PROVA SCRITTA DI ANALISI MATEMATICA 1,
CdS in Fisica e Astrofisica, 22 giugno 2020

Esercizio 1. Calcolare
$$\int \frac{1}{\cos^4 x}\, dx.$$

Esercizio 2. Determinare i valori di $\alpha \in \mathbb{R}$ per cui il limite
$$\lim_{x\to 0} \frac{\ln(1+x^2)+\cos(\alpha x)-1}{\cos x^2 - \mathrm{e}^{\alpha^2 x^4}},$$
risulta finito. Per tali valori di α calcolare il valore del limite.

Esercizio 3. Si consideri la funzione
$$F(x) = \int_1^x \frac{\Big((1+t)^{\sqrt{2}} - \sqrt{2}\,t - 1\Big)\sin t}{(\mathrm{e}^t - 1)\arctan^4(\sqrt{t})}\, dt.$$
(a) Determinare il dominio naturale di F.
(b) Discutere l'esistenza di eventuali asintoti per F.

ES1. $\int \frac{1}{\cos^4 x}\,dx =$

① $= \int \frac{1}{\cos^2 x}\cdot\frac{1}{\cos^2 x}\,dx \underset{P.P.}{\overset{\uparrow}{=}} \frac{tg x}{\cos^2 x} - 2\int \frac{tg x}{\cos^3 x}\sin x\,dx =$

$= \frac{tg x}{\cos^2 x} - 2\int tg^2 x \frac{1}{\cos^2 x}\,dx = \frac{tg x}{\cos^2 x} - \frac{2}{3}tg^3 x + C_1 \quad \forall C_1 \in \mathbb{R}$

② $= \int \frac{1+tg^2 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x} + \frac{tg^2 x}{\cos^2 x}\,dx = tg x + \frac{tg^3 x}{3} + C \quad \forall C \in \mathbb{R}$

③ $= \int (1+tg^2 x)(1+tg^2 x)\,dx \underset{\substack{tg x = t \\ (1+tg^2 x)dx = dt}}{\overset{\uparrow}{=}} \int (1+t^2_{(x)})\,dt_{(x)} =$

$= t_{(x)} + \frac{t^3_{(x)}}{3} + C = tg x + \frac{tg^3 x}{3} + C \quad \forall C \in \mathbb{R}$

④ $= \int \frac{\sin^2 x + \cos^2 x}{\cos^4 x}\,dx = \int \frac{\sin x}{\cos^4 x}\cdot\sin x\,dx + \int \frac{1}{\cos^2 x}\,dx =$

$\overset{P.P.}{=} \frac{1}{3\cos^3 x}\sin x - \frac{1}{3}\int \frac{\cos x}{\cos^3 x}\,dx + tg x =$

$= \frac{\sin x}{3\cos^3 x} - \frac{1}{3}tg x + tg x + C = \frac{\sin x}{3\cos^3 x} + \frac{2}{3}tg x + C \quad \forall C \in \mathbb{R}$

OSS: ① = ② = ③ = ④

$tg x + \frac{tg^3 x}{3} = \frac{\sin x}{\cos x} + \frac{\sin^3 x}{3\cos^3 x} = \frac{\sin x}{\cos x}\left(1 + \frac{\overset{1-\cos^2 x}{\sin^2 x}}{3\cos^2 x}\right) = \frac{\sin x}{\cos x}\left(1 - \frac{1}{3} + \frac{1}{3\cos^2 x}\right)$

$= \frac{2}{3}tg x + \frac{\sin x}{3\cos^3 x}$

$\frac{tg x}{\cos^2 x} - \frac{2}{3}tg^3 x = \frac{\sin x}{\cos^3 x} - \frac{2}{3}\frac{\sin^3 x}{\cos^3 x} = \frac{\sin x}{3\cos^3 x}(3 - 2\sin^2 x) = \frac{\sin x}{3\cos^3 x}(3 - 2(1-\cos^2 x)) =$

$= \frac{\sin x}{3\cos^3 x}(3 - 2 + 2\cos^2 x) = \frac{\sin x}{3\cos^3 x} + \frac{2}{3}\frac{\sin x}{\cos x}$ ✓

ES2. $\mathrm{NUM} = \ln(1+x^2) + \cos(\alpha x) - 1 \underset{\text{per } x\to 0}{=}$

$$= x^2\left(1-\frac{\alpha^2}{2}\right) + x^4\left(-\frac{1}{2}+\frac{\alpha^4}{24}\right) + o(x^4)$$

$$\mathrm{DEN} = \cos x^2 - e^{\alpha^2 x^4} = -x^4\left(\frac{1-\cos x^2}{x^4} + \frac{e^{\alpha^2x^4}-1}{\alpha^2x^4}\cdot\alpha^2\right)$$

Da cui

$$\frac{\mathrm{NUM}}{\mathrm{DEN}} = \frac{\frac{1}{x^2}\left(1-\frac{\alpha^2}{2}\right) + \left(-\frac{1}{2}+\frac{\alpha^4}{24}\right) + \frac{o(x^4)}{x^4}}{-\left(\frac{1-\cos x^2}{x^4} + \frac{e^{\alpha^2x^4}-1}{\alpha^2x^4}\,\alpha^2\right)}$$

da cui

$$\lim_{x\to 0}\frac{\mathrm{NUM}}{\mathrm{DEN}} = \begin{cases} \exists \text{ finito } = \dfrac{-\frac{1}{2}+\frac{4}{24}}{-(\frac{1}{2}+2)} & \text{se } \alpha^2=2 \text{ cioè se } \alpha=\sqrt{2} \vee \alpha=-\sqrt{2} \\ +\infty \quad \text{se } \alpha^2>2 & \text{cioè } \alpha>\sqrt{2} \vee \alpha<-\sqrt{2} \\ -\infty \quad \text{se } \alpha^2<2 & -\sqrt{2}<\alpha<\sqrt{2} \end{cases}$$

ES3. $F(x) = \int_1^x \frac{((1+t)^{\sqrt{2}} - \sqrt{2}t - 1)\cdot \sin t}{(e^t - 1)\, \operatorname{arctg}^4(\sqrt{t})}\, dt$

(a) Determinare il dominio naturale di F

(b) Discutere l'esistenza di eventuali asintoti per F.

$f(t) \doteq (((1+t)^{\sqrt{2}} - \sqrt{2}t - 1)\cdot \sin t)/((e^t-1)\operatorname{arctg}^4(\sqrt{t}))$

(a) OSS: $f(t)$ definita per $t \geq 0$ quindi F non può essere definita se $x < 0$. (o)

. sia $x > 0$ allora $f(t) \in C^0([1,x])$ se $x > 1$ e $f(t) \in C^0([x,1])$ se $0 < x < 1$.

$\Rightarrow f$ integrabile in $[1,x]$ (o $[x,1]$) e dunque F ben definita $\Rightarrow$ DOMINIO$(F) \supseteq (0,+\infty)$

. DOBBIAMO VALUTARE SE $0 \in$ DOMINIO F.

sia $x \to 0^+$ considero $t \to 0^+$

$$f(t) = \frac{\left(1+\sqrt{2}t + \frac{\sqrt{2}}{2}(\sqrt{2}-1)t^2 + o(t^2) - \cancel{\sqrt{2}t} - \cancel{1}\right)\cdot \frac{1}{t^2}\, \frac{\sin t}{t}}{\frac{e^t-1}{t}\left(\frac{\operatorname{arctg}\sqrt{t}}{\sqrt{t}}\right)^4}$$

da cui $\lim_{t\to 0^+} f(t) = \frac{\sqrt{2}}{2}(\sqrt{2}-1)$ e quindi

f integrabile in $\mathcal{U}_{0^+}$ intorno dx di 0.

$\Rightarrow$ DOMINIO $F = [0,+\infty)$ $\left(\begin{matrix}\text{Poiché da (o)}\\ (-\infty,0) \cap \text{DOM}F = \emptyset\end{matrix}\right)$

(b) Poiché f integrabile in $\mathcal{U}_{0^+}$ e $f \in C^0([1,x])$ $\forall x > 0$, si ha che $\nexists$ AS. verticali

. $\lim_{x\to+\infty} F(x)$. OSS per $t \to +\infty$ $|f(t)| \leq \frac{t^{\sqrt{2}}}{e^t}\cdot \Theta(\text{costante})$

quindi per confronto $\int_1^x |f(t)|\, dt$ converge

$\Rightarrow \int_1^x f(t)\,dt$ converge cioè $\exists$ finito $\lim_{x\to+\infty} F(x)$

cioè $F(x)$ ha un asintoto orizzontale per $x \to +\infty$.