Algoritmi e Modelli di Ottimizzazione

Optimization - Basic introduction

fabio.schoen@unifi.it
http://webgol.dinfo.unifi.it

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Definitions

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$.
Consider the optimization problem

$$\min_{x \in S} f(x)$$

f is the objective function, S the feasible set, $n \in \mathbb{N}$ is the dimension of the problem
If $S = \mathbb{R}^n \Rightarrow$ unconstrained problem
If $S = \emptyset \Rightarrow$ infeasible problem
Any $\bar{x} \in S$, $\bar{x}$ is a feasible solution
If $\forall \alpha \in \mathbb{R}$ a feasible solution $\bar{x} \in S$ exists : $f(\bar{x}) \leq \alpha$, then the problem is unbounded (from below)

Global and local optima

A feasible solution $x^\star \in S$ is a global minimum point if

$$f(x^\star) \leq f(x) \forall x \in S$$

$f^\star = f(x^\star)$ is the global minimum or global optimum. The set of global minima is denoted by

$$\arg\min_{x \in S} f(x)$$

A feasible solution $\bar{x} \in S$ is a local minimum point (or local optimum point) if $\exists \varepsilon > 0$:

$$f(\bar{x}) \leq f(x) \forall x \in S \bigcap B(\bar{x}, \varepsilon)$$

where $B(\bar{x}, \varepsilon) = \{x : \|x - \bar{x}\| \leq \varepsilon\}$ and $\|\cdot\|$ is any norm.

Convex optimization

A convex minimization problem is an optimization problem

$$\min_{x \in S} f(x)$$

in which S is a convex set and f is a convex function on S

Theorem

Any local minimum point in a convex optimization problem is also global

proof

Proof.

Let $y \in S$ be any feasible point and $x^\star \in S$ a local minimum point. From convexity (of both S and f):

$$f(\lambda y + (1-\lambda)x^\star) \leq \lambda f(y) + (1-\lambda)f(x^\star) \qquad \forall \lambda \in [0,1]$$

Chosing λ small enough

$$f(\lambda y + (1-\lambda)x^\star) \geq f(x^\star)$$

Thus

$$f(x^\star) \leq \lambda f(y) + (1-\lambda)f(x^\star)$$

from which $f(x^\star) \leq f(y)$ holds □

Unicity of optima for strictly convex problems

Recall: a function f is *strictly convex* if

$$f(\lambda x + (1-\lambda y)) < \lambda f(x) + (1-\lambda)f(y)$$

for all $x \neq y$ and $\lambda \in (0,1)$.

Theorem

Let S be a non empty convex set and f a strictly convex function on S. If $x^\star \in S$ is a local minimum for f in S, then $x^\star$ is also the unique global (and local) minimum of f in S

Sufficient existence conditions (constrained case)

Theorem (Weierstrass)

If f is continuous and S is compact, then there always exists a global minimum point for f in S

Definition

Let $\alpha \in \mathbb{R}$ be given. The *level set* for a function f at level α is defined as

$$\mathcal{L}_\alpha = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

The *contour* of f is instead the set

$$\mathcal{C}_\alpha = \{x \in \mathbb{R}^n : f(x) = \alpha\}$$

Sufficient existence conditions (unconstrained case)

Theorem

If there exists α so that the level set $\mathcal{L}_\alpha$ for a continuous function f is compact, then an unconstrained global minimum for f always exists.

Coercitivity

Definition

A function f is *coercive* iff for any sequence $\{x_k\}$ such that

$$\lim_{k\to\infty} \|x_k\| = \infty$$

it holds that

$$\lim_{k\to\infty} f(x_k) = +\infty$$

Theorem

Given a continuus function f, all of its level sets are compact if and only if f is coercive

Proof.

Assume all level sets are compact, but a sequence $\{x_k\}$ exists and a α exists:

$$\|x_k\| \to \infty \qquad f(x_k) \le \alpha$$

so every x_k is in $\mathcal{L}_\alpha$, but this is absurd as the level set, being compact, is also limited.
Assume now f coercive. As f is continuous, its level sets are closed. Assume there exists a α: $\{\mathcal{L}\}_\alpha$ is non empty but unlimited. Then a sequence $\{x_k\} \in \{\mathcal{L}\}_\alpha$ exists so that $\|x_k\| \to \infty$. Thus, for coercivity, $\lim f(x_k) = +\infty$. But this is absurd as $f(x_k) \le \alpha$ for all k. □