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ON COMMUTATORS IN GROUPS

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Dedicated to Hermann Heineken on the occasion of his 70th birthday

Abstract

Commutators originated over 100 years ago as a by-product of computing group characters of nonabelian groups. They are now an established and immensely useful tool in all of group theory. Commutators became objects of interest in their own right soon after their introduction. In particular, the phenomenon that the set of commutators does not necessarily form a subgroup has been well documented with various kinds of examples. Many of the early results have been forgotten and were rediscovered over the years. In this paper we give a historical overview of the origins of commutators and a survey of different kinds of groups where the set of commutators does not equal the commutator subgroup. We conclude with a status report on what is now called the Ore Conjecture stating that every element in a finite nonabelian simple group is a commutator.

1   Origins of commutators

> "In a group the product of two commutators need not be a commutator, consequently the commutator group of a given group cannot be defined as the set of all commutators, but only as the group generated by these. There seems to exist very little in the way of criteria or investigations on the question when all elements of the commutator group are commutators."

This is what Oystein Ore says in 1951 in the introduction to his paper "Some remarks on commutators" [57]. Since Ore made his comments, numerous contributions have been made to this topic and they are widely scattered over the literature. Many results have been rediscovered and republished. A case in point is Ore himself. The main result of [57] is that the alternating group on n letters, $n \geq 5$, consists entirely of commutators. This was already proved by G. A. Miller [54] over half a century earlier. The two authors of this paper almost got into a similar situation after rediscovering one of the major results in this area. Fortunately we realized this before publication and then concluded that a survey of the major questions and results in this area was needed, together with a historical overview of the origins of commutators.

Commutators came into the world 125 years ago as a by-product of Dedekind's first foray into determining group characters of nonabelian groups. In an 1896 letter to Frobenius, Dedekind revealed his ideas and results for the first time. Here is what Frobenius says in [16]:

> "Das Element F, das sich mittelst der Gleichung $BA = ABF$ aus A und B ergiebt, nenne ich nach DEDEKIND den Commutator von A und B." [1]

According to Frobenius, Dedekind proved in 1880 that the conjugate of a commutator is again a commutator, and therefore that the commutator subgroup generated by the commutators of a group is a normal subgroup of the group. Furthermore, Dedekind proved that any normal subgroup with abelian quotient contains the commutator subgroup, and that the commutator subgroup is trivial if and only if the group is abelian. However, these results were first published by G. A. Miller in [52].

The motivating force behind Dedekind's introduction of commutators was his goal of extending group characters from abelian to nonabelian groups. The central object of investigation for Frobenius and Dedekind was the group determinant and its factorization, out of which arose the theory of group characters. For the definition of the group determinant and further details we refer to [9], since we are only interested in a by-product of this concept, namely commutators.

Dedekind had spent the early years of his career at the ETH Zürich (1858-62). In 1880 he revisited Zürich and became personally acquainted with Frobenius, 18 years his junior, who was at the time a professor at the ETH. This was the starting point of an on-again-off-again, sometimes intense, correspondence between the two over many years as detailed by Hawkins in [32] and [33]. Around 1880 Dedekind was motivated by his studies of the discriminant in a normal field to consider the group determinant. One of the earliest results he obtained was that for a finite abelian group of order n the group determinant factored into n linear factors with the characters as coefficients of the linear factors. In his correspondence with Frobenius, Dedekind conjectured that for a nonabelian group G the number of linear factors of the group determinant was equal to the index of the commutator subgroup G' in G, with coefficients corresponding to those of the abelian group G/G', and in this context commutators and the commutator subgroup made their appearance.

A good deal of the correspondence between Dedekind and Frobenius deals with the group determinant, its factorization, and Dedekind's conjecture stated above. Dedekind determined the group determinant and its factorization for the symmetric group S_3 and the quaternions of order 8, and in turn, Frobenius did the same for the dihedral group of order 8. Finally, in his 1896 paper [16] Frobenius proves Dedekind's conjecture as part of the general theorem on the factorization of the group determinant for finite nonabelian groups. For the details of this result we refer the interested reader to Theorem 3.4 in [9].

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[1] "Following Dedekind, I am calling the element F, which is obtained from A and B with the help of the equation $BA = ABF$, the commutator of A and B."

Dedekind himself never published anything concerning the group determinant nor its connection with commutators. However, according to Hawkins [33], Dedekind decided to pursue some group theoretic research of his own that allowed him to use his commutators. Earlier, Dedekind had studied normal extensions of the rational field with all subfields normal. Some years later these investigations suggested to him the related problem: Characterize those groups with the property that all subgroups are normal – he called such groups Hamiltonian. Dedekind found, by making use of commutators, that determining the answer was relatively simple, and he communicated this to his friend Heinrich Weber, an editor of the Mathematische Annalen, who urged him to publish the result there. Dedekind eventually published his results in [11], but only after checking with Frobenius, who assured him that this result was significant and not a consequence of known results.

As was already mentioned, G. A. Miller was the first to publish the essential results on the commutator subgroup in [52]. However, he does not attach the label "commutator" to Dedekind's correction factor F. The headline of the section in which he deals with commutators is simply "On the operation $sts^{-1}t^{-1}$". Miller's motivation in [52] for using the commutator concept was the classification of groups of order less than 48 up to isomorphism. In his two later publications addressing commutators, namely [53] and [54], he uses the label commutator and attributes it to Dedekind. In [53] Miller further expands the basic properties of the commutator subgroup, and he introduces the derived series of a group. He also shows that the derived series is finite and ends with the identity if and only if G is solvable.

In his 1899 paper [54] Miller deals with commutators as objects that are of interest in their own right. He first develops a formula that shows that under certain conditions the product of two commutators is again a commutator. In modern notation, he is showing that $[tb, a][a, b] = [t^b, a^b]$ for a, b, t in a group G. With the help of this identity he shows in Theorem I that every element of the alternating group on n letters, $n \geq 5$, is a commutator, a result rediscovered over 50 years later by Ito [37] and Ore [57]. In Theorem II, Miller shows that in the holomorph of a cyclic group C_n the commutator subgroup consists entirely of commutators and is equal to C_n, if n is odd, and equal to the subgroup of index 2 in C_n in case n is even. This foreshadows later results by Macdonald [45] and others who investigate groups with cyclic commutator subgroup in which not all elements of the commutator subgroup are commutators.

In his publications Miller never addresses the central issue in our context of whether the commutator subgroup always consists entirely of commutators. In [53] he states that for generating the commutator subgroup not all commutators are needed, and he says that a rather small portion will suffice for this purpose. On the other hand, he shows in Theorem I and II of [54] that for certain groups the set of commutators is equal to the commutator subgroup. However, as we will see when discussing [15] below, this question can not have been far from his mind. The first explicit statement of this question is found in Weber's 1899 textbook [74], which is the first textbook to introduce commutators and the commutator subgroup. After referring to Dedekind's definition of commutators, Weber states that the set of commutators is not necessarily a subgroup, but does not provide

an example to prove his claim. He does prove that the commutator subgroup is generated by the set of commutators and this subgroup forms a normal subgroup. It is Fite [15] who provides the first such example in his paper "On metabelian groups". It should be mentioned that the metabelian groups in the title are what we now call groups of nilpotency class two, or, as Fite states it, a group with an abelian group of inner automorphisms. Fite constructs a group G of order 1024 and nilpotency class 2 in which not all elements of the commutator subgroup are commutators. He attributes this example to G. A. Miller. In addition he provides a homomorphic image of G that has order 256, in which the set of commutators is not equal to the commutator subgroup. We discuss this in detail in Section 5.

To conclude our early history of commutators we mention a 1903 paper by Burnside [5]. As detailed earlier, commutators arose out of the development of group characters. Burnside uses characters to obtain a criterion for when an element of the commutator subgroup is the product of two or more commutators. So we have come full circle! This criterion was later extended by Gallagher [17]. We discuss this in detail in Section 6.

There seemed to be little interest in the topic of commutators for the 30 years following 1903. It should be kept in mind that the familiar notation for commutators had not yet been developed and its absence apparently stifled further development. The first occurrence of the commutator notation we could find is in Levi and van der Waerden's seminal paper on the Burnside groups of exponent 3 [41]. They denote the commutator of two group elements i, j as $(i, j) = iji^{-1}j^{-1}$ and make creative use of this notation in their proofs. The first textbook using the new notation is by Zassenhaus [76]. There he gives familiar commutator identities, for example, the expansion formulas for products, but not the Jacobi identity. However, Zassenhaus states that in a group with abelian commutator subgroup the following "strange" (merkwürdig) rule holds: $(a, b, c)(b, c, a)(c, a, b) = e$. As the source for the definitions, notation and formulas in his section on commutators, Zassenhaus refers to Philip Hall's paper [31], which appeared after [41].

The new notation made it possible to develop a commutator calculus to solve a variety of group theoretic problems that had not been previously accessible. In turn, the extended use of commutators as a tool brought about renewed interest in questions about commutators themselves, in particular the question on when the set of commutators is a subgroup. The remainder of the paper focuses on this question.

There are significant topics about commutators that we do not cover in this paper. These topics include: viewing the commutator operation as a binary operation; Levi's characterization of groups in which the commutator operation is associative [40]; conditions for when a product of commutators is guaranteed to be a commutator [34]; and investigations into an axiomatic treatment of the commutator laws by Macdonald and Neumann ([49], [48], and [50]) and by Ellis [14].

In the following two sections we give a survey of conditions which imply that either the set of commutators is equal to the commutator subgroup or unequal to it. Sufficient conditions for equality are rather scarce and not very powerful. As Macdonald acknowledges in [43], a forerunner of [47], there are fundamental logical

difficulties in this area, for example, the main theorem of [2] implies that there is no effective algorithm for deciding whether an element is a commutator when G is a finitely presented group. However, there are necessary and sufficient conditions for an element of a finite group to be a commutator using the irreducible characters of the group. Hence, from the character table of a finite group we can read off if every element of the commutator subgroup is a commutator. Details of this can be found in Section 6.

We introduce the following notation to facilitate our discussion. For a group G let $K(G) = \{[g, h] \mid g, h \in G\}$ be the set of commutators of G and set $G' = \langle K(G) \rangle$, the commutator subgroup of G. We say that the group element g is a commutator if it is an element of $K(G)$ and a noncommutator otherwise.

In Section 4 we construct various minimal examples of groups such that the commutator subgroup contains a noncommutator. These examples are minimal with respect to the order of the group G and the order of G', respectively. With the help of GAP [19] we construct minimal examples G and H where G is a perfect group G such that $K(G) \neq G'$ and H is a group in which $H' \cap Z(H)$ is generated by noncommutators.

As mentioned earlier, the first examples of groups with the set of commutators not equal to the commutator subgroup are finite nilpotent 2-groups of class 2. In Section 5 we develop a general construction for nilpotent p-groups of class 2 such that the commutator subgroup contains a noncommutator. This construction is obtained by finding various *covering groups* $\tilde{A}$ of an elementary abelian p-group A of rank $n \geq 4$. By a counting argument it is always the case that $K(\tilde{A}) \neq \tilde{A}'$. We look at homomorphic images of two covering groups resulting in groups of order p^8 with exponent p and p^2, respectively, such that the set of commutators is unequal to the commutator subgroup. These groups appear in the literature ([47] and [67]) and various ad-hoc methods are used to show that the commutator subgroup contains a noncommutator. The question arises: What is the smallest integer n such that for a given prime p there exists a group G of order p^n with $G' \neq K(G)$? We conclude Section 5 with an answer to this question.

For a group G the function $\lambda(G)$ denotes the smallest integer n such that every element of G' is a product of n commutators. This function was introduced by Guralnick in his dissertation [23]. The statement $K(G) \neq G'$ is then equivalent to $\lambda(G) > 1$. In Section 6 we consider conditions for upper and lower bounds for $\lambda(G)$, as well as provide conditions and examples when $\lambda(G)$ can be specified exactly. Some of these results involve character theory, in particular, to provide a necessary and sufficient condition on a finite group G such that $\lambda(G) = n$. In this section we include a well known example by Cassidy [8]. This is a group of nilpotency class 2 and it is claimed there is no bound on the number of commutators in the product representing an element of the commutator subgroup. However, a typographical error impacts the verification of this claim made in [8]. We include a slightly more general proof of the claim.

For most problems one encounters in group theory the solution in the cyclic case is trivial. Not so here, where the situation for cyclic commutator subgroups is a microcosm for the complexity of the general case. In Section 7 we survey groups

with cyclic commutator subgroup in which the commutator subgroup contains a noncommutator.

Many results in Sections 2 through 7 have been extended to higher terms of the lower central series. A survey of these results is the topic of Section 8.

The topic of the final section is a report on the current status on what has been called in the literature the Ore Conjecture (see [1], [4], [13], [22], [72] and [73]), which states that every element in a nonabelian finite simple group is a commutator. The Ore Conjecture is still open for some of the finite simple groups of Lie type over small fields. The details are given in a table at the end of the paper. There are many contributions on the Ore Conjecture in the literature concerning various types of semisimple and infinite simple groups (see for example [58] and [59]). These contributions go beyond the scope of this survey.

2 Conditions for equality

In this section we discuss mostly conditions implying that the set of commutators is equal to the commutator subgroup. There are two types of such conditions. Those of the first type are conditions on the structure of the group or the commutator subgroup that allow us to conclude the commutator subgroup contains only commutators. Those of the second type are restrictions on the order of a group or its commutator subgroup. These restrictions are mainly derived from the structural conditions of the first type. Showing that the restrictions on the orders are best possible leads to the minimal examples discussed in Sections 4 and 5.

We start with conditions on the structure of the group. One of the most versatile results is due to Spiegel.

Theorem 2.1 ([68]) Suppose the group G contains a normal abelian subgroup A with cyclic factor group G/A. Then $K(G) = G'$.

Motivated by results in [44], Liebeck in [42] gives a necessary and sufficient condition that an element of the commutator subgroup is a commutator provided the group has nilpotency class 2. Using this condition, he shows that for a group G it follows $K(G) = G'$ whenever $G' \subseteq Z(G)$ and $d(G') \leq 2$, where $d(G')$ denotes the minimal number of generators of G', and he gives an example that this cannot be extended to rank 4 or greater. Rodney in [62] extends these results. In particular, he shows the following.

Theorem 2.2 ([62]) The following two conditions on a group G imply $G' = K(G)$:

(i) G is nilpotent of class two and the minimal number of generators of G' does not exceed three;

(ii) G' is elementary abelian of order p^3.

The following result by Guralnick generalizes one of Rodney in [62].

Theorem 2.3 ([28]) Let P be a Sylow p-subgroup of G with $P^* = P \cap G'$ abelian and $d(P^*) \leq 2$. Then $P^* \subseteq K(G)$.

Similarly, Guralnick obtains the following result if $p > 3$.

Theorem 2.4 ([28]) If G' is an abelian p-subgroup of G with $p > 3$ and $d(G') \leq 3$, then $G' = K(G)$.

For nilpotent groups, in particular for finite p-groups, the following conditions of the first type are useful results for arriving at sufficient conditions of the second type.

Theorem 2.5 ([61]) If G is nilpotent and G' is cyclic, then $G' = K(G)$.

Theorem 2.6 ([39]) Let G be a finite p-group with G' elementary abelian of rank less than or equal to three. Then $K(G) = G'$.

With the exception of some additional conditions of type one (in the case of cyclic commutator subgroups) that we will consider in a later section, Theorems 2.1 – 2.6 are the tools currently available for arriving at sufficient conditions on the orders of G and G' that imply $K(G) = G'$. We start with sufficient conditions on the orders of G and G', which are shown to be best possible in Section 4.

Theorem 2.7 ([26]) Let G be a group. If (i) G' is abelian and $|G| < 128$ or $|G'| < 16$ or (ii) G' is nonabelian and $|G| < 96$ or $|G'| < 24$, then $K(G) = G'$.

Many examples of groups whose commutator subgroup contains a noncommutator are groups of prime power order. The question arises: For a p-group G of order p^n, what is the largest n such that we can guarantee that $K(G) = G'$? As we show in Section 5, the following result is best possible.

Theorem 2.8 ([39]) Let p be a prime and G a group of order p^n. Then $G' = K(G)$ if $n \leq 5$ for odd p and $n \leq 6$ for $p = 2$.

3 Conditions for inequality

In this section we discuss conditions that lead to the conclusion that the set of commutators is not equal to the commutator subgroup. However, in some cases restrictions are imposed on the structure of the group or the commutator subgroup, and then the conditions for inequality turn out to be necessary and sufficient under these restrictions. These sufficient conditions often lead to the construction of families of groups in which the commutator subgroup always contains a noncommutator. Often the objective is to find minimal examples in a certain class of groups. The conditions discussed in this section come mainly from [47], [36], [29], and [39]. The selection is based on their relevance in the next two sections, which includes the discussion of minimal examples.

We start with an almost obvious criterion that one obtains by comparing the number of possible distinct commutators with the number of elements in the commutator subgroup. The condition is stated formally for the first time in [47], but earlier applications can be found in [44] and [18].

Theorem 3.1 ([47]) If G is any group and if $|G : Z(G)|^2 < |G'|$, then there are elements in G' that are not commutators.

As Macdonald observes, the criterion is very well suited for groups with central commutator subgroups. With the help of Theorem 3.1, Macdonald constructs a large family of groups of nilpotency class 2 with the property that the set of commutators is not equal to the commutator subgroup. Isaacs' motivation in [36] for stating his criterion is similar to Macdonald's. He says that it is well known for a group G that not every element of G' need be a commutator, but what is less well known is a convenient source of finite groups that are examples of this phenomenon. His examples are wreath products satisfying the following criterion.

Theorem 3.2 ([36]) Let U and H be finite groups with U abelian and H non-abelian. Let $G = U \wr H$ be the wreath product of U and H. Then G' contains a noncommutator if

$$\sum_{A \in \mathcal{A}} \left(\frac{1}{|U|} \right)^{[H:A]} \leq \frac{1}{|U|},$$

where $\mathcal{A}$ is the set of maximal abelian subgroups of H. In particular, this inequality holds whenever $|U| \geq |\mathcal{A}|$.

The above construction yields both solvable and nonsolvable groups with the set of commutators not equal to the commutator subgroup. Choosing H simple and U large enough leads to perfect groups, that is, groups such that $G' = G$, with the desired property.

Guralnick's goal in [29] is to determine bounds on a group G and its commutator subgroup G' such that $G' = K(G)$ always holds whenever the respective orders are below these bounds. The following criterion rules out groups with a "large" abelian commutator subgroup.

Theorem 3.3 ([29]) Suppose that A is an abelian group of even order. Then there exists a group G with $G' \cong A$ and $G' \neq K(G)$ if and only if $A \cong C_2 \times A_1 \times A_2 \times A_3$ or $A \cong C_{2^\alpha} \times A_1 \times A_2$, where the A_i are nontrivial abelian groups and $\alpha \geq 2$.

In [39] sufficient conditions on the nilpotency class and certain elements belonging to the center of a group G are established that guarantee that $K(G) \neq G'$. This leads to three classes of groups with the property that the commutator subgroup contains a noncommutator. The groups of smallest order in these classes are finite p-groups and appear as minimal examples in Section 5. As it turns out, we need to establish different criteria depending on whether $p > 3$, $p = 3$, or $p = 2$, respectively, which form the three classes of groups.

Proposition 3.4 ([39]) Let $p \geq 5$ be a prime and $H = \langle a, b \rangle$ be a nilpotent group of class exactly 4 with $[b, a, b] \in Z(H)$ and $\exp(H') = p$. Then $K(H) \neq H'$.

Proposition 3.5 ([39]) Let $H = \langle a, b \rangle$ be a nilpotent group of class exactly 4 with a^3, b^9, $[b, a, b] \in Z(H)$. Then $K(H) \neq H'$.

Proposition 3.6 ([39]) Let $H = \langle a, b, c \rangle$ be a group of class 3 precisely. If a^4, b^2, c^2, $[a, c]$, $[b, c]$ and $(ab)^2 \in Z(H)$, then $K(H) \neq H'$.

4   Some minimal examples

MacHale in [51] lists 47 conjectures about groups that are known to be false and asks for a minimal counterexample for each. Conjecture 7 in his paper states "In any group G, the set of all commutators forms a subgroup". MacHale indicates that a minimal counterexample is known. In fact this is the topic of Guralnick's Ph.D. dissertation [23] and several subsequent papers, in particular [26] and [28].

In [26] examples are constructed or cited to show that the conditions of Theorem 2.7 are tight. In this section and subsequent sections we consider the following classes of groups and find groups of minimal order in each:

Groups such that the commutator subgroup is not equal to the set of commutators and

(i)   the commutator subgroup is abelian;

(ii)   the commutator subgroup is abelian of order 16;

(iii)   the commutator subgroup is nonabelian;

(iv)   the commutator subgroup is nonabelian of order 24;

(v)   the intersection of the commutator subgroup and the center is generated by noncommutators;

(vi)   the group is perfect.

In Section 5 we construct a metabelian group G of order 2^7 such that G' has order 16 and $K(G) \neq G'$. This group is of minimal order in classes (i) and (ii). It follows from Theorem 2.6 that G' is not cyclic and in fact in any minimal example H in classes (i) and (ii) the commutator subgroup H' can not be cyclic. Hence we consider the following variant of classes (i) and (ii):

Groups such that the commutator subgroup is not equal to the set of commutators and

(i')   the commutator subgroup is cyclic;

(ii')   the commutator subgroup is cyclic of order 60.

The group G constructed in Example 7.10 has order 240 such that its commutator subgroup is cyclic of order 60 and $K(G) \neq G'$. This group is of minimal order in classes *(i')* and *(ii')*.

Of course MacHale was interested in the smallest group order such that the commutator subgroup contains a noncommutator. As we see below, such groups are in class *(iii)*. Minimal examples for each class *(i)* – *(vi)*, *(i')* and *(ii')* can be found using GAP by searching its small groups library. For example, Rotman [65] states that via computer search the smallest examples G such that $K(G) \neq G'$ have order 96. Rotman was apparently unaware of Guralnick's earlier work. The following example gives explicit constructions of the two nonisomorphic groups of order 96 whose commutator subgroups contain noncommutators.

Example 4.1 ([23]) There are exactly two nonisomorphic groups G of order 96 such that $K(G) \neq G'$. In both cases G' is nonabelian of order 32 and $|K(G)| = 29$.

(a) Let $G = H \rtimes \langle y \rangle$, where $H = \langle a \rangle \times \langle b \rangle \times \langle i, j \rangle \cong C_2 \times C_2 \times Q_8$ and $\langle y \rangle \cong C_3$. Let y act on H as follows: $a^y = b$, $b^y = ab$, $i^y = j$ and $j^y = ij$.

(b) Let $H = N \rtimes \langle c \rangle$, where $N = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_4$ and $\langle c \rangle \cong C_4$. Let c act on N by $a^c = a$ and $b^c = ab$. Let $G = H \rtimes \langle \gamma \rangle$ with $\langle \gamma \rangle \cong C_3$, where $a^\gamma = c^2 b^2$, $b^\gamma = cba$, $c^\gamma = ba$.

The group of Example 4.1 *(a)* appears in Dummit and Foote's textbook [12] as an example of a group G with $K(G) \neq G$. No claim of minimality is made.

Guralnick in [26] describes the following class of groups that yields many examples of groups G in which $K(G) \neq G'$. Let

$$G_1 = \langle a, b, x \mid a^4 = b^4 = x^3 = 1, xax^{-1} = b, xbx^{-1} = ab, \qquad (4.1.1)$$
$$a^2 = b^2, aba^{-1} = b^{-1} \rangle.$$

Then $G_1 = \langle H_1, x \rangle$, where $H_1 = G_1' = \langle a, b \rangle \cong Q_8$. Choose G_2 to be any nonabelian group with normal abelian subgroup H_2 of index 3. Then there exists $y \in G_2$ such that $G_2 = \langle H_2, y \rangle$. Let G be the subgroup of $G_1 \times G_2$ generated by $H_1 \times H_2$ and the element (x, y). Then G has order $24|H_2|$ and $K(G) \neq G'$. Note that $G' = G_1' \times G_2'$. In particular, for any $1 \neq g \in G_2'$ the element (a^2, g) is not a commutator. Our next three examples arise from this construction for particular choices of G_2.

Example 4.2 Let G_1 be defined as in (4.1.1) and take $G_2 = A_4$ and $H_2 = G_2' \cong C_2 \times C_2$. Then G has order 96 and $G' \cong Q_8 \times C_2 \times C_2$.

The group constructed in Example 4.2 is isomorphic to the group in Example 4.1 *(a)*. However, Example 4.1 *(b)* cannot be constructed in this manner, since there is not another nonabelian group of order 12 with a normal subgroup of order 4.

The following example gives a group of minimal order in class *(iv)*.

Example 4.3 Let G_1 be the group defined in (4.1.1). Take G_2 to be any non-abelian group of order 27 so that $|H_2| = 9$ and $G_2' \cong C_3$. Then $G' \cong Q_8 \times C_3$, which is nonabelian of order 24. The order of G is $24 \cdot 9 = 216$.

We finish this section by giving minimal examples of groups G such that $K(G) \neq G'$ with some additional property. The first example is a group G in which $G' \cap Z(G)$ is generated by noncommutators. The question of whether such a group exists was asked by R. Oliver and an example was given by Caranti and Scopolla [6]. Their example has order p^{14}, where p is an odd prime, but it is noted in the paper that a smaller example exists. Our example with this property has order 216 which by a search of the small groups library in GAP is the smallest such example.

Example 4.4 Set $G_1 = \langle H_1, x \rangle$ as in (4.1.1) and set

$$G_2 = \langle c, y \mid c^9 = y^3 = 1, y^{-1}cy = c^4 \rangle \cong C_9 \rtimes C_3$$

where $H_2 = C_9 = \langle c \rangle$ and $C_3 = \langle y \rangle$. Let $G = \langle H_1 \times H_2, \{(x,y)\} \rangle$. Then $Z(G) = \langle (a^2, c^3) \rangle \cong C_6$. However, since $c^3 \in G_2'$, we have that (a^2, c^3) is a noncommutator, as needed.

To find a perfect group G in which $K(G) \neq G'$ we can use the group construction and criterion from Theorem 3.2 due to Isaacs [36]. The smallest perfect group obtainable from this construction is the following.

Example 4.5 Let $G = C_2 \wr A_5$. Then G is a perfect group with $|G| = 2^{60} \cdot 60$. To see that $G' \neq K(G)$ we note that the maximal abelian subgroups of A_5 are its Sylow subgroups. There are ten maximal abelian subgroups of order 3, five maximal abelian subgroups of order 4, and six maximal abelian subgroups of order 5. This gives

$$10 \left(\frac{1}{3}\right)^{20} + 5 \left(\frac{1}{4}\right)^{15} + 6 \left(\frac{1}{5}\right)^{12} \leq \frac{1}{60},$$

which by Theorem 3.2 shows that $K(G) \neq G'$.

A search of the perfect groups in GAP shows that the smallest perfect group G that contains an element that is not a commutator has order 960. This group can be visualized in the following way. Let $H = C_2^5 \rtimes A_5$, where we think of A_5 having a "wreath action" on C_2^5. Set $G = H/Z(H)$. Then G is a perfect group of order $2^5 \cdot 60 \cdot \frac{1}{2}$ and has the property that $K(G) \neq G'$.

5 p-Groups

The earliest examples of groups G in the literature for which $K(G) \neq G'$ are 2-groups. Fite [15] attributes the following group of nilpotency class 2 and order 1024 to G. A. Miller, represented here as a subgroup of S_{24}:

$$\begin{aligned}
G = \langle &(1,3)(5,7)(9,11), (1,2)(3,4)(13,15)(17,19), \\
&(5,6)(7,8)(13,14)(15,16)(21,23), \\
&(9,10)(11,12)(17,18)(19,20)(21,22)(23,24) \rangle.
\end{aligned}$$

He states that G' contains 36 commutators and 28 noncommutators. Fite then considers a homomorphic image of G, represented here as a subgroup of S_{16}:

$$H = \langle (1,3)(5,7)(9,11), (1,2)(3,4)(13,15),$$
$$(5,6)(7,8)(13,14)(15,16), (9,10)(11,12) \rangle.$$

The group H has order 256 with $|K(H)| = 15$ and $|H'| = 16$. The group H of Fite is used in several textbooks, for example, Carmichael [7] and an early edition of Rotman [64], as an example of a group in which the commutator subgroup is not equal to the set of commutators.

Miller's group G above is the basis for our study of p-groups of nilpotency class 2 for which $K(G) \neq G'$. For some normal subgroups S of G' of order 4 the property $K(G/S) \neq (G/S)'$ holds; the group H given by Fite is an example. In fact there are several such subgroups of order 4 in G for which this is true, and explicit constructions of such quotients can be found in the literature, for example, in [46].

We now adapt the construction of the 2-group G above to a p-group $\tilde{A}$ that is the covering group of an elementary abelian p-group that has the property that $\tilde{A}' \neq K(\tilde{A})$. [2]

Definition 5.1 Let Q be a finite group and let $M(Q)$ be the Schur multiplier of Q. A group $\tilde{Q}$ is called the *covering group* of Q if $\tilde{Q}$ contains a normal subgroup N such that $N \subseteq Z(\tilde{Q}) \cap \tilde{Q}'$, $N \cong M(Q)$, and $\tilde{Q}/N \cong Q$.

Schur in [66] showed that every finite group has a covering group. Now let A be an elementary abelian p-group of rank $n \geq 2$. Then it can be shown that $M(A)$ is elementary abelian of order $p^{n(n-1)/2}$ and hence that the covering group $\tilde{A}$ has order $p^{n(n+1)/2}$. The center of $\tilde{A}$ equals $\tilde{A}'$; therefore $\tilde{A}$ is nilpotent of class 2 and $M(A) \cong \tilde{A}'$.

For $n \geq 6$, where n is the rank of A, we have $|\tilde{A}/Z(\tilde{A})| = p^n$. Hence in this case $|\tilde{A}/Z(\tilde{A})|^2 < |\tilde{A}'|$, and we can use Theorem 3.1 to see that $K(\tilde{A}) \neq \tilde{A}'$. We can actually take $n \geq 4$ and note that the covering group contains elements in $\tilde{A}'$ that are not in $K(\tilde{A})$. This is because the number of nonidentity commutators in $K(\tilde{A})$ is exactly

$$|K(\tilde{A})| - 1 = \frac{(p^n - 1)(p^{n-1} - 1)}{p^2 - 1}. \tag{5.1.1}$$

For $n \geq 4$ we have that $|K(\tilde{A})| < |\tilde{A}'|$. Moreover, there are homomorphic images $\mathcal{H}$ of $\tilde{A}$ such that $K(\mathcal{H}) \neq \mathcal{H}'$. This is exactly the case for Miller's group G and Fite's group H above for $p = 2$ with $G = \tilde{A}$ and $H = \mathcal{H}$.

An explicit construction of various covering groups of the elementary abelian p-group of rank $n \geq 6$ can be found in Macdonald [47]. Macdonald constructs these groups explicitly for the purpose of finding groups G for which $G' \neq K(G)$. As noted above, the covering groups for the elementary abelian p-group A of rank 4 have the property that $K(\tilde{A}) \neq \tilde{A}'$. In the following example we construct a

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[2]The authors would like to thank R. Gow and R. Quinlan for pointing out the connection between covering groups and this analysis of nilpotent groups of class 2 .

covering group of exponent p for $n = 4$ and $p > 2$. This group can be found in [67] and is attributed to W. P. Kappe. Because the group has exponent p it is straightforward to show explicitly that a particular element of the commutator subgroup is not a commutator and one does not need the counting argument of (5.1.1). The details can be found in [67].

Example 5.2 Let $G = \langle g_1, g_2, g_3, g_4 \rangle$ be the free nilpotent group of class 2 and exponent p, where $p > 2$ is a prime. The group G has order p^{10}, $G' = Z(G) \cong C_p^6$ and $G/G' \cong C_p^4$. Hence G is the covering group for C_p^4 as needed, and we have $K(G) \neq G'$ by (5.1.1).

Our next example is also a covering group of A for $n = 4$ and p a prime. This group has exponent p^2.

Example 5.3 Let $H = \langle g_1, g_2, g_3, g_4 \rangle$ be the free nilpotent group of class 2 and let p be an odd prime. Consider the following relations:

$$R = \{g_1^{p^2} = g_2^{p^2} = g_3^{p^2} = g_4^{p^2} = 1,$$
$$g_1^p = [g_1, g_3], g_2^p = [g_2, g_4], g_3^p = [g_1, g_2], g_4^p = [g_2, g_3]\}.$$

Let $G = H/N$, where N is the normal closure of R. Then G is a covering group for C_p^4, and $K(G) \neq G'$ by (5.1.1).

For Examples 5.2 and 5.3 we construct a homomorphic image of each group of order p^8. The first quotient, given in Example 5.4, is the smallest nilpotent group of class 2 and exponent p such that the set of commutators is unequal to the commutator subgroup. This group can also be found in [67].

Example 5.4 Let G be the group in Example 5.2 and let S be the normal subgroup of G generated by $[g_3, g_4]$ and $[g_2, g_4]$. The group $Q = G/S$ is a group of order p^8 and exponent p in which $K(Q) \neq Q'$.

The following group is a quotient of the group of Example 5.3. It is isomorphic to the group constructed by Macdonald in [46] as an example of a p-group in which the set of commutators is unequal to the commutator subgroup.

Example 5.5 Let G be the group in Example 5.3. Let S be the normal subgroup generated by $[g_1, g_4]$ and $[g_3, g_4]$. Then $Q = G/S$ has order p^8, $\exp(Q) = p^2$, and Q has the property $K(Q) \neq Q'$.

It is an open question whether every nilpotent group G of class 2 such that $K(G) \neq G'$ is a homomorphic image of a cover group of some abelian group.

The nilpotent groups of class 2 do not provide examples of p-groups of order p^n with n minimal such that $K(G) \neq G'$. As we can see from *(i)* of Theorem 2.2, any proper homomorphic image H/L of H of order less than p^8 has the property $K(H/L) = (H/L)'$. We finish this section with three examples to show that the bounds on n given in Theorem 2.8 are sharp.

Example 5.6 ([39]) Let p be a prime with $p > 3$ and let $V = \langle u \rangle \times \langle v \rangle \times \langle w \rangle \times \langle z \rangle$ be an elementary abelian p-group of rank 4. Let $B = V \rtimes \langle b \rangle$, the semidirect product of V with a cyclic group $\langle b \rangle$ of order p. The defining relations of B are those of V along with

$$b^p = 1, \quad [u, b] = w, \quad \text{and} \quad [v, b] = [w, b] = [z, b] = 1.$$

Similarly, let $G = B \rtimes \langle a \rangle$ be the semidirect product of B with a cyclic group $\langle a \rangle$ of order p. The defining relations of G are those of B along with

$$[b, a] = u, \quad [u, a] = v, \quad [v, a] = z, \quad a^p = [w, a] = [z, a] = 1.$$

It can be verified that G has order p^6, nilpotence class 4, and $\exp(G) = p$. Furthermore, $u = [b, a]$, $v = [b, a, a]$, $w = [b, a, b]$ and $z = [b, a, a, a]$. Thus G satisfies the conditions of Proposition 3.4. We conclude that $K(G) \neq G'$.

The next two examples are minimal examples from the classes of groups found in Propositions 3.5 and 3.6, respectively. These examples are different than those found in [39] and show that minimal examples in these classes are not unique.

Example 5.7 Let $A = \langle a \rangle \times \langle b \rangle \cong C_3 \times C_9$. Let $B = A \rtimes \langle x \rangle$ be the semidirect product of A with a cyclic group $\langle x \rangle$ of order 9. The defining relations of B are those of A along with $x^9 = [a, x] = 1$, $[b, x] = a^2$. We form $G = B \rtimes \langle y \rangle$ as a semidirect product of B with a cyclic group $\langle y \rangle$ of order 3. The relations of G are those of B along with $y^3 = [a, y] = 1$, $[b, y] = x$ and $[x, y] = b^6$. The order of G is 3^6 and $G = \langle b, y \rangle$, since $x = [b, y]$ and $a = [x, b]$. The group is nilpotent of class 4, since $[b, y, y, y] = [x, y, y] = [b^6, y] = x^6 \neq 1$. Now b^9 and y^3 are in the center of G because $b^9 = y^3 = 1$. It can be verified that $[b, y, b]$ is also in $Z(G)$. It follows by Proposition 3.5 that $K(G) \neq G'$.

Example 5.8 Let $A = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_8$. Let $G = A \rtimes \langle x, y \rangle$ be the semidirect product of A with $Q_8 = \langle x, y \rangle$, the group of quaternions. The defining relations of G are those of A and Q_8 along with $[a, x] = [a, y] = 1$, $[b, x] = b^6$, $[b, y] = ab^2$. The group G has order 128, is nilpotent of class 3 and $G = \langle x, bx^3, y \rangle$, since $[y, b] = a$. It is readily verified that G satisfies the conditions of Proposition 3.6, and hence $K(G) \neq G$.

The group G constructed in Example 5.8 is metabelian with $|G'| = 16$. The only groups G of order less than 128 for which $G' \neq K(G)$ are the two groups of order 96 found in Example 4.1. These two groups have nonabelian commutator subgroups. Hence the group constructed in Example 5.8 is a minimal group in the classes *(i)* and *(ii)* listed in Section 4.

6   The function $\lambda(G)$

We define the value $\lambda(G)$ for a group G to be the the smallest positive integer n such that every element of G' is a product of n commutators and if n is unbounded

then we define $\lambda(G) = \infty$ [23]. The condition $K(G) \neq G'$ is equivalent to $\lambda(G) > 1$. In this section we consider conditions for upper and lower bounds for $\lambda(G)$, as well as providing conditions and examples when $\lambda(G)$ can be specified exactly.

Obtaining an arbitrary lower bound for $\lambda(G)$ does not require a complicated structure for G'. The following result is due to Macdonald [45] and is a corollary to Theorem 7.1.

Theorem 6.1 ([45]) For any positive integer n there is a group G such that G' is cyclic and $\lambda(G) > n$.

Guralnick in [27] also investigates groups with cyclic commutator subgroup of order m and gives necessary and sufficient conditions such that $\lambda(G) > f(m)$. The details can be found in the next section (Theorem 7.8).

In [63] and [25] finite upper bounds on $\lambda(G)$ are applied to prove a classical result of Schur that if $[G : Z(G)]$ is finite then G' is finite. For a, b in G and u, v in $Z(G)$ we have $[au, bv] = [a, b]$. Hence G has at most $[G : Z(G)]^2$ commutators. When $[G : Z(G)]$ is finite it follows that G' is finite exactly when $\lambda(G)$ is finite. Setting $[G : Z(G)] = n$, Rosenlicht [63] shows $\lambda(G) \leq n^3$. Guralnick in [25] proves $\lambda(G) < \tau(n)/2$, where $\tau(n)$ is the number of divisors of n and improves this bound to $\lambda(G) < 3\rho(n)/2$, where $\rho(n)$ is the number of prime divisors of n (counting multiplicity).

Under certain conditions on the group G the bounds on $\lambda(G)$ can be improved as follows.

Theorem 6.2 ([25]) Let G be a group.

(i) If G is nilpotent and $G/Z(G)$ is generated by n elements, then $\lambda(G) \leq n$. If, in addition, $G' \subseteq Z(G)$, then $\lambda(G) \leq \lfloor \frac{n}{2} \rfloor$.

(ii) If G is finitely generated and is nilpotent-by-nilpotent, then $\lambda(G)$ is finite.

Nikolov and Segal in [56] prove that every subgroup of finite index in a (topologically) finitely generated profinite group is open. The next theorem is a special case of this result. [3]

Theorem 6.3 ([56]) There is a function g defined on the positive integers such that if G is a finite group generated by d elements, then $\lambda(G) \leq g(d)$.

In the following example Guralnick constructs groups G with $\lambda(G) = n$ for every positive integer n.

Example 6.4 ([25]) Let p be a prime and let n be a positive integer. Let

$$H = H(n, p) = \langle x_1, \ldots, x_{2n} \mid x_i^p = [x_i, [x_j, x_k]] = 1, 1 \leq i, j, k \leq 2n \rangle$$

and

$$N = N(n, p) = \langle [x_i, x_j] \mid i + j > 2n + 1 \rangle.$$

Set $G = G(n, p) = H/N$. Then $\lambda(G) = n$.

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[3]The authors would like to thank R. Guralnick for pointing out this result to us.

Proof We observe that H is nilpotent of class 2 and $|H| = p^{2n+\binom{2n}{2}}$ with $\exp H = p$ for p odd. We have $N \triangleleft H$, since $N \subseteq Z(H)$ and $|N| = p^{n^2-n}$. Thus $|G| = |H/N| = p^{2n+n^2}$. We observe $H' = Z(H)$ and $G' = Z(G)$, as well as $[H : Z(H)] = [G : Z(G)] = p^{2n}$. Thus by (i) of Theorem 6.2 it follows that $\lambda(G) \leq n$. Lemma 5.1 in [25] yields that there exists an element in G' which is the product of exactly n nontrivial commutators. $\square$

We conclude this discussion on bounds for $\lambda(G)$ by considering a well-known example of Cassidy [8] that replaced Fite's example of order 256 [15] as an example of a group in which the set of commutators is not a subgroup (see e.g. [65]). Our context here is different. It is well known that in free groups, even of finite rank, $\lambda(G)$ is not bounded (see [60]). Cassidy's example, which is not finitely generated, but is nilpotent of class 2, shows that the assumption of being finitely generated cannot be omitted from Theorem 6.2 *(ii)*. As already mentioned in the introduction, the proof given in [8] contains a major typographical error. Our example below addresses a slightly more general situation. [4]

Example 6.5 Let f and g be polynomials over a field K in x and y, respectively, and let h be a polynomial in x and y with coefficients in K. Let $m(f, g, h)$ be the matrix
$$\begin{pmatrix} 1 & f(x) & h(x,y) \\ 0 & 1 & g(y) \\ 0 & 0 & 1 \end{pmatrix}.$$
Then the set of matrices $m(f, g, h)$ forms a group G under matrix multiplication. The group G is nilpotent of class 2 with $G' = Z(G)$ and $\lambda(G) = \infty$.

Proof It is easy to see that the set of matrices forms a group as claimed and that the center of G consists of the matrices of the form $m(0, 0, h)$. An arbitrary commutator in G has the form
$$[m(f_1, g_1, h_1), m(f_2, g_2, h_2)] = m(0, 0, f_1 g_2 - f_2 g_1). \qquad (6.5.1)$$
It follows that $G' \subseteq Z(G)$. Conversely, we have
$$m(0, 0, \Sigma a_{ij} x^i y^j) = \prod [m(a_{ij} x^i, 0, 0), m(0, y^j, 0)],$$
hence $Z(G) \subseteq G'$, and our claim follows.

To show $\lambda(G) = \infty$, let n be a positive integer and $h(x, y) = \sum_{i=0}^{2n} x^i y^{2n-i}$. We will show that $m(0, 0, h(x, y))$ is not the product of n commutators. Assume otherwise. Without loss of generality we can write
$$m(0, 0, h(x, y)) = \prod_{j=1}^{n} [m(s_j(x), t_j(x), 0), m(u_j(x), v_j(y), 0)],$$

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[4]Special thanks go to W. P. Kappe for providing the authors with this proof.

where $s_j(x) = \sum_i a_{ij} x^i$, $u_j(x) = \sum_i b_{ij} x^i$, and the a_{ij} and b_{ij} are elements of K. By (6.5.1) it follows that

$$h(x,y) = \sum_{j=1}^{n} (s_j(x) v_j(y) - u_j(x) t_j(y)). \qquad (6.5.2)$$

In (6.5.2) we compare the coefficients of the powers of x which are polynomials in y and obtain the following $2n+1$ linear relations in the vector space of polynomials in y over K

$$y^{2n-i} = \sum_{j=1}^{n} (a_{ij} v_j(y) - b_{ij} t_j(y)) \text{ for } i = 0, 1, \ldots, 2n. \qquad (6.5.3)$$

Set $V = \text{span}\{1, y, \ldots, y^{2n}\}$ and $W = \text{span}\{t_1(y), \ldots, t_n(y), v_1(y), \ldots, v_n(y)\}$. By (6.5.3) it follows $V \subseteq W$. We observe $\dim W \leq 2n$ and $\dim V = 2n+1$, since $\{1, y, \ldots, y^{2n}\}$ is a linearly independent set, so $2n + 1 \leq 2n$, a contradiction. $\quad\square$

A few remarks on the nature of the typographical error in [8] are in order. The critical element in the commutator subgroup is defined as $m(0,0,h)$, where $h(x,y) = \sum_{i=0}^{2n+1} x^i y^j$. No specification for j is given. According to P. J. Cassidy [5], the j should be replaced by i. After some minor adjustments of indices, the proof can be carried out as indicated in [8].

Character theory for finite groups provides necessary and sufficient conditions as to when an element is a commutator, and allows one to compute $\lambda(G)$. Oft quoted is Honda [35], but these ideas were known to Burnside [5] and, even earlier, to Frobenius.

Theorem 6.6 Let G be a finite group and g an element of G. Consider the following function $\sigma : G \mapsto \mathbb{C}$ defined by

$$\sigma(g) = \sum_{\chi \in \text{Irr}(G)} \frac{\chi(g)}{\chi(1)}. \qquad (6.6.1)$$

Then g is a commutator if and only if $\sigma(g) \neq 0$.

A brute force method for testing whether or not $\lambda(G) > 1$ is to compute the set $S = \{g \in G \mid \sigma(g) \neq 0\}$ and check whether $|S| = |G'|$. Since the irreducible characters are functions on the conjugacy classes, we know that all elements of a conjugacy class are either commutators or are not commutators. Hence in general we can refine our test by computing the following sum:

$$\sum_{c \in \mathcal{C}} |c| \cdot \sigma'(c(g)),$$

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[5]Personal communication with P. J. Cassidy by the authors.

where $\mathcal{C}$ is the set of conjugacy classes, $c(g)$ is a representative of the conjugacy class c, and

$$\sigma'(g) = \begin{cases} 0 & \text{if } \sigma(g) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Our observations now lead to the following necessary and sufficient criterion such that $\lambda(G) > 1$ in a finite group G. This result is a consequence of Theorem 6.6 and until now has not been explicitly stated in the literature.

Corollary 6.7 Let G be a finite group. Then $\lambda(G) > 1$ if and only if

$$\sum_{c \in \mathcal{C}} |c| \sigma'(c(g)) \neq |G'|. \tag{6.7.1}$$

For perfect groups the test is even simpler. If G is a perfect group, then $\lambda(G) > 1$ if and only if any $\sigma(c(g))$ is equal to zero.

Let f_χ be the degree of the irreducible character χ. Define $m(G)$ to be the cardinality of the set $\{f_\chi \mid \chi \in \text{Irr}(G)\}$. Guralnick gives an upper bound for $\lambda(G)$ using $m(G)$.

Theorem 6.8 ([25]) If G is finite, then $\lambda(G) < m(G)$.

Guralnick also states a lemma based on a result in [17], from which $\lambda(G)$ can be exactly computed for a finite group G.

Lemma 6.1 ([25]) *Let G be a finite group with irreducible characters $\chi_1, \ldots, \chi_h$. Consider the expression*

$$S(k, g) = \sum_{i=1}^{h} f_i^{1-2k} \chi_i(g). \tag{6.8.1}$$

Then $\lambda(G) = n$ if and only if $S(n, g) = 0$ for all $g \in G'$ and $S(n-1, g) \neq 0$ for some $g \in G'$.

The criterion of Lemma 6.1 can be implemented in GAP. It turns out that every group G of order less than 1000 has $\lambda(G) \leq 2$. The question arises: What is the smallest group G such that $\lambda(G) = 3$, and more generally, for which $\lambda(G) = n$?

7   Cyclic commutator subgroups

In [45], I. D. Macdonald begins the discussion on groups with cyclic commutator subgroups by showing that the commutator subgroup of such groups need not have a generating commutator. There are three follow-up papers, [61], [20] and [27], with more or less the same title, which expand on Macdonald's earlier result. These four papers are the topic of discussion in this section.

Macdonald's result is summarized in the following theorem.

Theorem 7.1 ([45]) If G' is cyclic and either G nilpotent or G' is infinite, then G' is generated by a suitable commutator. However, for any given positive integer n there is a group G in which G' is cyclic and generated by no set of fewer than n commutators.

It should be mentioned here that by a result of Honda [35], for any $g \in G'$ every generator of $\langle g \rangle$ is the product of the same number of commutators.

The main result of Rodney's paper is a sufficient condition for every element of G' to be a commutator, where G is a group with finite cyclic commutator subgroup generated by a commutator.

Theorem 7.2 ([61]) Let G' be cyclic of finite order with $4 \nmid |G'|$. Suppose $G' = \langle c \rangle$ with $c = [a, b]$. Let μ and ν be integers such that $c^a = c^\mu$ and $c^b = c^\nu$. If one of the following four conditions fails to hold for every prime divisor p of $|G'|$, then G' consists entirely of commutators:

   I. $\mu - 1 \equiv 0 \mod p, \nu - 1 \equiv 0 \mod p$;

  II. $\mu - 1 \equiv 0 \mod p, \nu - 1 \not\equiv 0 \mod p$;

 III. $\mu - 1 \not\equiv 0 \mod p, \nu - 1 \equiv 0 \mod p$;

 IV. $\mu - 1 \not\equiv 0 \mod p, \nu - 1 \not\equiv 0 \mod p$.

As one can observe, the four conditions are mutually exclusive and generate a partition on the set of primes dividing $|G'|$. The statement of the theorem then says that $K(G) = G'$ if at least one of the equivalence classes is empty. This leads to the following corollary not stated in [61].

Corollary 7.3 If $G' = \langle [a, b] \rangle$, $|G'|$ is finite and at most three primes divide $|G'|$, but $4 \nmid |G'|$, then $K(G) = G'$.

We observe that the assumption of being generated by a commutator can be dropped in the case G' is cyclic of p-power order. Finally, Rodney obtains the following corollary to Theorems 7.1 and 7.2.

Corollary 7.4 ([61]) If G' is cyclic and either G is nilpotent or G' is infinite, then $G' = K(G)$.

In [20], Gordon, Guralnick and Miller determine all integers n such that there is a group G in which the set of commutators has fewer than n elements and that set generates a cyclic subgroup of order n. At the same time, they obtain a sufficient condition on n such that for cyclic G' of order n we have the equality $K(G) = G'$. Both results are under the assumption that G' is generated by a commutator.

Theorem 7.5 ([20]) Let p and q_i be primes. Suppose that either

  (i) $n = p^\alpha \cdot q_0^{\alpha_0} \cdot q_1^{\alpha_1} \cdots q_h^{\alpha_h}$, $h \geq p$ and $q_i \equiv 1 \mod p$ for $i = 0, 1, \ldots, p$, or

  (ii) $n = 2^\alpha \cdot q_0^{\alpha_0} \cdot q_1^{\alpha_1} \cdots q_h^{\alpha_h}$, $\alpha \geq 2$, $h \geq 1$.

Then there exists a group G such that $G' = \langle a \rangle$ is cyclic of order n, a is a commutator but not every element of G' is a commutator.

Theorem 7.6 ([20]) Let p and p_i be primes. Suppose that $G' = \langle a \rangle$, where a is a commutator. Assume that a has order n, where either

(i) $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $4 \nmid n$ and $|\mathcal{J}_i| \leq p_i$, where $\mathcal{J}_i = \{j \mid p_j \equiv 1 \mod p_i\}$, or

(ii) $n = 2^\alpha \cdot p^\beta$, $\alpha \geq 2$, $\beta \geq 0$.

Then every element of G' is a commutator.

Unaware of [61], Gordon et al. show in [20] that for cyclic G' either of infinite or p-power order, or for G nilpotent, it follows that $K(G) = G'$. That paper concludes with the following interesting result.

Corollary 7.7 ([20]) Suppose that G is a commutator subgroup of a group H and that G' is cyclic. Then every element of G' is a commutator.

In [27] all pairs of integers (m, n) are determined for which there exists a group G with G' cyclic of order n and $\lambda(G) > m$. The results are summarized in the following theorem.

Theorem 7.8 ([27]) Let p and p_i be primes.

(a) Given the ordered pair (n, m) with $m \geq 2$, there exists a group G with G' cyclic of order n and $\lambda(G) > m$ if and only if

(i) $n = p_1^{\alpha_1} \cdots p_\nu^{\alpha_\nu}$, $\nu \geq 2^{2m+1} - 1$, or

(ii) $n = 2^\alpha p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$, $\nu \geq 2^{2m+1} - 3$.

(b) There exists a group G with G' cyclic of order n and $\lambda(G) > 1$ if and only if

(i) $n = p_1^{\alpha_1} \cdots p_\nu^{\alpha_\nu}$, $\nu \geq 7$,

(ii) $n = 3^\alpha p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$, $p_i \equiv 1 \pmod 3$ for $i = 1, 2, 3, 4$,

(iii) $n = 2 p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$, $\nu \geq 3$, or

(iv) $n = 2^\alpha p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$, $\nu \geq 2$, $\alpha \geq 2$.

In this context Guralnick takes up the original theme of Macdonald [Mac63], that is the study of groups with cyclic commutator subgroup in which the generators are noncommutators.

Theorem 7.9 ([27]) Let p and p_i be primes. Consider a natural number n satisfying one of the following conditions:

(i) $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $r \geq 2^{2m+1} - 1$;

(ii) $n = 2^\alpha p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $r \geq 2^{2m+1} - 3$.

Then there exists a group G with G' cyclic of order n such that no generator of G' is a product of m commutators. Conversely, let $H = \langle a \rangle$ be cyclic of order n. If there exists a group G with $G' = H$ and no generator of H is a product of m commutators, then n satisfies condition (i) or (ii).

According to Theorem 7.9 the minimal order for a cyclic commutator subgroup with none of the generators being a commutator is 2810. The smallest n and the smallest group order for which there exists a group with cyclic commutator subgroup of order n, where the set of commutators is not equal to the commutator subgroup, is given in [20], and follows from Theorem 7.8. Similar examples are given in [61], but no claim of minimality is made.

Example 7.10 ([20]) Let $G = \langle x, y, a \rangle$ with relations $a^{60} = x^8 = y^6 = 1$, $x^{-1}ax = a^{29}$, $y^{-1}ay = a^{19}$, $[x, y] = a$, $x^2 = a^{15}$, $y^2 = a^{40}$. Then $|G| = 240$, $G' = \langle a \rangle$ and $|G'| = 60$. This group has minimal order in both classes *(i')* and *(ii')* defined in Section 4.

8   Higher commutators

The topic of this section is the relationship between the set of r-fold simple commutators and the r-th term of the lower central series, the subgroup generated by them. This relationship is studied in [18], [38], [10], [29], and [30]. Since each paper extends and generalizes some of the results of the preceding ones, we discuss them in chronological order. The notation in these papers is not uniform. Thus we adopt the following notation. For $x_1, x_2, \ldots, x_r \in G$, let $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and recursively define $[x_1, \ldots, x_{r-1}, x_r] = [[x_1, \ldots, x_{r-1}], x_r]$ as the r-fold simple commutator of $x_1, \ldots, x_r$. Let $K_r(G) = \{[x_1, \ldots, x_r] \mid x_1, \ldots, x_r \in G\}$, the set of r-fold simple commutators of G, and let $\gamma_r(G) = \langle K_r(G) \rangle$, which is the r-th term of the lower central series. Observe that $K_2(G) = K(G)$ and $G' = \gamma_2(G)$. In this section we deal with the case when $r > 2$. We also extend the function $\lambda(G)$, introduced in Section 6, to higher terms of the lower central series. We say that $\lambda_r(G)$ is the smallest integer n such that every element in $\gamma_r(G)$ is a product of n elements in $K_r(G)$ and if n is unbounded we define $\lambda_r(G) = \infty$. Note that $\lambda(G) = \lambda_2(G)$.

In [18], Gallagher extends results obtained in [17] for the commutator subgroup to higher terms of the lower central series. As in [17], the proofs involve intricate but elementary character calculations. His main result is the following.

Theorem 8.1 ([18]) Let G be a group. If

$$n > \left(\frac{2^{r-1}}{3} \right)^{1/2} \log(2|\gamma_r(G)| - 2),$$

then each element of $\gamma_r(G)$ is a product of n commutators.

In [38], the results of Macdonald [45] on cyclic commutator subgroups are extended to the terms of the lower central series. Specifically, the following analogue of Theorem 7.1 is proved.

Theorem 8.2 ([38]) If $\gamma_r(G)$ is cyclic and either G is nilpotent or $\gamma_r(G)$ is infinite, then $\gamma_r(G)$ is generated by a suitable commutator of weight r. For any given integer n, however, there is a group G in which $\gamma_r(G)$ is cyclic and generated by no set of fewer than n commutators of weight r.

The examples mentioned in the above theorem are the same as given by Macdonald in [45]. The next theorem is a partial extension of Spiegel's result [68] (see Theorem 2.1) to higher terms of the lower central series.

Theorem 8.3 ([38]) Let G be a metabelian group. If $\gamma_r(G)$ and the automorphism group induced on $\gamma_r(G)$ are both cyclic, then $K_r(G) = \gamma_r(G)$.

In [10] Dark and Newell extend some of the results in [45], [61], [42], [24], and [38] on commutator subgroups with a small number of generators to the higher terms of the lower central series. They give examples to show that some results for $\gamma_2(G)$ do not necessarily hold for higher terms of the lower central series. The next theorem extends results of [61] and [38] (see Corollary 7.4 and Theorem 8.2).

Theorem 8.4 ([10]) If $\gamma_r(G)$ is cyclic and either G is nilpotent or $\gamma_r(G)$ is infinite, then $\gamma_r(G) = K_r(G)$.

It should be mentioned here that, as shown in [29], the assumption in the above theorem that G is nilpotent can be replaced by the weaker condition that $\gamma_\infty(G)$ is finite and cyclic of p-power order, where

$$\gamma_\infty(G) = \bigcap_{i=2}^{\infty} \gamma_i(G).$$

The next theorem establishes that Rodney's results [62] (see Theorem 2.2) can only be extended in a limited way to higher terms of the lower central series. It is obvious from Theorem 8.4 that for cyclic and central $\gamma_r(G)$ we have $\gamma_r(G) = K_r(G)$ for all r. However, as the next theorem shows, this cannot be extended to the case that $d(\gamma_r(G)) \geq 2$, if $r \geq 3$. (Recall that for a group G we denote the minimal number of generators of G by $d(G)$.)

Theorem 8.5 ([10]) For every integer $r \geq 3$ there is a metabelian group G such that $\gamma_{r+1}(G) = 1$, $d(\gamma_r(G)) = 2$, and $\gamma_r(G) \neq K_r(G)$.

If $|\gamma_r(G)|$ is finite and central, then (i) of Theorem 2.2 can be extended to $d(\gamma_r(G)) = 2$, if $r \geq 3$. But the theorem is not true if $d(\gamma_r(G)) = 3$, in particular (ii) of Theorem 2.2 does not hold, if $r > 2$.

Theorem 8.6 ([10]) If $\gamma_{r+1}(G) = 1$ and $\gamma_r(G)$ is finite with $d(\gamma_r(G)) = 2$, then $\gamma_r(G) = K_r(G)$. However, for every integer $r \geq 3$, and every prime p, there is a metabelian group G such that $\gamma_{r+1}(G) = 1$, $\gamma_r(G)$ is an elementary abelian p-group of rank 3, and $\gamma_r(G) \neq K_r(G)$.

Guralnick in [29] quantifies the results for $r \geq 3$ of [10] in the same way as this is done for $r = 2$ in [20] and [27] for Macdonald's results in [45]. The main result of [29] is the following.

Theorem 8.7 ([29]) Suppose $r \geq 3$. There exists a group G with $\gamma_r(G)$ cyclic of order n and $\lambda_r(G) > k$ if and only if $n = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$, where the p_i are distinct primes and $m \geq 2^{k+1} - 1$.

The conditions for the case $r = 2$, discussed in [27], are much more complicated than those of the above theorem for $r > 2$ (see Theorem 7.8). As shown in [20], a generating commutator for $\gamma_2(G)$ does not imply that $\lambda_2(G) = 1$ (see Theorem 7.5). Similar examples can be constructed for $r > 2$. The following result is of interest in this context.

Theorem 8.8 ([29]) For any $r \geq 2$, if $\gamma_r(G) = \langle a \rangle$ and $a \in (K_r(G))^e$, then $\gamma_r(G) = (K_r(G))^{e+1}$.

Guralnick in [30] extends various results on Sylow subgroups of the commutator subgroup to the Sylow subgroups S of higher terms of the lower central series. The main result is summarized in the following theorem.

Theorem 8.9 ([30]) Suppose $\gamma_r(G)$ is finite and $P \in \mathrm{Syl}_p(\gamma_r(G))$ with P abelian of rank at most 2. If any of the following conditions hold then $P \subseteq K_r(G)$:

  (i) $p \geq 5$;

 (ii) P is cyclic;

(iii) $\exp(P) = p$;

(iv) $P \cap \gamma_\infty(G) \neq 1$;

 (v) $P \cap \gamma_{r+1}(G) = 1$;

(vi) $r \leq 2$.

The result for $r = 2$ can be found in [62] and [28] (see Theorems 2.3 and 2.4). The main idea of the proof of Theorem 8.9 is to reduce to the case where $P = \gamma_r(G)$. With this additional hypothesis, the proofs of (iii) and (iv) are given in [29], while the proofs for (ii) and (v) can be found in [10]. The condition (i) is new here. An example is given in [30] showing that the condition $p \geq 5$ cannot be replaced by $p = 2$, and possibly not by $p = 3$. Examples in [10] and [29] show that rank 2 cannot be replaced by rank 3. Guralnick's paper concludes with some results on the more general problem of when $P \subseteq (K_r(G))^k$.

9   Ore's conjecture

After proving that every element in the alternating group A_n, $n \geq 5$, is a commutator, Ore [57] states the following: "It is possible that a similar theorem holds for any simple group of finite order, but it seems that at present we do not have the necessary methods to investigate the question." Now over fifty years later, the question is still not answered, no counterexample has been found, but for most families of finite simple groups what is now called Ore's Conjecture has been verified. The open cases are finite simple groups of Lie type over small fields. In this section we give a full account of which cases are settled and which are still open.

Although the classification of finite simple groups did not result in a general method for proving his conjecture, as Ore had hoped, it led to a better understanding of finite simple groups, allowing for a piecemeal approach and the potential to know that at some point all cases have been covered. Investigations of finite simple groups led to the following stronger conjecture attributed to John Thompson (see e.g. [1] and [13]), which states that every finite simple group G contains a conjugacy class C such that $C^2 = G$. Thompson's Conjecture implies Ore's Conjecture but the converse does not hold. To see that Ore's Conjecture is weaker, note that the infinite restricted alternating group A has no class C such that $C^2 = A$. But every element of A has finite support and thus is clearly a commutator [4]. The status of Thompson's Conjecture is the same as that of Ore's Conjecture.

The fact that every element in the alternating group on five or more letters is a commutator seems to be one of the most rediscovered and republished results in this area. As already mentioned in the introduction, G. A. Miller [54] proved this result in 1899. Ito [37] published it simultaneously with Ore in 1951. Yet another proof can be found in [73]. Hsü (Xu) in [75] proved Thompson's Conjecture for the alternating groups.

There are some early results for certain sporadic simple groups, for example, the Ore conjecture is verified for the Mathieu groups in [73]. As announced in [55], Neubüser et al. verified Thompson's (and consequently Ore's) Conjecture for all sporadic simple groups using computer aided calculations. A year after [55], the Ore Conjecture was verified in [1] for the sporadic groups using classical methods.

This leaves the finite simple groups of Lie type to be discussed. In our account we use the notation found in [21]. This differs slightly from the one used in [13] with regards to norming of the parameter q. This difference is only an issue when we have to identify cases for which Ore's Conjecture is open. Fortunately, for those families where the notations differ, there are no open cases and so we do not have to address this issue.

R.C. Thompson in [69], [70], and [71] proves Ore's Conjecture for the entire family $A_n(q)$, $n \geq 1$. Tseng and Hsü in [72] do the same for the Suzuki groups, that is the entire class $^2B_2(q)$, $q = 2^{2m+1}$.

There are various partial solutions of the Thompson Conjecture for some families of finite simple groups that have since been overridden by more general results in [13]. In a recent paper Gow derives sufficient conditions in terms of character theory that a simple group of Lie type must satisfy so that the Ore conjecture holds [22].

We should mention here that with the help of computer calculations Karni verified Thompson's Conjecture for all finite simple groups of order less than 10^6 [1].

In his dissertation Bonten [3] proves the following interesting result, which gives an asymptotic solution to Ore's Conjecture: Let $G(q) = X_n(q)$, $^l X_n(q)$ be a series of groups of Lie type. Then there exists a constant q_0, depending on n and l, such that every element in $G(q)$ is a commutator if $q > q_0$. In [3] only the existence of such numbers q_0 is proved, but the methods allowed Bonten to calculate an estimate for q_0 in some cases. These were good enough for groups of small Lie rank so that, together with computer calculations for small q, Bonten was able to prove Ore's Conjecture for the following families of finite simple groups of Lie type: $G_2(q)$, $^2G_2(q)$, $^3D_4(q)$, $F_4(q)$ and $^2F_4(q)$.

The most far reaching results on Ore's Conjecture to date were obtained by Ellers and Gordeev in [13]. They prove Thompson's Conjecture for all finite simple groups of Lie type over fields with more than 8 elements. In fact, the result is somewhat stronger than this, since for most of the families of these groups, field sizes smaller than 8 suffice as the lower bound for establishing the validity of the conjecture. The details can be found in Table 1, which gives the current status of Ore's Conjecture for finite simple groups of Lie type.

Name	Verified cases	References	Open cases
$A_n(q)$, $n \geq 1$	all cases	[69], [70], [71]	none
$B_2(q)$	all cases	[13], [1]	none
$B_n(q)$, $n \geq 3$	$q \geq 7$	[13]	$q = 2, 3, 4, 5$
$C_n(q)$, $n > 2$	$q \geq 4$	[13]	$q = 2, 3$
$D_{2n}(q)$, $n \geq 2$	$q \geq 5$	[13]	$q = 2, 3, 4$
$D_{2n+1}(q)$, $n \geq 2$	$q \geq 4$	[13]	$q = 2, 3$
$G_2(q)$	all cases	[3]	none
$F_4(q)$	all cases	[3]	none
$E_6(q)$	$q \geq 7$	[13]	$q = 2, 3, 4, 5$
$E_7(q)$	$q \geq 5$	[13]	$q = 2, 3, 4$
$E_8(q)$	$q \geq 7$	[13]	$q = 2, 3, 4, 5$
$^2A_{2l-1}(q)$, $l > 1$	$q \geq 8$	[13]	$q = 2, 3, 4, 5, 7$
$^2A_{2l}(q)$, $l \geq 1$	$q \geq 4$	[13]	$q = 2, 3$
$^2B_2(q)$, $q = 2^{2m+1}$	all cases	[73]	none
$^2D_n(q)$, $n > 3$	$q \geq 7$	[13]	$q = 2, 3, 4, 5$
$^3D_4(q)$	all cases	[3]	none
$^2G_2(q)$	all cases	[3]	none
$^2F_4(q)$	all cases	[3]	none
$^2E_6(q)$	$q \geq 8$	[13]	$q = 2, 3, 4, 5, 7$

Table 1. Status of Ore's Conjecture for finite simple groups of Lie Type

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