ES 1. $\lim_{x\to 0} \dfrac{\ln(\cos(\sin(x^{3/2}))) + x^3/2}{(\sin(x^2) - tg(x^2))\cos(\sin^3 x)}$ [*]

oss: $\sin x^2 - tg\, x^2 = \sin x^2 \left(1 - \dfrac{1}{\cos x^2}\right) = -\dfrac{\sin x^2}{\cos x^2}(1-\cos x^2)$

quindi [*] $= \lim_{x\to 0} \dfrac{\cos x^2 (-1)}{\cos(\sin^3 x)} \cdot \dfrac{f(x)}{g(x)}$

dove $f(x) = \ln(\cos(\sin(x^{3/2}))) + \dfrac{x^3}{2}$ è infinitesima per $x \to 0$

$g(x) = \sin x^2 (1-\cos x^2)$ è infinitesima per $x\to 0$

studiamo l'andamento di $f(x)$ e $g(x)$ per x vicino a 0:

$g(x)$: $\sin x^2 = x^2 - \dfrac{x^6}{6} + o(x^8)$

$1-\cos x^2 = \dfrac{x^4}{2} - \dfrac{x^8}{24} + o(x^{10})$

da cui $g(x) = \dfrac{x^6}{2} - \dfrac{1}{8}x^{10} + o(x^{12})$

$f(x)$: $\sin(x^{3/2}) = x^{3/2} - \dfrac{x^{9/2}}{6} + o(x^{9/2})$

$$\cos(\sin(x^{3/2})) - 1 = -\frac{1}{2}\left(x^{\frac{3}{2}} - \frac{x^{\frac{9}{2}}}{6} + o(x^{\frac{9}{2}})\right)^2 + \frac{1}{24}\left(x^{\frac{3}{2}} - \frac{x^{\frac{9}{2}}}{6} + o(x^{\frac{9}{2}})\right)^4 + o((\dots)^5) =$$

$$= -\frac{1}{2}x^3 + \frac{1}{6}x^6 + o(x^6) + \frac{1}{24}x^6 = -\frac{1}{2}x^3 + \left(\frac{5}{24}\right)x^6 + o(x^6)$$

$$\ln(\cos(\sin(x^{3/2}))) = \ln(1 + [\cos(\sin(x^{3/2})) - 1]) =$$

$$= \left(-\frac{x^3}{2} + \frac{5}{24}x^6 + o(x^6)\right) - \frac{1}{2}\left(-\frac{x^3}{2} + \frac{5}{24}x^6 + o(x^6)\right)^2 + o((\dots)^2) =$$

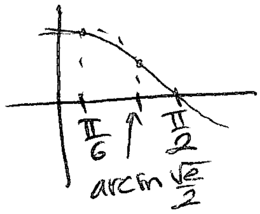
$$= -\frac{x^3}{2} + \frac{5}{24}x^6 - \frac{1}{8}x^6 + o(x^6) = -\frac{x^3}{2} + \frac{2}{24}x^6 + o(x^6)$$

da cui

$$[*] = \lim_{x\to 0} -\frac{f(x)}{g(x)} = -\lim_{x\to 0} \frac{\cancel{-\frac{x^3}{2}} + \frac{1}{12}x^6 + o(x^6) \cancel{+\frac{x^3}{2}}}{\frac{x^6}{2} - \frac{1}{8}x^{10} + o(x^{12})} = -\frac{1}{12}\cdot 2 = -\frac{1}{6}.$$

ES. 2. $\displaystyle\int_0^{\frac{1}{2}} \frac{e^{6x}}{\sqrt{4-e^{2x}}}\,dx = \int_0^{\frac{1}{2}} \frac{1}{2}\,\frac{e^{5x}}{\sqrt{1-\left(\frac{e^x}{2}\right)^2}}\,e^x dx =$

$$\underset{\substack{e^x=2t \\ \frac{e^x}{2}dx=dt}}{\overset{\uparrow}{=}} \int_{\frac{1}{2}}^{\frac{\sqrt{e}}{2}} \frac{1}{2}\,\frac{32\,t^5}{\sqrt{1-t^2}}\cdot 2dt \underset{\substack{t=\sin\theta \\ dt=\cos\theta\, d\theta}}{\overset{\uparrow}{=}} 32\int_{\frac{\pi}{6}}^{\arcsin\left(\frac{\sqrt{e}}{2}\right)} \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta \overset{\uparrow}{=}$$



$$= 32\int_{\frac{\pi}{6}}^{\theta_0} \frac{\sin^5\theta}{\cancel{\cos\theta}_{1}}\,\overset{1}{\cancel{\cos\theta}}\,d\theta =$$

$$= 32\int_{\frac{\pi}{6}}^{\theta_0} \sin^5\theta\, d\theta = 32\left[-\frac{1}{5}\sin^4\theta\cos\theta - \frac{4}{15}\cos\theta\sin^2\theta + \theta\right]\Bigg|_{\frac{\pi}{6}}^{\theta_0}$$

ES. 3. $f(x) = \operatorname{arctg}\left(\frac{a}{x}\right) + x$

• $\underline{a=0}$ $f(x) = x$ funzione lineare $\in C^0(\mathbb{R})$, $f \nearrow$, invertibile, $f^{-1} \equiv f = x$ funzione identità

f è una retta ($\Rightarrow$ as. obl. di se stessa)

• $\underline{a \neq 0}$ f dispari $\Rightarrow$ il comportamento per $x<0$ è analogo a quello di $-f(x)$ per $x>0$.

(a) $\mathbb{D} = (-\infty, 0) \cup (0, +\infty)$

(b) $\lim_{x \to +\infty} f(x) = +\infty$; $\lim_{x \to +\infty} \frac{f(x)}{x} = 1$; $\lim_{x \to +\infty} f(x) - x = 0$ $\forall a \in \mathbb{R}^*$

$\Rightarrow$ la retta $y = x$ è as. obliquo per $x \to +\infty$ e $x \to -\infty$.

(quindi $\nexists$ as. oriz.)

$\lim_{x \to 0^+} f(x) = \operatorname{sgn}(a) \frac{\pi}{2}$; $\lim_{x \to 0^-} f(x) = -\operatorname{sgn}(a) \frac{\pi}{2}$ $\Rightarrow \nexists$ as. verticali

(c) $\forall a \neq 0$ $f \in C^0(\mathbb{R} \setminus \{0\})$ in $x=0$ si ha una discontinuità di salto.
(essendo $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$ entrambi finiti)

(d) $\forall x \neq 0$ $f(x) = -\operatorname{arctg}\left(\frac{x}{a}\right) + \operatorname{sgn} x \cdot \frac{\pi}{2} + x$

$f'(x) = \frac{x^2 + (a^2 - a)}{x^2 + a^2}$ $\underline{\text{p.ti critici}}$ $f' = 0 \iff x^2 = -a(a-1)$

• se $\boxed{0<a<1}$ allora $\exists$ 2 p.ti critici $x_1 = -\sqrt{a(1-a)} \doteq -b$ $x_2 = \sqrt{a(1-a)} \doteq b$

		$-b$		0		$+b$	
segno f': num	$+$		$-$		$-$		$+$
den	$+$		$+$		$+$		$+$
	$+$		$-$		$-$		$+$
monotonia di f:	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

in $x = -b$ max relativo
$x = +b$ min relativo

• se $\boxed{a \geq 1}$ (siamo nel caso $a \neq 0$) $\nexists$ p.to critico. $f \nearrow$ in $(-\infty, 0)$, $f \nearrow$ in $(0, +\infty)$
$\lim_{x \to 0^+} f(x) = \frac{\pi}{2} > \lim_{x \to 0^-} f(x) = -\frac{\pi}{2}$
quindi $f \nearrow$ in $\mathbb{D}$.

• se $\boxed{a<0}$ $\nexists$ p.to critico. $f \nearrow$ in $(-\infty, 0)$, $f \nearrow$ in $(0, +\infty)$ $\underline{\text{ma}}$
$\lim_{x \to 0^+} f(x) = -\frac{\pi}{2} < \lim_{x \to 0^-} f(x) = \frac{\pi}{2}$ $\Rightarrow f$ $\underline{\text{non}}$ crescente in $\mathbb{D}$.

(e) $\forall a \geq 1$ ($\vee$ $a = 0$) f monotona $\nearrow$ in $\mathbb{D}$ $\Rightarrow f$ invertibile

sia $g(y) = f^{-1}(y)$ $\Rightarrow g'(y) = \frac{1}{f'(x(y))}$ dove y t.c. $f(x(y)) = y$

quindi, poiché $f(a) = \frac{\pi}{4} + a$ si ha: $g'\left(\frac{\pi}{4} + a\right) = \frac{1}{f'(a)} = \frac{2a^2}{2a^2 - a}$.

ES.4.

$$\int_{-\infty}^{+\infty} |x|^\alpha |\sin(\tfrac{1}{x})|^\beta dx \qquad \begin{matrix}\beta>0\\ \alpha,\beta\in\mathbb{R}\end{matrix}$$

oss: . E' integrale improprio per $x\to 0$, $x\to\pm\infty$
. l'andamento a $\underset{0^+}{-\infty}$ e' analogo all'andamento a $\underset{0^-}{-\infty}$

$$f(x) \doteq |x|^\alpha |\sin(\tfrac{1}{x})|^\beta$$

- $x\to+\infty$: $\lim_{x\to+\infty} f(x)\cdot\frac{|x|^\beta}{|x|^\alpha} = \lim_{x\to+\infty} \cancel{|x|^\alpha}\left|\frac{\sin(\frac{1}{x})}{\frac{1}{x}}\right|^\beta \frac{1}{\cancel{|x|^\alpha}} = 1$

quindi $\int_a^{+\infty} f(x)dx$ ha lo stesso comportamento di $\int_a^{+\infty}|x|^{\alpha-\beta}dx$ che converge $\iff$ $\alpha-\beta<-1$ cioè $\boxed{\beta>\alpha+1}$
(confronto asintotico)

- $x\to-\infty$ analogo

- $x\to 0$
  - se $\alpha>0$ $\lim_{x\to 0} f(x) = 0 \Rightarrow f(x)$ è limitata in $x=0 \Rightarrow$ integrabile in senso classico.
  - se $\alpha=0$ e $\beta\le 1$ allora poiché non converge $\int_a^{+\infty} f$ si può non considerare.
  e $\beta>1$ allora $\int_0^1 |\sin(\frac{1}{x})|^\beta dx = \int_{+\infty}^1 -|\sin t|^\beta \frac{1}{t^2}dt = \underbrace{\int_1^{+\infty}|\sin t|^\beta\frac{1}{t^2}dt}$
  $\frac{1}{x}=t$, $x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$
  è integrale convergente essendo $|\sin t|^\beta\frac{1}{t^2}\le\frac{1}{t^2}$.

  - se $\alpha<0$ allora $f(x)\le\frac{1}{|x|^{-\alpha}} \Rightarrow \int_0^1 f(x)dx$ converge se $-\alpha<1$ cioè $\alpha>-1$

se $\alpha\le-1$ si ha $\lim_{x\to 0}\frac{f(x)}{|x|^{\alpha+\varepsilon}} = +\infty$ $\forall\varepsilon>0 \Rightarrow |f(x)|>|x|^{\alpha+\varepsilon}$

$\Rightarrow$ se $\int_0^1 |x|^{\alpha+\varepsilon}dx$ diverge allora $\int_0^1 f(x)dx$ diverge

cioè se $\alpha+\varepsilon\le-1$ cioè per $\alpha\le-1-\varepsilon$ cioè per $\alpha<-1$

se $\alpha=-1$ [!!!] non converge

$\longrightarrow$ per $\alpha<0$ $\int_0^1 f(x)dx$ converge $\iff$ $\alpha>-1$

RICAPITOLANDO: $\int_1^{+\infty} f(x)$ converge $\iff$ $\beta>\alpha+1$

$\int_0^1 f(x)$ converge $\iff$ $\alpha>0 \lor \alpha=0\land\beta>1 \lor \alpha>-1$
(escluso il caso $\alpha=0$ e $\beta\le1$)

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)dx \text{ converge} \iff \begin{cases}\beta>\alpha+1\\ \alpha>0 \lor (\alpha=0\lor\beta>1)\lor\alpha>-1\end{cases} \iff \begin{cases}\beta>\alpha+1\\ \alpha>-1\end{cases}$$