$

$\cos(180-\alpha) = -\cos(\alpha)$

$sen(180-\alpha) = sen(\alpha)$





$\cos(\alpha) = \cos(-\alpha)$

$sen(-\alpha) = -sen(\alpha)$

$\cos(\alpha) = \cos(360-\alpha)$

$sen(360-\alpha) = -sen(\alpha)$