On the reversal bias of the Minimax social choice correspondence*

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October 27, 2017

Abstract

We introduce three different qualifications of the reversal bias in the framework of social choice correspondences. For each of them, we prove that the Minimax social choice correspondence is immune to it if and only if the number of voters and the number of alternatives satisfy suitable arithmetical conditions. We prove those facts thanks to a new characterization of the Minimax social choice correspondence and using a graph theoretical approach. We discuss the same issue for the Borda and Copeland social choice correspondences.

Keywords: reversal bias; Minimax social choice correspondence; directed graphs.

JEL classification: D71.
MSC classification: 05C20.

1 Introduction

Consider a committee having $h \geq 2$ members who have to select one or more elements within a set of $n \geq 2$ alternatives. Usually, the procedure used to make that choice only depends on committee members' preferences on alternatives. We assume that preferences of committee members are expressed as strict rankings (linear orders) on the set of alternatives, and call preference profile any list of h preferences, each of them associated with one of the individuals in the committee. Thus, a procedure to choose, whatever individual preferences are, one or more alternatives as social outcome can be represented by a social choice correspondence (SCC), that is, a function from the set of preference profiles to the set of nonempty subsets of the set of alternatives.

*We are grateful to two anonymous referees and an anonymous associate editor for providing useful suggestions for highly improving the readability of the paper. Daniela Bubboloni was partially supported by GNSAGA of INdAM.

The assessment of different SCCs and their comparison is usually based on which properties, among the ones considered desirable or undesirable under a social choice viewpoint, those SCCs fulfil. Moving from the ideas originally proposed by Saari (1994) and then deepened by Saari and Barney (2003), we focus here on a quite unpleasant property that a SCC may meet and that, in our opinion, hasn't deserved the right attention yet.

In order to describe such a property, recall that the reversal of a preference profile is the preference profile obtained by it assuming a complete change in each committee member's mind about her own ranking of alternatives (that is, the best alternative gets the worst, the second best alternative gets the second worst, and so on). Assume now that a given SCC associates with a certain preference profile a singleton, that is, it selects a unique alternative. If we next consider the outcome determined by the reversal of the considered preference profile, we would expect to have something different from the previous singleton as it seems natural to demand a certain degree of difference between the outcomes associated with a preference profile and its reversal. As suggested by Saari and Barney (2003, p.17),

> suppose after the winner of an important departmental election was announced, it was discovered that everyone misunderstood the chair's instructions. When ranking the three candidates, everyone listed his top, middle, and bottom-ranked candidate in the natural order first, second, and third. For reasons only the chair understood, he expected the voters to vote in the opposite way. As such, when tallying the ballots, he treated a first and last listed candidate, respectively, as the voter's last and first choice. Imagine the outcry if after retallying the ballots the chair reported that [...] the same person won.

In other words, common sense suggests that we should express doubts about the quality of a SCC which associates the same singleton both with a preference profile and with its reversal, that is, which suffers what we are going to call the reversal bias.

Among the classical SCCs, such a bias is experienced by the Minimax SCC, also known as Simpson-Kramer or Condorcet SCC, that is, the SCC which selects those alternatives whose greatest pairwise defeat is minimum. Indeed, assume that a committee having six members ($h = 6$) has to select some alternatives within a set of four alternatives denoted by 1, 2, 3 and 4 ($n = 4$). Consider then a preference profile represented by the matrix

$$\begin{bmatrix} 4 & 4 & 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 1 & 2 & 4 & 4 & 4 \end{bmatrix}$$

where, for every $i \in \{1, 2, 3, 4, 5, 6\}$, the i-th column represents the i-th member's preferences according to the rule that the higher the alternative is, the better it is. A simple check shows that the Minimax SCC associates both with that preference profile and with its reversal the same set $\{4\}$. On the other hand, if we consider two alternatives only, then the Minimax SCC agrees with the simple majority and it is immediate to verify that it is immune to the reversal bias whatever the number of committee members is.

For such a reason, we address the problem of finding conditions on the number of individuals and on the number of alternatives that make the Minimax SCC immune to the reversal bias. Our main result[1] is the following theorem.

Theorem A. *The Minimax* SCC *is immune to the reversal bias if and only if* $h \leq 3$ *or* $n \leq 3$ *or* $(h, n) \in \{(4, 4), (5, 4), (7, 4), (5, 5)\}$.

Theorem A shows, in particular, that the Minimax SCC does no exhibit the reversal bias not only when there are two alternatives but also in other cases. Remarkably, that property holds true

[1] Theorem A is a rephrase of Theorem 2 for $j = 1$.

when alternatives are three, independently on the number of individuals, and when individuals are three, independently on the number of alternatives.

The proof of Theorem A requires a certain amount of work and the use of language and methods taken from graph theory[2]. Indeed, standard social choice theoretical arguments naturally allow one to prove that, for lots of pairs (h, n), the Minimax SCC suffers the reversal bias[3]. On the other hand, except for the trivial case $n = 2$, they turn out to be difficult to apply to prove that, for the remaining pairs, the Minimax SCC is immune to the reversal bias. In particular, no simple intuition indicates how to treat the cases $(h, n) \in \{(4, 4), (5, 4), (7, 4), (5, 5)\}$. For such a reason, we first propose a new characterization of the Minimax SCC showing that, for every preference profile, an alternative x is selected by the Minimax SCC if and only if, for every majority threshold μ not exceeding the number of individuals but exceeding half of it, if there is an alternative which is preferred by at least μ individuals to x, then, for every alternative, there is another one which is preferred by at least μ individuals to it (Proposition 1). We then associate with each preference profile p and each majority threshold μ a directed graph $\Gamma_\mu(p)$, called a majority graph, whose vertices are the alternatives and whose arcs are the μ-majority relations among alternatives (Section 5.2). By the analysis of connection and acyclicity properties of those graphs, we find out a general and unified method to approach the proof of Theorem A. That allows, in particular, to avoid the repetition of similar arguments and the discussion of very long lists of cases and subcases. The geometric representation of the graph $\Gamma_\mu(p)$ is also a useful mental guidance in the tricky steps needed to carry on such an analysis as well as the proof of Theorem A. We emphasise that the results related to graph theory deal with quite general majority issues so that they are not limited, in their meaning, to the specific problem considered in the paper. We are confident that those results could be a smart tool to manage, in the future, many other problems.

We also introduce two weaker versions of reversal bias. Namely, we say that a SCC suffers the reversal bias of type 2 if there exists a preference profile such that the outcomes associated with it and its reversal are not disjoint and one of the two is a singleton; we say instead that a SCC suffers the reversal bias of type 3 if there exists a preference profile such that the outcomes associated with it and its reversal are not disjoint and none of the two is the whole set of the alternatives. It is immediate to observe that the reversal bias (also called reversal bias of type 1) implies the reversal bias of type 2 which in turn implies the reversal bias of type 3. Using the same tools and techniques used to prove Theorem A, we get the following results[4].

Theorem B. *The Minimax* SCC *is immune to the reversal bias of type 2 if and only if* $h = 2$ *or* $n \leq 3$ *or* $(h, n) = (4, 4)$.

Theorem C. *The Minimax* SCC *is immune to the reversal bias of type 3 if and only if* $n = 2$ *or* $(h, n) = (3, 3)$.

We emphasize that there is an interesting link between the different qualifications of reversal bias above described and the concept of Condorcet loser. Indeed, let C be a SCC satisfying the Condorcet principle, that is, always selecting the Condorcet winner as unique outcome when it exists. If C is immune to the reversal bias of type 1, then it never selects the Condorcet loser as the unique outcome, that is, C fulfils the weak Condorcet loser property; if C is immune to the reversal bias of type 2, then it never selects the Condorcet loser, that is, C fulfils the Condorcet loser property. Thus, since the Minimax SCC satisfies the Condorcet principle, Theorems A and B provide, in particular, conditions on (h, n) that are sufficient to make the Minimax SCC satisfy the weak Condorcet loser property and the Condorcet loser property, respectively. Certainly, as it is not known whether such conditions are also necessary, determining all the pairs (h, n) making the Minimax SCC satisfy those properties is an interesting problem which, in our opinion, can be

[2]Note that the use of graphs in social choice theory is well established (see, for instance, Laslier (1997)).
[3]See Propositions 23 and 24 and related comments.
[4]Theorems B and C are rephrases of Theorem 2 for $j = 2$ and $j = 3$, respectively.

fruitfully attacked using the methods described in this paper. Finally note that, given a SCC C always selecting the Condorcet winner (not necessarily as the unique outcome) when it exists, we have that if C is immune to the reversal bias of type 2, then it fulfils the weak Condorcet loser property; if C is immune to the reversal bias of type 3, then it never selects the Condorcet loser when the set of outcomes is different from the whole set of alternatives.

Observe now that, even though the main concepts of our paper are mainly inspired by the ideas of Saari and Barney (2003), the framework we consider, as well as the terminology we use, is different from the one they used. Indeed, they deal with election methods, namely, functions from the set of finite sequences of individual preferences (still called preference profiles) to the set of complete and transitive relations on the set of alternatives. In that framework, they say that an election method suffers the reversal bias if it associates the same relation with a preference profile and its reversal, provided that such a relation is not a complete tie, so that in their paper the expression reversal bias is used with a different meaning. For every $k \leq n-1$, they also introduce the concept of k-winner reversal bias (called top-winner bias when $k=1$), the phenomenon that occurs when an election method associates with a preference profile and its reversal two relations having the same k top ranked alternatives[5]. Anyway, despite the differences, it is obvious that any result of theirs about the top-winner reversal bias of a certain election method implies some information about the reversal bias of type 1 for the SCC generated by that method restricting its domain to those sequences of individual preferences having h terms and looking only at those alternatives that are top ranked. On the other hand, it is clear that none of their theorems implies a result about the reversal biases of type 2 and 3 as an immediate by-product. In particular, from Theorem 8 in Saari and Barney (2003), we deduce that the Borda and Copeland SCCs are immune to the reversal bias of type 1, but nothing can be deduced about the other types of reversal bias. That makes interesting the following result[6].

Theorem D. *The Borda and Copeland* SCC*s are immune to the reversal bias of type 3.*

We conclude with an observation. Recall that a positional method is an election method where each time an alternative is ranked k-th by one individual it obtains w_k points and alternatives are then ranked according to the final score they get; the vector $w = (w_k)_{k=1}^n \in \mathbb{R}^n$ associated with the method is called its voting vector and is assumed to satisfy $w_1 \geq w_2 \geq \ldots \geq w_n$ and $w_1 > w_n$. Assume now that $n \geq 3$ and consider a voting vector w such that there exist $k_1, k_2 \in \{1, \ldots, n\}$ with $w_{k_1} + w_{n-k_1} \neq w_{k_2} + w_{n-k_2}$. Then, from Theorem 1 in Saari and Barney (2003), we deduce that the SCC generated by the positional method associated with w suffers the reversal bias of type 1, provided that the number h of individuals is large enough. That fact is remarkable because it implies the existence of many SCCs different from the Minimax SCC, like plurality and anti-plurality SCCs, which suffer that bias. Certainly, as the considered theorem gives no information about the exact values of h for which the reversal bias of type 1 really occurs, finding those values of h is an interesting issue that deserves to be carefully investigated. More generally, given any classical SCC C and any social choice property, one can consider the problem of determining conditions on the number of individuals and alternatives which are necessary and sufficient to make C fulfil the property. We believe that investigating those problems is an interesting and promising research project since, as particularly shown by our results on the reversal bias, comparing different SCCs on the basis of their properties cannot ignore how many individuals and alternatives are involved in the decision process.

[5]Saari (1994) introduces for election methods another interesting concept, called reversal symmetry, which is related to the ones now discussed. Namely, an election method is said to be reversal symmetric if the outcomes associated with any preference profile and its reversal are one the reversal of the other. Of course, if an election method is reversal symmetric it cannot suffer either the reversal bias or the k-winner reversal bias. Reversal symmetry has been recently studied by Llamazares and Peña (2015) for positional methods and by Bubboloni and Gori (2015) for social welfare functions with values in the set of linear orders.

[6]Theorem D is a rephrase of Proposition 3.

2 Preliminary definitions

Let $\mathbb{N}_\diamond = \{a \in \mathbb{N} : a \geq 2\}$. From now on, let $n, h \in \mathbb{N}_\diamond$ be fixed, and let $N = \{1, \ldots, n\}$ be the set of alternatives and $H = \{1, \ldots, h\}$ be the set of individuals.

A *preference relation* on N is a linear order on N, that is, a complete, transitive and antisymmetric binary relation on N. The set of linear orders on N is denoted by $\mathcal{L}(N)$. Let $q \in \mathcal{L}(N)$ be fixed. Given $x, y \in N$, we usually write $x \geq_q y$ instead of $(x, y) \in q$, and $x >_q y$ instead of $(x, y) \in q$ and $x \neq y$. The function $\mathrm{rank}_q : N \to \{1, \ldots, n\}$ defined, for every $x \in N$, by $\mathrm{rank}_q(x) = |\{y \in N : y >_q x\}| + 1$, is bijective. We identify q with the function rank_q^{-1} and denote it still by q. We also identify q with the column vector $[q(1), \ldots, q(n)]^T$. Moreover, we define q^r as the element in $\mathcal{L}(N)$ such that, for every $x, y \in N$, $(x, y) \in q^r$ if and only if $(y, x) \in q$. Of course, $(q^r)^r = q$. For instance, let $n = 3$ and $q \in \mathcal{L}(N)$ be such that $2 >_q 1 >_q 3$. Then $q(1) = 2$, $q(2) = 1$, $q(3) = 3$ and we identify q with $[2, 1, 3]^T$ and q^r with $[3, 1, 2]^T$.

A *preference profile* is an element of $\mathcal{L}(N)^h$. The set $\mathcal{L}(N)^h$ is denoted by $\mathcal{P}$. Let $p \in \mathcal{P}$ be fixed. Given $i \in H$, the i-th component of p is denoted by p_i and represents the preferences of individual i. The preference profile p can be naturally identified with the matrix whose i-th column is $[p_i(1), \ldots, p_i(n)]^T$. Define $p^r \in \mathcal{P}$ as the preference profile such that, for every $i \in H$, $(p^r)_i = (p_i)^r$. Of course, $(p^r)^r = p$. We will write the i-th component of p^r simply as p_i^r, instead of $(p^r)_i$. Given $\mu \in \mathbb{N} \cap (h/2, h]$ and $x, y \in N$, we write $x >_\mu^p y$ if $|\{i \in H : x >_{p_i} y\}| \geq \mu$. Note that $x >_\mu^p y$ if and only if $y >_\mu^{p^r} x$. Elements in $\mathbb{N} \cap (h/2, h]$ are called majority thresholds. We call *minimal majority threshold* the integer $\mu_0 = \lceil \frac{h+1}{2} \rceil$. Further details about preference relations and preference profiles can be found in Bubboloni and Gori (2015).

A *social choice correspondence* (SCC) is a function from $\mathcal{P}$ to the set of the nonempty subsets of N. The set of SCCs is denoted by $\mathfrak{C}$. Let $C \in \mathfrak{C}$. We say that C suffers the *reversal bias (of type 1)* if, there exists $p \in \mathcal{P}$ and $x \in N$ such that

$$C(p) = C(p^r) = \{x\};$$

the *reversal bias of type 2* if, there exists $p \in \mathcal{P}$ such that

$$|C(p)| = 1 \text{ and } C(p) \cap C(p^r) \neq \varnothing;$$

the *reversal bias of type 3* if, there exists $p \in \mathcal{P}$ such that

$$|C(p)| < n \text{ and } C(p) \cap C(p^r) \neq \varnothing.$$

Clearly if C suffers the reversal bias of type 1, then C suffers also the reversal bias of type 2, and if C suffers the reversal bias of type 2 then C suffers also the reversal bias of type 3. For every $j \in \{1, 2, 3\}$, we say that C is immune to the reversal bias of type j if C does not suffer the reversal bias of type j.

We define, for every $j \in \{1, 2, 3\}$, the sets

$$\mathfrak{C}^j = \{C \in \mathfrak{C} : C \text{ is immune to the reversal bias of type } j\}.$$

Note that $\mathfrak{C}^3 \subseteq \mathfrak{C}^2 \subseteq \mathfrak{C}^1$.

3 The Minimax SCC

In this section we focus on the Minimax SCC, denoted by M and defined, for every $p \in \mathcal{P}$, by[7]

$$M(p) = \underset{x \in N}{\operatorname{argmin}} \max_{y \in N \setminus \{x\}} |\{i \in H : y >_{p_i} x\}|.$$

According to the above definition, $M(p)$ is then the set of those alternatives which minimize the greatest pairwise defeat, with respect to the individual preferences described by p. However, the outcomes of the Minimax SCC admit an alternative interpretation in terms of majority thresholds.

Given $\mu \in \mathbb{N} \cap (h/2, h]$, define, for every $p \in \mathcal{P}$, the set

$$D_\mu(p) = \{x \in N : \forall y \in N,\, |\{i \in H : y >_{p_i} x\}| < \mu\}.$$

Thus, an alternative x belongs to $D_\mu(p)$ if and only if it cannot be found another alternative which is preferred to x by at least μ individuals, according to the preference profile p. Note that the set $D_\mu(p)$ corresponds to the set of μ-majority equilibria associated with p as defined by Greenberg (1979) in the more general setting where individual preferences are represented via complete and transitive relations. Observe that if $\mu \leq \mu'$, then $D_\mu(p) \subseteq D_{\mu'}(p)$ for all $p \in \mathcal{P}$. Moreover, as an immediate consequence of Corollary 3 in Greenberg (1979) and its proof, for every $\mu \in \mathbb{N} \cap (h/2, h]$, we have that

$$D_\mu(p) \neq \varnothing \text{ for all } p \in \mathcal{P} \text{ if and only if } \mu > \frac{n-1}{n}h. \tag{1}$$

Since $h \in \mathbb{N} \cap (h/2, h]$ and $h > \frac{n-1}{n}h$, it is well defined the *Greenberg majority threshold* given by

$$\mu_G = \min\left\{m \in \mathbb{N} \cap (h/2, h] : m > \frac{n-1}{n}h\right\}.$$

For every $p \in \mathcal{P}$, we consider the integer

$$\mu(p) = \min\{\mu \in \mathbb{N} \cap (h/2, h] : D_\mu(p) \neq \varnothing\}.$$

Note that, since (1) implies $D_{\mu_G}(p) \neq \varnothing$, we have that $\mu(p)$ is well defined and $\mu_0 \leq \mu(p) \leq \mu_G$.

We can now prove the following proposition.

Proposition 1. *For every $p \in \mathcal{P}$, $M(p) = D_{\mu(p)}(p)$.*

Proof. We show first that $M(p) \subseteq D_{\mu(p)}(p)$ proving that $N \setminus D_{\mu(p)}(p) \subseteq N \setminus M(p)$. Let $x_0 \in N \setminus D_{\mu(p)}(p)$. Then there exists $y_0 \in N \setminus \{x_0\}$ such that $|\{i \in H : y_0 >_{p_i} x_0\}| \geq \mu(p)$. Picking now $x_1 \in D_{\mu(p)}(p)$, we have that, for every $y \in N \setminus \{x_1\}$, $|\{i \in H : y >_{p_i} x_1\}| \leq \mu(p) - 1$. Thus,

$$\max_{y \in N \setminus \{x_1\}} |\{i \in H : y >_{p_i} x_1\}| \leq \mu(p) - 1 < |\{i \in H : y_0 >_{p_i} x_0\}| \leq \max_{y \in N \setminus \{x_0\}} |\{i \in H : y >_{p_i} x_0\}|,$$

which says $x_0 \notin M(p)$.

We next show that $D_{\mu(p)}(p) \subseteq M(p)$. Let $x_0 \in D_{\mu(p)}(p)$. Then, we have that

$$\max_{y \in N \setminus \{x_0\}} |\{i \in H : y >_{p_i} x_0\}| \leq \mu(p) - 1.$$

Assume, by contradiction, that there exists $x_1 \in N$ such that

$$\max_{y \in N \setminus \{x_1\}} |\{i \in H : y >_{p_i} x_1\}| < \max_{y \in N \setminus \{x_0\}} |\{i \in H : y >_{p_i} x_0\}|.$$

[7] Fishburn (1977) presents the equivalent definition

$$M(p) = \underset{x \in N}{\operatorname{argmax}} \min_{y \in N \setminus \{x\}} |\{i \in H : x >_{p_i} y\}|.$$

Then $x_0 \neq x_1$ and, for every $y \in N \setminus \{x_1\}$, we have $|\{i \in H : y >_{p_i} x_1\}| < \mu(p) - 1$. If $\mu(p)-1 > h/2$, that says $x_1 \in D_{\mu(p)-1}(p) = \varnothing$ and the contradiction is found. Assume instead that $\mu(p) - 1 \leq h/2$. Then $\mu(p) = \mu_0 = \lceil \frac{h+1}{2} \rceil \leq \frac{h+2}{2}$ and since $|\{i \in H : x_0 >_{p_i} x_1\}| \leq \mu_0 - 2$, we get $|\{i \in H : x_1 >_{p_i} x_0\}| \geq h - \mu_0 + 2$. Now we observe that, due to $\mu_0 \leq \frac{h+2}{2}$, we have $h - \mu_0 + 2 \geq \mu_0$, against $x_0 \in D_{\mu_0}(p)$. $\square$

Define the sets

$$T_1 = \{(h,n) \in \mathbb{N}^2_\diamond : h \leq 3\} \cup \{(h,n) \in \mathbb{N}^2_\diamond : n \leq 3\} \cup \{(4,4),(5,4),(7,4),(5,5)\},$$

$$T_2 = \{(h,n) \in \mathbb{N}^2_\diamond : h = 2\} \cup \{(h,n) \in \mathbb{N}^2_\diamond : n \leq 3\} \cup \{(4,4)\},$$

$$T_3 = \{(h,n) \in \mathbb{N}^2_\diamond : n = 2\} \cup \{(3,3)\}.$$

and note that $T_3 \subsetneq T_2 \subsetneq T_1$. We can now state the main result of the paper. Its proof is technical and will be presented in Section 5. We stress that it relies on Proposition 1 and the use of language and methods of graph theory (Sections 5.1 and 5.2).

Theorem 2. *Let $j \in \{1,2,3\}$. Then, $M \in \mathfrak{C}^j$ if and only if $(h,n) \in T_j$.*

4 The Borda and Copeland SCCs

In this section we show that, as distinguished from the case of the Minimax SCC, the analysis of the reversal bias is easy for the Borda and Copeland SCCs. Those SCCs are respectively denoted by *Bor* and *Cop*, and defined[8], for every $p \in \mathcal{P}$, as

$$Bor(p) = \underset{x \in N}{\operatorname{argmax}} \textstyle\sum_{i=1}^h \big(n - \operatorname{rank}_{p_i}(x)\big),$$

$$Cop(p) = \underset{x \in N}{\operatorname{argmax}} \big(|\{y \in N : x >^p_{\mu_0} y\}| - |\{y \in N : y >^p_{\mu_0} x\}|\big).$$

The following results show that they are immune to the reversal bias of type 3.

Proposition 3. *Bor, Cop* $\in \mathfrak{C}^3$.

Proof. We start considering the Borda SCC. We need to show that, for every $p \in \mathcal{P}$, $Bor(p) \cap Bor(p^r) \neq \varnothing$ implies $Bor(p) = N$. Fix then $p \in \mathcal{P}$ and $x_0 \in N$ such that $x_0 \in Bor(p) \cap Bor(p^r)$. Let f, g and u be the functions from N to $\mathbb{R}$ defined, for every $x \in N$, by

$$f(x) = \sum_{i=1}^h \big(n - \operatorname{rank}_{p_i}(x)\big), \quad g(x) = \sum_{i=1}^h \big(n - \operatorname{rank}_{p_i^r}(x)\big), \quad u(x) = \sum_{i=1}^h \operatorname{rank}_{p_i}(x).$$

Note that

$$Bor(p) = \underset{x \in N}{\operatorname{argmax}}\, f(x), \quad Bor(p^r) = \underset{x \in N}{\operatorname{argmax}}\, g(x),$$

and that, for every $x \in N$, $f(x) = hn - u(x)$ and $g(x) = u(x) - h$, due to the fact that $\operatorname{rank}_{p^r}(x) = n + 1 - \operatorname{rank}_p(x)$. Then x_0 realises both the minimum and the maximum of u, so that u is constant. It follows that f is constant too and therefore $Bor(p) = N$.

[8]With Borda SCC we mean the well-known Borda count. The definition of the Copeland SCC can be found, for instance, in Fishburn (1977).

We next consider the Copeland SCC. We need to show that, for every $p \in \mathcal{P}$, $Cop(p) \cap Cop(p^r) \neq \varnothing$ implies $Cop(p) = N$. Fix then $p \in \mathcal{P}$ and $x_0 \in N$ such that $x_0 \in Cop(p) \cap Cop(p^r)$. Let f and g be functions from N to $\mathbb{R}$ defined, for every $x \in N$, by

$$f(x) = |\{y \in N : x >^p_{\mu_0} y\}| - |\{y \in N : y >^p_{\mu_0} x\}|, \quad g(x) = |\{y \in N : x >^{p^r}_{\mu_0} y\}| - |\{y \in N : y >^{p^r}_{\mu_0} x\}|.$$

Note that

$$Cop(p) = \operatorname*{argmax}_{x \in N} f(x), \quad Cop(p^r) = \operatorname*{argmax}_{x \in N} g(x).$$

Moreover, since $x >_{p_i^r} y$ is equivalent to $y >_{p_i} x$ for all $x, y \in N$ and $i \in H$, we have that, for every $x \in N$, $g(x) = -f(x)$. Then x_0 realises both the minimum and the maximum of f. It follows that f is constant and therefore $Cop(p) = N$. □

The next corollaries show how the results on the reversal bias of M, Bor and Cop can be used to establish conditions on the number of individuals and alternatives that are necessary and sufficient to have the equalities $M = Bor$ and $M = Cop$.

Corollary 4. *$M = Bor$ if and only if $n = 2$.*

Proof. If $n = 2$, then we surely have $M = Bor$. Assume now that $n \geq 3$. If $(h, n) \notin T_3$, then, by Theorem 2 and Proposition 3, $M \notin \mathfrak{C}^3$ and $Bor \in \mathfrak{C}^3$ so that $M \neq Bor$. If $(h, n) \in T_3$, then $(h, n) = (3, 3)$ and we still have $M \neq Bor$ since the two SCCs differ, for instance, on the preference profile

$$p = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

□

Corollary 5. *$M = Cop$ if and only if $(h, n) \in T_3$.*

Proof. If $n = 2$, then we surely have $M = Cop$. Assume now that $n \geq 3$. If $(h, n) \notin T_3$, then, by Theorem 2 and Proposition 3, $M \notin \mathfrak{C}^3$ and $Cop \in \mathfrak{C}^3$ so that $M \neq Cop$. If $(h, n) \in T_3$, then $(h, n) = (3, 3)$. Consider $p \in \mathcal{P}$. Since both M and Cop are neutral, without loss of generality, we can assume that $p_1 = [1, 2, 3]^T$. Recalling that both M and Cop satisfy the Condorcet principle, they surely coincide when a Condorcet winner exists. Thus, we can assume that the alternatives ranked first are all different. Since both M and Cop are anonymous, we can assume that $p_2(1) = 2$ and $p_3(1) = 3$. That leaves just four possibilities for p and, by a case by case consideration, one finally proves that $M = Cop$. □

5 Proof of Theorem 2

From Proposition 1 we immediately have that Theorem 2 is implied by the following three propositions. Their tricky proofs, based on graph theory, are presented in the Sections 5.3, 5.4 and 5.5.

Proposition 6. *There exist $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$ if and only if $(h, n) \in \mathbb{N}^2_\diamond \setminus T_1$.*

Proposition 7. *There exist $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = \{x\} \subseteq D_{\mu(p^r)}(p^r)$ if and only if $(h, n) \in \mathbb{N}^2_\diamond \setminus T_2$.*

Proposition 8. *There exists $p \in \mathcal{P}$ such that $D_{\mu(p)}(p) \neq N$ and $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) \neq \varnothing$ if and only if $(h, n) \in \mathbb{N}^2_\diamond \setminus T_3$.*

5.1 Graphs

In this section, we recall some basic facts and notation from graph theory, which we are going to use in the sequel[9]. All the considered graphs are directed. A *graph* is a pair (V, A), where V is a nonempty set called vertex set and A is a subset of $\{(x, y) \in V^2 : x \neq y\}$ called arc set. Note that if $\Gamma = (V, A)$ is a graph and $|V| = 1$, then $A = \varnothing$. Given two graphs $\Gamma_1 = (V_1, A_1)$ and $\Gamma_2 = (V_2, A_2)$, we say that Γ_2 is a *subgraph* of Γ_1 if $V_2 \subseteq V_1$ and $A_2 \subseteq A_1$. If Γ_2 is a subgraph of Γ_1, we write $\Gamma_2 \leq \Gamma_1$.

Let now $\Gamma = (V, A)$ be a graph. Γ is called *complete* if for every $x, y \in V$ with $x \neq y$, we have $(x, y) \in A$ or $(y, x) \in A$. We say that $x \in V$ is *maximal* [*minimal*] for Γ if there exists no $y \in V$ such that $(y, x) \in A$ [$(x, y) \in A$]. We denote by $\max(\Gamma)$ [$\min(\Gamma)$] the set of maximal [minimal] vertices for Γ. Note that those sets may be empty. We say that $x \in V$ is a *maximum* [*minimum*] of Γ if, for every $y \in V \setminus \{x\}$, we have that $(x, y) \in A$ [$(y, x) \in A$][10]. We denote by $\mathrm{Max}(\Gamma)$ [$\mathrm{Min}(\Gamma)$] the set of maxima [minima] of Γ. We say that $x \in V$ is *isolated* in Γ if, for every $y \in V \setminus \{x\}$, $(x, y), (y, x) \notin A$. We denote by $\mathrm{I}(\Gamma)$ the set of the isolated vertices of Γ. It is useful to note that

$$\max(\Gamma) \cap \min(\Gamma) = \mathrm{I}(\Gamma). \tag{2}$$

Note also that if $x \in \mathrm{Max}(\Gamma) \cup \mathrm{Min}(\Gamma)$ and $|V| \geq 2$, then $x \notin \mathrm{I}(\Gamma)$.

Γ is said to be *connected* if, for every $x, y \in V$ with $x \neq y$, there exist $k \geq 2$ and an ordered sequence $x_1, \ldots, x_k$ of distinct elements of V such that $x_1 = x$, $x_k = y$, and, for every $j \in \{1, \ldots, k-1\}$, $(x_j, x_{j+1}) \in A$ or $(x_{j+1}, x_j) \in A$. Note that if Γ has a maximum [minimum], then Γ is connected. It is well known that there exist a uniquely determined $c \in \mathbb{N}$ and connected subgraphs $\Gamma_1 = (V_1, A_1), \ldots, \Gamma_c = (V_c, A_c)$ of Γ such that $\cup_{i=1}^c V_i = V$, $\cup_{i=1}^c A_i = A$, and for every $i, j \in \{1, \ldots, c\}$ with $i \neq j$, $V_i \cap V_j = A_i \cap A_j = \varnothing$. Those subgraphs $\Gamma_1, \ldots, \Gamma_c$ are called the *connected components* of Γ. They are maximal among the connected subgraphs of Γ, that is, if $\Gamma' \leq \Gamma$ is connected and $\Gamma' \geq \Gamma_i$ for some $i \in \{1, \ldots, c\}$, then $\Gamma' = \Gamma_i$. In particular, for every $i \in \{1, \ldots, c\}$, $x \in V_i$ and $y \in V \setminus V_i$ imply $(x, y), (y, x) \notin A$; $x, y \in V_i$ and $(x, y) \in A$ imply $(x, y) \in A_i$. Note that $x \in N$ is isolated in Γ if and only if the connected component of Γ containing x is $(\{x\}, \varnothing)$. Given $l \geq 2$, Γ is said to be a *l-cycle* if $|V| = l$ and there exists an ordered sequence $x_1, \ldots, x_l$ of the elements of V such that, once defined $x_{l+1} = x_1$, we have that $A = \{(x_j, x_{j+1}) \ : 1 \leq j \leq l\}$. Γ is said to be a *cycle* if it is a l-cycle for some $l \geq 2$. Fixed $l \geq 2$, Γ is said to be *l-cyclic* if there exists a l-cycle $\Gamma_1 \leq \Gamma$, and *l-acyclic* otherwise. Γ is said to be *acyclic* if it is l-acyclic for all $l \geq 2$. Note that if $|V| = 1$, then Γ is acyclic.

5.2 Majority graphs and their properties

Let $p \in \mathcal{P}$ and $\mu \in \mathbb{N} \cap (h/2, h]$. In a natural way, we associate with the relation on N given by $\Sigma_\mu(p) = \{(x, y) \in N \times N : x >_\mu^p y\}$, the graph $\Gamma_\mu(p) = (N, \Sigma_\mu(p))$, called the *$\mu$-majority graph* of p. Note that, if $\mu, \mu' \in \mathbb{N} \cap (h/2, h]$ with $\mu' \leq \mu$, then $\Gamma_\mu(p) \leq \Gamma_{\mu'}(p)$. In particular, $\Gamma_\mu(p) \leq \Gamma_{\mu_0}(p)$ holds for all $\mu \in \mathbb{N} \cap (h/2, h]$. The concept of majority graph has been considered by many authors essentially in relation to the case when h is odd and $\mu = \mu_0 = \frac{h+1}{2}$ (see, for instance, Miller (1977)). For the purpose of our paper that case is interesting because $\Gamma_{\frac{h+1}{2}}(p)$ is complete (see Lemma 12), but we are not focussed only on that particular majority graph.

The properties of the relation $\Sigma_\mu(p)$ translates easily into graph theoretical properties for $\Gamma_\mu(p)$. Moreover, considering $\Gamma_\mu(p)$ we gain the advantage of using concepts like l-acyclicity and connectedness which typically belong to graph theory. That gives easily a better comprehension of the sets $D_\mu(p)$ and $D_{\mu(p)}(p) = M(p)$.

[9] All unexplained notation is standard. See, for instance, Diestel (2010).

[10] Note that if x is a maximum [minimum] of Γ it is not necessarily maximal [minimal] for Γ. In fact, given $\Gamma = (\{1, 2\}, \{(1, 2), (2, 1)\})$, we have that 1 and 2 are both a maximum [minimum] but none of them is maximal [minimal].

Lemma 9. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. Then $D_\mu(p) = \max(\Gamma_\mu(p)) = \min(\Gamma_\mu(p^r))$. Moreover $D_\mu(p) \cap D_\mu(p^r) = \mathrm{I}(\Gamma_\mu(p)) = \mathrm{I}(\Gamma_\mu(p^r))$.*

Proof. The equalities $D_\mu(p) = \max(\Gamma_\mu(p)) = \min(\Gamma_\mu(p^r))$ follow from the definitions of $D_\mu(p)$ and p^r. As a consequence, since $(p^r)^r = p$, we also have $D_\mu(p^r) = \max(\Gamma_\mu(p^r)) = \min(\Gamma_\mu(p))$, so that (2) completes the proof. □

Lemma 10. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. Then $\Gamma_\mu(p)$ is 2-acyclic and $\Gamma_\mu(p)$ has at most one maximum. Moreover, if $\Gamma_\mu(p)$ has a maximum $x \in N$, then $D_\mu(p) = \mathrm{Max}(\Gamma_\mu(p)) = \{x\}$.*

Proof. The 2-acyclicity follows immediately from $\mu > h/2$.

Assume, by contradiction, that there exist distinct $x, y \in \mathrm{Max}(\Gamma_\mu(p))$. Then $(x, y) \in \Sigma_\mu(p)$ and $(y, x) \in \Sigma_\mu(p)$, so that $\Gamma_1 = (\{x, y\}, \{(x, y), (y, x)\}) \leq \Gamma_\mu(p)$. Since Γ_1 is a 2-cycle, that contradicts the fact that $\Gamma_\mu(p)$ is 2-acyclic.

Assume now that there exists $x \in \mathrm{Max}(\Gamma_\mu(p))$. We must show that $x \in \max(\Gamma_\mu(p)) = D_\mu(p)$. By contradiction, let $x \notin \max(\Gamma)$. Then there is $y \in N$ such that $(y, x) \in \Sigma_\mu(p)$. Since also $(x, y) \in \Sigma_\mu(p)$, the 2-cycle $\Gamma_1 = (\{x, y\}, \{(x, y), (y, x)\})$ is a subgraph of $\Gamma_\mu(p)$ and the contradiction is found. We complete the proof simply noticing that, being x a maximum of $\Gamma_\mu(p)$, for every $y \in N \setminus \{x\}$, we have that $(x, y) \in \Sigma_\mu(p)$ so that $y \notin \max(\Gamma_\mu(p))$. □

Lemma 11. *Let $\mu \in \mathbb{N} \cap (h/2, h]$. Then $\Gamma_\mu(p)$ is acyclic for all $p \in \mathcal{P}$ if and only if $\mu \geq \mu_G$. In particular $\Gamma_h(p)$ is acyclic for all $p \in \mathcal{P}$.*

Proof. It is an immediate consequence of Proposition 6 and 7 in Bubboloni and Gori (2014). □

Lemma 12. *If h is odd, then, for every $p \in \mathcal{P}$, $\Gamma_{\mu_0}(p)$ is complete and $\mathrm{I}(\Gamma_{\mu_0}(p)) = \varnothing$. Moreover, if $D_{\mu_0}(p) \neq \varnothing$ then $\mu(p) = \mu_0$, $\Gamma_{\mu_0}(p)$ admits maximum $x \in N$ and $D_{\mu_0}(p) = \{x\}$.*

Proof. Let us fix $p \in \mathcal{P}$ and note that, being h odd, we have $\mu_0 = \frac{h+1}{2}$. Assume now, by contradiction, that there exist $x, y \in N$ with $x \not>^p_{\mu_0} y$ and $y \not>^p_{\mu_0} x$. Then, we get the impossible relation

$$h = |\{i \in H : x >_{p_i} y\}| + |\{i \in H : y >_{p_i} x\}| \leq \mu_0 - 1 + \mu_0 - 1 = 2\left(\frac{h+1}{2}\right) - 2 = h - 1.$$

Thus $\Gamma_{\mu_0}(p)$ is complete and, as an immediate consequence, $\mathrm{I}(\Gamma_{\mu_0}(p)) = \varnothing$.

In order to prove the second part, assume that $D_{\mu_0}(p) \neq \varnothing$. Then $\mu(p) \leq \mu_0$ and so $\mu(p) = \mu_0$. Next, pick $x \in D_{\mu_0}(p)$. Since $\Gamma_{\mu_0}(p)$ is complete, then we have $x >^p_{\mu_0} y$ for all $y \in N \setminus \{x\}$, that is, x is a maximum in $\Gamma_{\mu_0}(p)$. Then, by Lemma 10, $D_{\mu_0}(p) = \{x\}$. □

Let us denote by $\mathcal{C}(\Gamma_\mu(p))$ the set of the connected components of $\Gamma_\mu(p)$ and define $\mathcal{A}(\Gamma_\mu(p)) = \{\Gamma \in \mathcal{C}(\Gamma_\mu(p)) : \Gamma \text{ is acyclic}\}$. We are ready for a key proposition giving a lower bound for $|D_\mu(p)|$ and leading to some interesting consequences.

Proposition 13. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. Then*

$$D_\mu(p) = \bigcup_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \bigcup_{\Gamma \in \mathcal{A}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \mathrm{I}(\Gamma_\mu(p)),$$

and

$$|D_\mu(p)| = \sum_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} |\max(\Gamma)| \geq |\mathcal{A}(\Gamma_\mu(p))| \geq |\mathrm{I}(\Gamma_\mu(p))|.$$

Proof. Let $\Gamma = (V, A) \in \mathcal{C}(\Gamma_\mu(p))$. Then $\Gamma \leq \Gamma_\mu(p)$, so that $V \subseteq N$ and $A \subseteq \Sigma_\mu(p)$. Since Γ is a connected component of $\Gamma_\mu(p)$, we have that, for every $x \in V$ and $y \in N \setminus V$, $y \not>_\mu^p x$. This immediately gives that each $x \in \max(\Gamma)$ belongs to $D_\mu(p)$, so that $D_\mu(p) \supseteq \bigcup_{\Gamma\in\mathcal{C}(\Gamma_\mu(p))} \max(\Gamma)$. The other inclusion is trivial and thus $D_\mu(p) = \bigcup_{\Gamma\in\mathcal{C}(\Gamma_\mu(p))} \max(\Gamma)$. Since $\mathcal{A}(\Gamma_\mu(p)) \subseteq \mathcal{C}(\Gamma_\mu(p))$ and, for every $x \in \mathrm{I}(\Gamma_\mu(p))$, $(\{x\}, \varnothing) \in \mathcal{A}(\Gamma_\mu(p))$, we also get

$$\bigcup_{\Gamma\in\mathcal{C}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \bigcup_{\Gamma\in\mathcal{A}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \mathrm{I}(\Gamma_\mu(p)).$$

In particular, since there is no overlap between vertices of different connected components, we deduce $|D_\mu(p)| = \sum_{\Gamma\in\mathcal{C}(\Gamma_\mu(p))} |\max(\Gamma)|$. We complete the proof showing that for every $\Gamma \in \mathcal{A}(\Gamma_\mu(p))$, we have $\max(\Gamma) \neq \varnothing$. Pick $x_1 \in V$. If $y \not>_\mu^p x_1$ for all $y \in V$, then we have $x_1 \in \max(\Gamma)$ and we have finished. Assume instead there exists $x_2 \in V$ with $x_2 >_\mu^p x_1$. Obviously, we have $x_2 \neq x_1$. Then, repeat the argument for x_2. Since the set N is finite and Γ contains no cycle, in a finite number $k \leq n$ of steps, we obtain an element $x_k \in \max(\Gamma)$. □

Corollary 14. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\Gamma_\mu(p)$ admits at least an acyclic connected component, then $\mu(p) \leq \mu$.*

Proof. By Proposition 13, we have $|D_\mu(p)| \geq 1$, so that $D_\mu(p) \neq \varnothing$. □

Corollary 15. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\Gamma_\mu(p)$ is acyclic and $D_\mu(p)$ is a singleton, then $\Gamma_\mu(p)$ is connected.*

Proof. Since $\Gamma_\mu(p)$ is acyclic, we have that $\mathcal{C}(\Gamma_\mu(p)) = \mathcal{A}(\Gamma_\mu(p))$. Then, using Proposition 13, we get $1 = |D_\mu(p)| \geq |\mathcal{C}(\Gamma_\mu(p))| \geq 1$. That implies $|\mathcal{C}(\Gamma_\mu(p))| = 1$, that is, $\Gamma_\mu(p)$ is connected. □

Lemma 16. *Let $p \in \mathcal{P}$ such that $\mu(p^r) \leq \mu(p)$. Then:*

(i) $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) \subseteq \mathrm{I}(\Gamma_{\mu(p)}(p))$. *In particular, if $\Gamma_{\mu(p)}(p)$ is connected, then $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$.*

(ii) If $|D_{\mu(p)}(p)| = 1$ and $\Gamma_{\mu(p)}(p)$ is acyclic, then $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$.

Proof. (*i*) From $\mu(p^r) \leq \mu(p)$ we get $D_{\mu(p^r)}(p^r) \subseteq D_{\mu(p)}(p^r)$ and thus, by Lemma 9, we deduce $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) \subseteq D_{\mu(p)}(p) \cap D_{\mu(p)}(p^r) = \mathrm{I}(\Gamma_{\mu(p)}(p))$. If $\Gamma_{\mu(p)}(p)$ is connected, then $\mathrm{I}(\Gamma_{\mu(p)}(p))$ is empty and thus also $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$.

(*ii*) By Corollary 15, (i) applies giving $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$. □

Corollary 17. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\Gamma_\mu(p)$ is acyclic, then, for every $x \in N$, we do not have $D_\mu(p) = D_\mu(p^r) = \{x\}$.*

Proof. Assume by contradiction that $D_{\mu_0}(p) = D_{\mu_0}(p^r) = \{x\}$, for some $x \in N$. Then, by Lemma 9, we have that x is isolated in $\Gamma_\mu(p)$. On the other hand, by Corollary 15, $\Gamma_\mu(p)$ is connected so that its only vertex is x, against $n \geq 2$. □

Lemma 18. *Let $p \in \mathcal{P}$ and assume that both $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ admit an acyclic connected component. Then:*

(i) $\mu(p) = \mu(p^r)$.

(ii) If $\Gamma_{\mu(p)}(p)$ is connected, then $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$.

(iii) If $\Gamma_{\mu(p)}(p)$ is acyclic, then there exists no $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$.

Proof. (i) The fact that $\Gamma_{\mu(p)}(p)$ admits an acyclic connected component implies that also $\Gamma_{\mu(p)}(p^r)$ admits an acyclic connected component and therefore Corollary 14 gives $\mu(p^r) \leq \mu(p)$. The same argument applied to $\Gamma_{\mu(p^r)}(p^r)$ gives $\mu(p) \leq \mu(p^r)$.

(ii) Assume that $\Gamma_{\mu(p)}(p)$ is connected. Since, by (i), we have that $\mu(p) = \mu(p^r)$, then Lemma 16 applies, giving $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$.

(iii) Let $\Gamma_{\mu(p)}(p)$ be acyclic and assume, by contradiction, that there exists $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$. By Corollary 15, $\Gamma_{\mu(p)}(p)$ is connected and thus, by (ii), we have that $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$, a contradiction. □

Lemma 19. *If h is odd, then, for every $p \in \mathcal{P}$, $\mu(p) = \mu_0$ and $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r)$ imply $\mu(p^r) > \mu_0$.*

Proof. Let $p \in \mathcal{P}$ and assume that $\mu(p) = \mu_0$ and $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r)$. Then $D_{\mu_0}(p) \neq \varnothing$ and, using Lemma 12, $\Gamma_{\mu(p)}(p)$ has a maximum $x \in N$ and $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$. Assume by contradiction that $\mu(p^r) = \mu_0$. By Lemma 9, we get that x is isolated in $\Gamma_{\mu(p)}(p)$, against the fact that x is the maximum of $\Gamma_{\mu(p)}(p)$. □

Corollary 20. *Let $p \in \mathcal{P}$ such that $\mu(p) = \mu_0$. If $\Gamma_{\mu(p)}(p)$ is acyclic, then there exists no $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$.*

Proof. The acyclicity of $\Gamma_{\mu(p)}(p)$ implies that of $\Gamma_{\mu(p)}(p^r)$, so that, by Corollary 14, we have $\mu_0 \leq \mu(p^r) \leq \mu(p) = \mu_0$. It follows that $\mu(p^r) = \mu(p) = \mu_0$ and Corollary 17 applies. □

Due to the previous results, it is important to understand which conditions guarantee the acyclicity of $\Gamma_{\mu(p)}(p)$. By Lemma 10, we know that, for every $\mu \in \mathbb{N} \cap (h/2, h]$, $\Gamma_\mu(p)$ is 2-acyclic. Anyway, it can admit l-cycles for some $l \geq 3$. We explore this possibility through Propositions 6 and 7 in Bubboloni and Gori (2014).

Proposition 21. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $l \in \mathbb{N} \cap [2, n]$. Then there exists $p \in \mathcal{P}$ such that $\Gamma_\mu(p)$ is l-cyclic if and only if $\mu \leq \frac{l-1}{l}h$.*

Proof. Consider $\mu > \frac{l-1}{l}h$ and assume by contradiction that there exists $p \in \mathcal{P}$ and an l-cycle $\Gamma \leq \Gamma_\mu(p)$ with vertex set V. Then $V \subseteq N$ and $|V| = l \leq n$. Consider the preference profile p' on the set of l alternatives V obtained from p eliminating (if any) those entries in $N \setminus V$. By Proposition 6 in Bubboloni and Gori (2014), we have that $\Gamma_\mu(p')$ is acyclic, against the fact that $\Gamma \leq \Gamma_\mu(p')$.

Let now $\mu \leq \frac{l-1}{l}h$ and let $V \subseteq N$ with $|V| = l$. By Proposition 7 in Bubboloni and Gori (2014), there exists a preference profile p' on the set of alternatives V such that $\Gamma_\mu(p')$ contains an l-cycle Γ whose set of vertices is V. Consider a preference profile p on the set of alternatives N, in which every individual $i \in H$ ranks in the first l positions the alternatives in V as p'_i and those in $N \setminus V$ as she likes. Then $\Gamma \leq \Gamma_\mu(p)$. □

Let us consider now

$$\mu_a = \min\left\{ m \in \mathbb{N} \cap (h/2, h] : m > \frac{n-2}{n-1}h \right\},$$

and note that μ_a is well defined because $h \in \mathbb{N} \cap (h/2, h]$ and $h > \frac{n-2}{n-1}h$. Moreover, we have that $\mu_0 \leq \mu_a \leq \mu_G$ and, when $n \in \{2, 3\}$, $\mu_a = \mu_0$.

Corollary 22. *Let $p \in \mathcal{P}$. If $\mu(p) \geq \mu_a$, then $\Gamma_{\mu(p)}(p)$ is acyclic. In particular, for every $n \in \{2, 3\}$, $\Gamma_{\mu(p)}(p)$ is acyclic.*

Proof. Consider $\Gamma_{\mu(p)}(p)$. It admits no n-cycle, because having such a cycle obviously implies the contradiction $D_{\mu(p)}(p) = \varnothing$. On the other hand, by Proposition 21, it does not have l-cycles for all $l \in \{2, \ldots, n-1\}$ because $\mu(p) > \frac{n-2}{n-1}h \geq \frac{l-1}{l}h$. Finally note that, if $n \in \{2, 3\}$, then $\mu(p) \geq \mu_0 = \mu_a$. □

Due to the previous result, we call μ_a the *acyclicity threshold*.

We present now two results, namely Propositions 23 and 24 below, which are crucial for proving Proposition 6. Indeed, they allow to determine a very large set of pairs (h,n) for which there exists $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$. Even though those propositions will be proved exploiting the theory developed until now, it is interesting to note that they could be proved using a standard social choice approach by means of a suitable arithmetical reasoning applied to special preference profiles.

Consider at first $(h,n) = (11,4)$ and note that h is odd and $\mu_0 = 6$. Consider then the preference profile

$$p = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 & 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 2 & 3 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 3 & 1 & 2 & 4 & 4 & 4 & 4 & 4 \end{bmatrix} \tag{3}$$

Observe that in p the arrangement of the alternatives 1, 2 and 3 is inspired by the paradox of voting, while alternative 4 is top ranked μ_0 times and bottom ranked $\mu_0 - 1$ times. A simple computation shows that, both in p and in p^r, each alternative in the set $\{1,2,3\}$ is beaten by another alternative in the same set at least $\mu_0 + 1$ times, while 4 is beaten at most μ_0 times by any other alternative. As a consequence, $\mu(p) = \mu_0$, $\mu(p^r) = \mu_0 + 1$, and $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{4\}$. Given now (h,n) with h odd and $n \geq 4$, using the ideas underlying the proof of Proposition 7 in Bubboloni and Gori (2014), it can be easily proved that a preference profile p having the same structure as (3) is such that $D_{\mu(p)}(p)$ and $D_{\mu(p^r)}(p^r)$ are equal to the same singleton if and only if the inequality in Proposition 23 holds true.

Consider now $(h,n) = (8,4)$ and note that h is even and $\mu_0 = 5$. Consider then the preference profile

$$p = \begin{bmatrix} 4 & 4 & 4 & 4 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 & 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \end{bmatrix} \tag{4}$$

where the arrangement of the alternatives 1, 2 and 3 is still inspired by the paradox of voting, but now alternative 4 is top ranked $\mu_0 - 1$ times and bottom ranked $\mu_0 - 1$ times. A simple computation shows that $\mu(p) = \mu(p^r) = \mu_0$, and $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{4\}$. As before, given (h,n) with h even and $n \geq 4$, it can be easily proved that a preference profile p having the same structure as (4) is such that $D_{\mu(p)}(p)$ and $D_{\mu(p^r)}(p^r)$ are equal to the same singleton if and only if the inequality in Proposition 24 holds true.

Proposition 23. *If $n \geq 4$ and h is odd and such that $h \geq \frac{3(n-1)}{n-3}$, then there exists $p \in \mathcal{P}$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{n\}$.*

Proof. First of all, note that $\mu_0 = \frac{h+1}{2}$. Define then $\mu = \frac{h+3}{2} = \mu_0 + 1$ and $V = N \setminus \{n\}$. The assumption $h \geq \frac{3(n-1)}{n-3}$ is equivalent to $\mu \leq \frac{(n-1)-1}{n-1} h$ and thus, by Proposition 21, there exists p', a preference profile on the set of alternatives V, such that $\Gamma_\mu(p')$ has an $(n-1)$-cycle Γ. We define now the preference profile $p \in \mathcal{P}$ defining, for every $i \in H$, the preference p_i as follows. If $i \leq \mu_0$, then let $p_i(1) = n$ and $p_i(j) = p'_i(j-1)$ for all $j \in \{2, \dots, n\}$; if $\mu_0 < i \leq h$, then let $p_i(j) = p'_i(j)$ for all $j \in \{1, \dots, n-1\}$ and $p_i(n) = n$. Note that in p, the alternative n is ranked first μ_0 times and last $h - \mu_0$ times. Thus, n is a maximum in $\Gamma_{\mu_0}(p)$. By Lemma 12, we then get $\mu(p) = \mu_0$ and $D_{\mu(p)}(p) = \{n\}$. Moreover $\Gamma \leq \Gamma_\mu(p)$ so that also $\Gamma_\mu(p^r)$ contains an $(n-1)$-cycle Γ^r, with inverted orientation, whose vertex set is V. That implies that $D_{\mu_0}(p^r) = \varnothing$. Indeed, n is not maximal in $\Gamma_{\mu_0}(p^r)$, being beaten μ_0 times by any other alternative, and each alternative in V is not maximal in $\Gamma_{\mu_0}(p^r)$ because, due to the presence of the cycle Γ^r, it is beaten $\mu > \mu_0$ times by a suitable alternative in V. Anyway $D_\mu(p^r) = \{n\}$, because n is isolated and thus maximal in

$\Gamma_\mu(p^r)$, by (2); no other alternative is maximal because involved in Γ^r. It follows that $\mu(p^r)=\mu$ and $D_{\mu(p)}(p)=D_{\mu(p^r)}(p^r)=\{n\}$. $\square$

Proposition 24. *If $n\geq 4$ and h is even and such that $h\geq \frac{2(n-1)}{n-3}$, then there exists $p\in\mathcal{P}$ such that $D_{\mu(p)}(p)=D_{\mu(p^r)}(p^r)=\{n\}$.*

Proof. First of all, note that $\mu_0=\frac{h+2}{2}$ and define $V=N\setminus\{n\}$. The assumption $h\geq\frac{2(n-1)}{n-3}$ is equivalent to $\mu_0\leq\frac{(n-1)-1}{n-1}h$ and thus, by Proposition 21, there exists a preference profile p' on the set of alternatives V such that $\Gamma_{\mu_0}(p')$ has an $(n-1)$-cycle Γ. We define now the preference profile $p\in\mathcal{P}$, defining, for every $i\in H$, the preference p_i as follows. If $i\leq\frac{h}{2}$, then let $p_i(1)=n$ and $p_i(j)=p'_i(j-1)$ for all $j\in\{2,\ldots,n\}$; if $\frac{h}{2}<i\leq h$, then let $p_i(j)=p'_i(j)$ for all $j\in\{1,\ldots,n-1\}$ and $p_i(n)=n$. Note that in p, the alternative n is ranked first $\frac{h}{2}$ times and last $\frac{h}{2}$ times. Thus, by (2), n is isolated and maximal both in $\Gamma_{\mu_0}(p)$ and in $\Gamma_{\mu_0}(p^r)$. Moreover, no further alternative is maximal in $\Gamma_{\mu_0}(p)$ because each element in V is involved in the cycle $\Gamma\leq\Gamma_{\mu_0}(p)$. Since each cycle in $\Gamma_{\mu_0}(p)$ determines a cycle with inverted orientation in $\Gamma_{\mu_0}(p^r)$, the same consideration holds for $\Gamma_{\mu_0}(p^r)$, as well. Then, we conclude that $\mu(p)=\mu(p^r)=\mu_0$ and $D_{\mu(p)}(p)=D_{\mu(p^r)}(p^r)=\{n\}$. $\square$

In the next section we are going to prove that if (h,n) does not satisfy the conditions stated in Propositions 23 and 24, then we cannot find p such that $D_{\mu(p)}(p)$ and $D_{\mu(p^r)}(p^r)$ are equal to the same singleton. As already explained in the introduction, such a proof does not seem to us to be based on a simple and intuitive argument. That fact suggested developing an alternative approach based on majority graphs.

We conclude the section with a lemma useful to manage the case with three individuals and three alternatives.

Lemma 25. *Let $(h,n)=(3,3)$ and $p\in\mathcal{P}$. Then:*

(i) the two following conditions are equivalent:

(a) the alternatives ranked first as well as those ranked third in p are distinct;

(b) $\Gamma_2(p)$ is a 3-cycle.

Moreover, if one of the above conditions holds true, then the arc set of $\Gamma_3(p)$ is empty.

(ii) $\mu(p)=\mu(p^r)$.

Proof. (i) We start showing that (a) implies (b). Assume that $p_i(1)\neq p_j(1)$ and $p_i(3)\neq p_j(3)$ for all $i,j\in H=\{1,2,3\}$ with $i\neq j$. Without loss of generality we can assume that $p_1(1)=1,p_2(1)=2,p_3(1)=3$. Thus $p_1(3)\in\{2,3\}$. If $p_1(3)=2$, then, since the alternatives ranked third are distinct, we necessarily have $p_2(3)=3$ and $p_3(3)=1$. That gives

$$p=\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix}$$

Similarly, if $p_1(3)=3$, we get

$$p=\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}$$

In both cases we have that $\Gamma_2(p)$ is a 3-cycle and the arc set of $\Gamma_3(p)$ is empty.

We next show that (b) implies (a). Assume that there exists $x\in N$ such that, in p, x is ranked first by at least two individuals or x is ranked third by at least two individuals. Then x is a maximum or a minimum for $\Gamma_2(p)$ and so it cannot be involved in a cycle of $\Gamma_2(p)$. Thus, $\Gamma_2(p)$ is not a 3-cycle.

(ii) By contradiction, assume that $\mu(p^r) \neq \mu(p)$, say $\mu(p^r) > \mu(p)$. Then $\mu(p) = 2$ and $\mu(p^r) = 3$. Thus $\Gamma_2(p^r)$ admits a cycle. By Lemma 10, we then get that $\Gamma_2(p^r)$ is a 3-cycle. Using (i) we deduce that the alternatives ranked first as well as those ranked third in p^r are distinct. But then, the same property holds for p, so that also $\Gamma_2(p)$ is a 3-cycle. Thus $D_2(p) = \varnothing$, against $\mu(p) = 2$. □

5.3 Proof of Proposition 6

First of all, let us prove that if $(h, n) \in \mathbb{N}^2_\diamond \setminus T_1$, then there exist $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$. We obtain the proof showing that the assumptions of Propositions 23 or 24 hold true. First of all, note that $(h, n) \in \mathbb{N}^2_\diamond \setminus T_1$ implies $h \geq 4$ and $n \geq 4$. If $n = 4$, then either h is even with $h \geq 6$ and so satisfies $h \geq \frac{2(n-1)}{n-3}$, or h is odd with $h \geq 9$ and so satisfies $h \geq \frac{3(n-1)}{n-3}$. If $n = 5$, then the same argument applies. If $n \geq 6$, then we have $\frac{2(n-1)}{n-3} \leq 4 \leq h$ for all h even, as well as $\frac{3(n-1)}{n-3} \leq 5 \leq h$ for all h odd.

Assume now that $(h, n) \in T_1$ and prove that it cannot be found $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$. Consider then $(h, n) \in T_1$ and assume, by contradiction, that there exist $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{x\}$. Since the Minimax SCC is neutral, we can assume that $x = n$ so that

$$D_{\mu(p)}(p) = D_{\mu(p^r)}(p^r) = \{n\}. \tag{5}$$

There are several cases to study.

If $n \in \{2, 3\}$, then, by Corollary 22, we have that $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic so that Lemma 18 (iii) applies contradicting (5).

If $h = 2$, then $\mu(p) = \mu(p^r) = \mu_a = 2$ and, by Corollary 22, we have that $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic, so that Lemma 18 (iii) applies contradicting (5).

If $h = 3$, then $\mu_0 = 2$ and $\mu(p), \mu(p^r) \in \{2, 3\}$. If $\mu(p) = 2$, then, by Lemma 19, we have that $\mu(p^r) = 3$, so that $D_{\mu(p^r)}(p^r) = \{n\}$ and $D_{\mu(p)}(p^r) = \varnothing$. Let $V = N \setminus \{n\}$. Since n is the only maximal element in $\Gamma_{\mu(p^r)}(p^r)$, for every $x \in V$, there exists $y \in N$ with $y >^{p^r}_{\mu(p^r)} x$. Note that if y were equal to n, then from $n >^{p^r}_{\mu(p^r)} x$ we would get $x >^{p}_{\mu(p^r)} n$ against the maximality of n in $\Gamma_{\mu(p)}(p)$. Thus, there exists a cycle in $\Gamma_{\mu(p^r)}(p^r)$ involving some vertices of V. That leads to a contradiction since, by Lemma 11, $\Gamma_{\mu(p^r)}(p^r)$ is acyclic. If $\mu(p^r) = 2$, then the previous argument applies to p^r. If $\mu(p) = \mu(p^r) = 3$, then we reach a contradiction applying Lemma 11 and Corollary 17.

If $(h, n) = (4, 4)$, then $\mu_0 = \mu_a = 3$ and $\mu(p),\ \mu(p^r) \in \{3, 4\}$. Thus, by Corollary 22, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic, so that Lemma 18 (iii) applies contradicting (5).

If $(h, n) = (5, 4)$, then $\mu_0 = 3$, $\mu_a = \mu_G = 4$, and $\mu(p),\ \mu(p^r) \in \{3, 4\}$. If $\mu(p) = \mu(p^r) = 4$, then by Corollary 22, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic and we contradict (5), using Lemma 18 (iii). If $\mu(p) = 3 = \mu_0$, then, by Lemma 19, $\mu(p^r) = 4$. By Corollary 22 we have that $\Gamma_{\mu(p^r)}(p^r)$ is acyclic and then, by Corollary 15, connected. Assume there exists $x \in V = \{1, 2, 3\}$ such that $4 >^{p^r}_{\mu(p^r)} x$. Then $x >^{p}_{\mu(p^r)} 4$, against $4 \in D_{\mu(p)}(p)$. So, we have $4 \not>^{p^r}_{\mu(p^r)} x$, for all $x \in V$. On the other hand, from $4 \in D_{\mu(p^r)}(p^r)$, we deduce that $x \not>^{p^r}_{\mu(p^r)} 4$. Thus, 4 is isolated in $\Gamma_{\mu(p^r)}(p^r)$, against the connection of $\Gamma_{\mu(p^r)}(p^r)$. If $\mu(p^r) = 3 = \mu_0$, then the previous argument applies to p^r.

If $(h, n) = (7, 4)$, then $\mu_0 = 4$, $\mu_a = 5$, $\mu_G = 6$ and $\mu(p),\ \mu(p^r) \in \{4, 5, 6\}$. If $\mu(p),\ \mu(p^r) \in \{5, 6\}$, then by Corollary 22, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic and we contradict (5), using Lemma 18 (iii). If $\mu(p) = 4$, then, by Lemma 19, $\mu(p^r) \in \{5, 6\}$. By Corollary 22, $\Gamma_{\mu(p^r)}(p^r)$ is acyclic and then, by Corollary 15, connected. Assume there exists $x \in V = \{1, 2, 3\}$ such that $4 >^{p^r}_{\mu(p^r)} x$. Then $x >^{p}_{\mu(p^r)} 4$, against $4 \in D_{\mu(p)}(p)$. So, we have $4 \not>^{p^r}_{\mu(p^r)} x$ for all $x \in V$. On the other hand, from

$4 \in D_{\mu(p^r)}(p^r)$ we deduce that $x \not>^{p^r}_{\mu(p^r)} 4$ for all $x \in V$. Thus, 4 is isolated in $\Gamma_{\mu(p^r)}(p^r)$, against the connection of $\Gamma_{\mu(p^r)}(p^r)$. If $\mu(p^r) = 4$, then the previous argument applies to p^r.

If $(h, n) = (5, 5)$, then $\mu_0 = 3$, $\mu_a = 4$, $\mu_G = 5$ and $\mu(p)$, $\mu(p^r) \in \{3, 4, 5\}$. If $\mu(p)$, $\mu(p^r) \in \{4, 5\}$, then by Corollary 22, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic and we contradict (5), using Lemma 18 (iii). If $\mu(p) = 3$, then, by Lemma 19, $\mu(p^r) \in \{4, 5\}$. By Corollary 22, $\Gamma_{\mu(p^r)}(p^r)$ is acyclic and then, by Corollary 15, connected. Assume there exists $x \in V = \{1, 2, 3, 4\}$ such that $5 >^{p^r}_{\mu(p^r)} x$. Then $x >^{p}_{\mu(p^r)} 5$, against $5 \in D_{\mu(p)}(p)$. On the other hand, from $5 \in D_{\mu(p^r)}(p^r)$ we deduce that $x \not>^{p^r}_{\mu(p^r)} 5$ for all $x \in V$. Thus, 5 is isolated in $\Gamma_{\mu(p^r)}(p^r)$, against the connection of $\Gamma_{\mu(p^r)}(p^r)$. If $\mu(p^r) = 3$, then the previous argument applies to p^r.

5.4 Proof of Proposition 7

First of all, let us prove that if $(h, n) \in \mathbb{N}^2_\diamond \setminus T_2$, then there exists $p \in \mathcal{P}$ such that $D_{\mu(p)}(p) = \{1\} \subseteq D_{\mu(p^r)}(p^r)$. If $(h, n) \in \mathbb{N}^2_\diamond \setminus T_1$, then we can apply Proposition 6. Assume then that $(h, n) \in T_1 \setminus T_2$ and note that

$$T_1 \setminus T_2 = \{(h, n) \in \mathbb{N}^2_\diamond : h = 3,\ n \geq 4\} \cup \{(5, 4), (5, 5), (7, 4)\}.$$

If $(h, n) \in \mathbb{N}^2_\diamond$ is such that $h = 3$ and $n \geq 4$, then consider $p \in \mathcal{P}$ defined by

$$p_1 = [1, (5), \dots, (n), 2, 3, 4]^T, \quad p_2 = [1, (5), \dots, (n), 3, 4, 2]^T, \quad p_3 = [4, 2, 3, (n), \dots, (5), 1]^T$$

Thus, $\mu(p) = 2$ and $D_{\mu(p)}(p) = \{1\}$, while $\mu(p^r) = 3$ and $D_{\mu(p^r)}(p^r) = N$.

If $(h, n) = (5, 4)$, then consider $p \in \mathcal{P}$ defined by

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 \\ 3 & 4 & 2 & 4 & 2 \\ 4 & 2 & 3 & 1 & 1 \end{bmatrix}$$

Thus, $\mu(p) = 3$ and $D_{\mu(p)}(p) = \{1\}$, while $\mu(p^r) = 4$ and $D_{\mu(p^r)}(p^r) = \{1, 2, 4\}$.

If $(h, n) = (5, 5)$, then consider $p \in \mathcal{P}$ defined by

$$\begin{bmatrix} 1 & 1 & 1 & 5 & 2 \\ 2 & 3 & 4 & 2 & 3 \\ 3 & 4 & 5 & 3 & 4 \\ 4 & 5 & 2 & 4 & 5 \\ 5 & 2 & 3 & 1 & 1 \end{bmatrix}$$

Thus, $\mu(p) = 3$ and $D_{\mu(p)}(p) = \{1\}$, while $\mu(p^r) = 4$ and $D_{\mu(p^r)}(p^r) = \{1, 5\}$.

If $(h, n) = (7, 4)$, then consider $p \in \mathcal{P}$ defined by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 & 4 & 2 & 3 \\ 3 & 4 & 2 & 3 & 2 & 3 & 4 \\ 4 & 2 & 3 & 4 & 1 & 1 & 1 \end{bmatrix}$$

Thus, $\mu(p) = 4$ and $D_{\mu(p)}(p) = \{1\}$, while $\mu(p^r) = 5$ and $D_{\mu(p^r)}(p^r) = \{1, 2, 4\}$.

Assume now that $(h, n) \in T_2$. We prove that it cannot be found $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = \{x\} \subseteq D_{\mu(p^r)}(p^r)$. By Lemmata 18(i) and 16(ii), it is enough to show that $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^r)}(p^r)$ are both acyclic. This comes applying Corollary 22 in all the possible cases. The application is obvious when $n \in \{2, 3\}$; if $h = 2$ note that $\mu(p) = \mu(p^r) = \mu_a = 2$; if $(h, n) = (4, 4)$ note that $\mu_0 = \mu_a = 3$ and thus $\mu(p), \mu(p^r) \geq 3$.

5.5 Proof of Proposition 8

First of all, let us prove that if $(h,n) \in \mathbb{N}_\diamond^2 \setminus T_3$, then there exists $p \in \mathcal{P}$ such that $D_{\mu(p)}(p) \neq N$ and $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) \neq \varnothing$. If $(h,n) \in \mathbb{N}_\diamond^2 \setminus T_2$ then we can apply Proposition 7. Assume then that $(h,n) \in T_2 \setminus T_3$ and note that

$$T_2 \setminus T_3 = \{(h,n) \in \mathbb{N}_\diamond^2 : h = 2,\, n \geq 3\} \cup \{(h,n) \in \mathbb{N}_\diamond^2 : h \neq 3,\, n = 3\} \cup \{(4,4)\}.$$

If $(h,n) \in \mathbb{N}_\diamond^2$ is such that $h = 2$ and $n \geq 3$, then consider $p \in \mathcal{P}$ defined by

$$p_1 = [1,2,3,\dots,n-1,n]^T, \quad p_2 = [n,1,2,\dots,n-1]^T.$$

Thus, $\mu(p) = \mu(p^r) = 2$ and, since $n \geq 3$, we have $D_{\mu(p)}(p) = \{1,n\} \neq N$. Moreover $D_{\mu(p^r)}(p^r) = \{n-1,n\}$, so that $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \{n\}$.

If $(h,n) \in \mathbb{N}_\diamond^2$ is such that $h \neq 3$ and $n = 3$, then consider the partition of $\mathbb{N}_\diamond \setminus \{3\}$ given by $H_1 = \{h = 2+3k : k \geq 0\}$, $H_2 = \{h = 1+3k : k \geq 1\}$, $H_3 = \{h = 3+3k : k \geq 1\}$. If $h \in H_1$, then consider any $p \in \mathcal{P}$ such that

$$\begin{aligned}&|\{i \in H : p_i = [1,2,3]^T\}| = 1+k, \quad |\{i \in H : p_i = [3,1,2]^T\}| = 1+k,\\ &|\{i \in H : p_i = [2,3,1]^T\}| = k.\end{aligned}$$

If $h \in H_2$, consider any $p \in \mathcal{P}$ such that

$$\begin{aligned}&|\{i \in H : p_i = [1,2,3]^T\}| = k, \quad |\{i \in H : p_i = [2,3,1]^T\}| = k,\\ &|\{i \in H : p_i = [3,1,2]^T\}| = k, \quad |\{i \in H : p_i = [1,3,2]^T\}| = 1.\end{aligned}$$

If $h \in H_3$, consider any $p \in \mathcal{P}$ such that

$$\begin{aligned}&|\{i \in H : p_i = [1,2,3]^T\}| = k, \quad |\{i \in H : p_i = [3,1,2]^T\}| = k,\\ &|\{i \in H : p_i = [2,3,1]^T\}| = k+1, \quad |\{i \in H : p_i = [1,3,2]^T\}| = 2.\end{aligned}$$

In all the above situations, it is easily checked that $D_{\mu(p)}(p) = \{1,3\} \neq N$ and $D_{\mu(p^r)}(p^r) = \{2,3\}$ so that $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \{3\} \neq \varnothing$.

If $(h,n) = (4,4)$, then consider $p \in \mathcal{P}$ defined by

$$\begin{bmatrix} 1 & 1 & 4 & 4 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 1 & 1 \end{bmatrix}$$

Thus $\mu(p) = \mu(p^r) = 3$, $D_{\mu(p)}(p) = \{1,2,4\} \neq N$ and $D_{\mu(p^r)}(p^r) = \{1,3,4\}$.

Assume now that $(h,n) \in T_3$. We prove that it cannot be found $p \in \mathcal{P}$ such that $D_{\mu(p)}(p) \neq N$ and $D_{\mu(p)}(p) \cap D_{\mu(p^r)}(p^r) = \varnothing$.

If $n = 2$, then the condition $D_{\mu(p)}(p) \neq N$ is equivalent to $|D_{\mu(p)}(p)| = 1$. Since $(h,n) \in T_2$ Proposition 7 applies.

Finally let $(h,n) = (3,3)$. We show that, for every $p \in \mathcal{P}$, we have $M(p) \cap M(p^r) = \varnothing$ or $M(p) = N$. Fix $p \in \mathcal{P}$ and note that $\mu_0 = 2$. Assume first that there exists $x \in N$ such that $\{i \in H : p_i(1) = x\}$ has at least two elements. By Lemma 25 (i) and Lemma 10, $\Gamma_2(p)$ is acyclic. Thus $\mu(p) = 2$ and $M(p) = D_2(p) = \{1\}$. By Lemma 25 (ii) we also have $\mu(p^r) = 2$. Since in p^r the alternative 1 is beaten by the alternative 2 at least two times, we have that $1 \notin D_2(p^r) = M(p^r)$ and so $M(p) \cap M(p^r) = \varnothing$. If there exists $x \in N$ such that $\{i \in H : p_i(3) = x\}$ has at least two elements we apply the argument above to p^r, obtaining again $M(p) \cap M(p^r) = \varnothing$. We are then left with assuming that the alternatives ranked first as well as those ranked third are distinct in p. In this case, by Lemma 25 (i), $\Gamma_2(p)$ is a 3-cycle and $\mu(p) = 3$. Moreover, the arc set of $\Gamma_3(p)$ is empty so that $M(p) = D_3(p) = N$.

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