Orders of units in group rings and blocks of defect 1

Leo Margolis

Abstract: Over the decades that $U(\mathbb{Z}G)$, the unit group of an integral group ring of a finite group G, has been studied, many conjectures have been raised on how the structure of G influences the structure of subgroups of $U(\mathbb{Z}G)$. Though it often took considerable time, the strongest of these conjectures found counterexamples in the class of solvable groups. Contrary to this the arithmetic properties of finite subgroups of $U(\mathbb{Z}G)$ are very restricted for solvable G. For instance the orders of group elements and orders of torsion units $u \in U(\mathbb{Z}G)$ coincide, under the natural assumption that u has augmentation 1.

A problem on these arithmetic properties, the Prime Graph Question for integral group rings, asks: Is is true that whenever $\mathrm{U}(\mathbb{Z}G)$ contains an element of augmentation 1 and order pq, where p and q are different primes, also G must contain an element of order pq? In contrast to other problems in the area, this question is known to have a reduction to almost simple groups.

Employing Young tableaux combinatorics and Brauer's theory of blocks of defect 1 we show that when the Sylow p-subgroup of G has order p then $U(\mathbb{Z}G)$ contains an element of augmentation 1 and order pq, for any prime q, if and only if G contains an element of order pq. This directly answers the Prime Graph Question in particular for 22 sporadic simple groups, symmetric groups and also for infinite series' of groups of Lie type.

This is joint work with M. Caicedo.