1) $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-y \\ -x+y+2z \\ y+4z \end{pmatrix}$$

f E' LINEARE:

· $\forall \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}^3$

$$f\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = f\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} 2(x_1+x_2)-(y_1+y_2) \\ -(x_1+x_2)+(y_1+y_2)+2(z_1+z_2) \\ (y_1+y_2)+4(z_1+z_2) \end{pmatrix} =$$

$$= f\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2x_1-y_1 \\ -x_1+y_1+2z_1 \\ y_1+4z_1 \end{pmatrix} + \begin{pmatrix} 2x_2-y_2 \\ -x_2+y_2+2z_2 \\ y_2+4z_2 \end{pmatrix}$$

· $\forall \lambda \in \mathbb{R}, \ \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

$$f\left(\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = f\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} = \begin{pmatrix} 2\lambda x - \lambda y \\ -\lambda x + \lambda y + 2\lambda z \\ \lambda y + 4\lambda z \end{pmatrix} = \lambda f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} 2x-y \\ -x+y+2z \\ y+4z \end{pmatrix}$$

$$Ker(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ \middle| \ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x-y \\ -x+y+2z \\ y+4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x-y = 0 \\ -x+y+2z = 0 \\ y+4z = 0 \end{cases}$$

GAUSS

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \operatorname{Ker}(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y = -4z,\; x = -2z,\; z \in \mathbb{R} \right\} =$$

$$= \left\{ \begin{pmatrix} -2z \\ -4z \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z \in \mathbb{R} \right\} = \left\{ z \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} =$$

$$= \operatorname{SPAN} \left\{ \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix} \right\} \qquad \dim(\operatorname{Ker}(f)) = 1$$

~~Im(f) = SPAN~~

f LINEARE $\Rightarrow$ $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

CON $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}$

~~Im(f) = SPAN~~ $\cancel{\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \right\}}$

~~COLONNE DI A~~

TEOREMA DELLA DIMENSIONE:

$$\dim(\operatorname{Im}(f)) = \dim(\mathbb{R}^3) - \dim(\operatorname{Ker}(f)) = 3 - 1 = 2$$

$$\operatorname{Im}(f) = \operatorname{SPAN} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \right\} = \operatorname{SPAN} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \right\}$$

COLONNE DI A

$$\det(A-\lambda \mathbb{I}_3) = \det\begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 1-\lambda & 2 \\ 0 & 1 & 4-\lambda \end{pmatrix} =$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda = -\lambda(\lambda^2 - 7\lambda + 11) = 0$$

$$\Rightarrow \quad \lambda = 0, \quad \lambda^2 - 7\lambda + 11 = 0 \qquad \Delta = 5, \quad \lambda_{1,2} = \frac{7 \pm \sqrt{5}}{2}$$

AUTOVALORI : $0, \ \frac{7+\sqrt{5}}{2}, \ \frac{7-\sqrt{5}}{2}$

2) $a \in \mathbb{R}$, $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax^2 - ax + 3x^2 - y \\ -x + y \end{pmatrix}$$

<u>METODO 1</u>

f_a LINEARE $\Rightarrow$ $f_a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f_a\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = f_a\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$, $\forall \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

$$f_a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f_a\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} ax_1^2 - ax_1 + 3x_1^2 - y_1 \\ -x_1 + y_1 \end{pmatrix} + \begin{pmatrix} ax_2^2 - ax_2 + 3x_2^2 - y_2 \\ -x_2 + y_2 \end{pmatrix} =$$

$$= \begin{pmatrix} a(x_1^2 + x_2^2) - a(x_1 + x_2) + 3(x_1^2 + x_2^2) - (y_1 + y_2) \\ -(x_1 + x_2) + (y_1 + y_2) \end{pmatrix}$$

$$f_a\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + x_2)^2 - a(x_1 + x_2) + 3(x_1 + x_2)^2 - (y_1 + y_2) \\ -(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} =$$

$$= \begin{pmatrix} a(x_1^2 + x_2^2) + 2ax_1x_2 - a(x_1 + x_2) + 3(x_1^2 + x_2^2) + 6x_1x_2 - (y_1 + y_2) \\ -(x_1 + x_2) + (y_1 + y_2) \end{pmatrix}$$

$$f_a\begin{pmatrix}x_1\\y_1\end{pmatrix}+f_a\begin{pmatrix}x_2\\y_2\end{pmatrix}=f_a\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix} \Leftrightarrow$$

$$0 = 2a x_1 x_2 + 6 x_1 x_2 \qquad \forall x_1, x_2 \in \mathbb{R}$$

$\Leftrightarrow$ ($x_1 = x_2 = 1$) $\quad 0 = 2a+6 \quad \Rightarrow \quad a = -3$

f_{-3} È LINEARE E $f_{-3}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x-y\\-x+y\end{pmatrix}$

<u>METODO 2</u>

f_a LINEARE $\Rightarrow$ $f_a\begin{pmatrix}x_1\\y_1\end{pmatrix}+f_a\begin{pmatrix}x_2\\y_2\end{pmatrix}=f_a\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix} \quad \forall \begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}\in\mathbb{R}^2$

SCELGO $\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$

$$f_a\begin{pmatrix}2\\1\end{pmatrix}+f_a\begin{pmatrix}-1\\0\end{pmatrix}=f_a\left(\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\0\end{pmatrix}\right)$$

$$\begin{pmatrix}4a-2a+12-1\\-2+1\end{pmatrix}+\begin{pmatrix}a+a+3\\1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$$

$$\begin{pmatrix}4a+14\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix} \Rightarrow 4a+14=2 \Rightarrow a=-3$$

$\Rightarrow f_{-3}$ È LINEARE, $f_{-3}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3&-1\\-1&+1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x-y\\-x+y\end{pmatrix}$

$A_{-3}=\begin{pmatrix}3&-1\\-1&+1\end{pmatrix}$ SIMMETRICA

A_{-3} AMMETTE BASE DI AUTOVETTORI PER $\mathbb{R}^3$

COME CONSEGUENZA DEL TEOREMA SPETTRALE

3) $p(x,y,z) = 2x^2 + 4xy + 6yz - 8xz + 2y^2 + 2z^2$

TROVARE A E Q_A TALI CHE

$$Q_A\begin{pmatrix}x\\y\\z\end{pmatrix} = (x\ \ y\ \ z)\, A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}^t A\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Q_A = FORMA QUADRATICA ASSOCIATA AL POLINOMIO p

$$A = \begin{pmatrix} 2 & 2 & -4 \\ 2 & 2 & 3 \\ -4 & 3 & 1 \end{pmatrix}$$

$$\Rightarrow \quad Q_A\begin{pmatrix}x\\y\\z\end{pmatrix} = (x\ \ y\ \ z)\, A\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\det(A - \lambda \mathbb{I}_3) = \det\begin{pmatrix} 2-\lambda & 2 & -4 \\ 2 & 2-\lambda & 3 \\ -4 & 3 & 1-\lambda \end{pmatrix} =$$

$$= (2-\lambda)(2-\lambda)(1-\lambda) - 24 - 24 - 16(2-\lambda) - 9(2-\lambda) - 4(1-\lambda) =$$

$$= (4+\lambda^2-4\lambda)(1-\lambda) - 48 - 32 + 16\lambda - 18 + 9\lambda - 4 + 4\lambda =$$

$$= 4 + \lambda^2 - 4\lambda - 4\lambda - \lambda^3 + 4\lambda^2 - 80 + 16\lambda - 18 + 9\lambda - 4 + 4\lambda =$$

$$= -\lambda^3 + 5\lambda^2 + 21\lambda - 98$$

$$-\lambda^3 + 5\lambda^2 + 21\lambda - 98 = 0$$

SICCOME A È SIMMETRICA, LE SOLUZIONI DI

$-\lambda^3 + 5\lambda^2 + 21\lambda - 98 = 0$ SONO TUTTE REALI E AL

MASSIMO 3 (PERCHÈ POLINOMIO DI TERZO GRADO)

- 0 NON È AUTOVALORE (SOLUZIONE DI $\det(A - \lambda I_3) = 0$) PERCHÉ IL TERMINE NOTO DI $-\lambda^3 + 5\lambda^2 + 21\lambda - 98$ È DIVERSO DA ZERO

- VARIAZIONI DI SEGNO

$-\lambda^3 + 5\lambda^2 + 21\lambda - 98$

CONSIDERO ADESSO I COEFFICIENTI DEL POLINOMIO

$-1, +5, +21, -98$

$- \quad + \qquad + \quad -$

2 VARIAZIONI DI SEGNO

2 AUTOVALORI POSITIVI

1 AUTOVALORE NEGATIVO

Q_A È INDEFINITA

ESEMPIO

$Q_A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 - 8 + 1 = -5 < 0$

$Q_A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 > 0$

4) $M = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$

M È SIMMETRICA $\Rightarrow$ M È DIAGONALIZZABILE E

$\exists$ C MATRICE INVERTIBILE E ORTOGONALE
TALE CHE $C^{-1}MC =$ MATRICE DIAGONALE

$$\det(M - \lambda I_4) = \det\begin{pmatrix} 2-\lambda & 2 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 3 \\ 0 & 0 & 3 & 3-\lambda \end{pmatrix} =$$

$$= \lambda^2(\lambda-4)(\lambda-6)$$

AUTOVALORI DI M

- 0 CON MOLTEPLICITÀ (ALGEBRICA) 2
- 4 " " " 1
- 6 " " " 1

$$V_0 = \operatorname{Ker}(L_M) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; M\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

GAUSS

$$\left(\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

II^a RIGA: $3z + 3w = 0 \to z = -w$

I^a RIGA: ~~2z + 2w = 0~~
$2x + 2y = 0 \to x = -y$

$$V_0 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x = -y \\ z = -w \end{matrix} \right\} = \left\{ \begin{pmatrix} -y \\ y \\ -w \\ w \end{pmatrix} \;\middle|\; y, w \in \mathbb{R} \right\} =$$

$$= \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; y, w \in \mathbb{R} \right\} =$$

$$= \text{SPAN} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$(v_1, v_2) = \left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right) = (-1) \cdot 0 + 1 \cdot 0 + 0 \cdot (-1) + 0 \cdot 1 = 0$$

v_1, v_2 ORTOGONALI

$$\|v_1\| = \left\| \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\|v_2\| = \left\| \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$w_1 = \frac{1}{\sqrt{2}} v_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = \frac{1}{\sqrt{2}} v_2 = \begin{pmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

SONO ORTOGONALI

E DI NORMA 1

(ORTONORMALI)

5) $L_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

SP(A) = SPETTRO DI A = { AUTOVALORI DI A }

$$\det(A - \lambda I_4) = \det\begin{pmatrix} 2-\lambda & -1 & 1 & 0 \\ 0 & 1-\lambda & 4 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} =$$

$$= (2-\lambda)(1-\lambda)(1-\lambda)(2-\lambda) = (1-\lambda)^2(2-\lambda)^2$$

AUTOVALORI DI A

- 1 CON MOLTEPLICITA' 2
- 2 " " 2

$SP(A) = \{1, 2\} \Rightarrow |SP(A)|$ = CARDINALITA' DI $SP(A) = 2$

$$V_1 = \operatorname{Ker}(L_A - I_4) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$w = 0$

$4z = 0 \rightarrow z = 0$

$x - y + z = 0 \Rightarrow x - y = 0$

$$V_1 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{c} w = 0 = z \\ x = y \end{array} \right\} = \left\{ \begin{pmatrix} x \\ x \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x \in \mathbb{R} \right\} =$$

$$= \left\{ x \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} = \operatorname{SPAN}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\dim V_1 = 1$ (= MOLTEPLICITÀ GEOMETRICA DI $\lambda = 1$) < 2

(MOLTEPLICITÀ ALGEBRICA DI $\lambda = 1$)

$\Downarrow$

L_A NON È DIAGONALIZZABILE $\Leftrightarrow$

NON ESISTE BASE DI AUTOVETTORI DI $\mathbb{R}^4$ DI L_A.

6) $X = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -3 \end{pmatrix}$

$$\|X\| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\|Y\| = \sqrt{1^2 + 1^2 + (-3)^2} = \sqrt{11}$$

$$d(X,Y) = \|X - Y\| = \left\| \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ -3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -2 \\ 0 \\ 0 \\ 6 \end{pmatrix} \right\| =$$

$$= \sqrt{(-2)^2 + 6^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

PROIEZIONE DI X SU Y

$$P_Y(X) = \frac{\langle X, Y \rangle}{\langle Y, Y \rangle} Y$$

$$(Y,Y) = \|Y\|^2 = 11, \quad (X,Y) = (-1)\cdot 1 + 1\cdot 1 + 0\cdot 0 + 3\cdot(-3) =$$

$$= -1 + 1 - 9 = -9$$

$$P_Y(X) = \frac{-9}{11} Y = \begin{pmatrix} -9/11 \\ -9/11 \\ 0 \\ \frac{27}{11} \end{pmatrix}$$

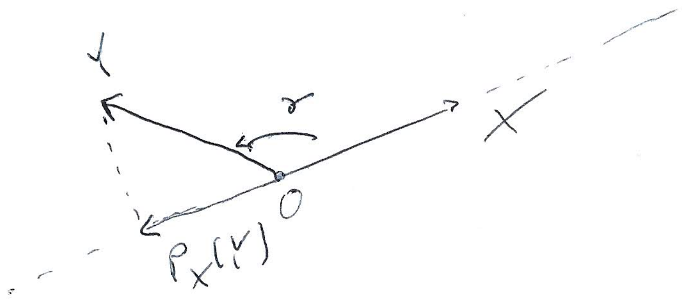
$$P_X(Y) = \frac{(Y,X)}{(X,X)} X$$

$(Y, X) = (X, Y) = -9$

$(X, X) = \|X\|^2 = 11$

$\Rightarrow P_X(Y) = -\frac{9}{11} X = \begin{pmatrix} +9/11 \\ -9/11 \\ 0 \\ -\frac{27}{11} \end{pmatrix}$

$\cos(\gamma) = \frac{(X, Y)}{\|X\| \cdot \|Y\|} = \frac{-9}{11 \cdot 11} = -\frac{9}{121} < 0 \Rightarrow \gamma$ E' OTTUSO



7) $X, Y \in \mathbb{R}^n \Rightarrow X - P_Y(X), \; Y$ ORTOGONALI

DIM

$X - P_Y(X)$ E Y ORTOGONALI $\Leftrightarrow (X - P_Y(X), Y) = 0$

$P_Y(X) = \frac{(X, Y)}{\|Y\|^2} Y$

$\left(X - \frac{(X, Y)}{\|Y\|^2} Y, Y\right) = (X, Y) - \frac{(X, Y)}{\|Y\|^2} (Y, Y) =$

$= (X, Y) - \frac{(X, Y)}{\cancel{\|Y\|^2}} \cancel{\|Y\|^2} = (X, Y) - (X, Y) = 0$

8) ~~[illegible]~~

$$B = \begin{pmatrix} 2 & -1 \\ 6 & 1 \end{pmatrix} \in M_2(\mathbb{Q}) \qquad \det(B) = 2+6 = 8$$

$$A \in M_2(\mathbb{Q}) \Rightarrow AB = B^{-1}A$$

$$\Rightarrow \det(AB) = \det(B^{-1}A)$$

$$\det(A)\det(B) = \det(B^{-1})\det(A)$$

$$8\det(A) = \frac{1}{8}\det(A)$$

$$\det(A)\left(8-\frac{1}{8}\right) = 0 \Rightarrow \det(A)\frac{63}{8} = 0 \Rightarrow \det(A) = 0$$

$\Rightarrow$ A NON È INVERTIBILE

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow AB = B^{-1}A \iff BAB = A$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 6 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2a - c & 2b - d \\ 6a + c & 6b + d \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 6 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4a - 2c + 12b - 6d & -2a + c + 2b - d \\ 12a + 2c + 36b + 6d & -6a - c + 6b + d \end{pmatrix}$$

$$\begin{cases} a = 4a - 2c + 12b - 6d \\ b = -2a + c + 2b - d \\ c = 12a + 2c + 36b + 6d \\ d = -6a - c + 6b + d \end{cases} \Rightarrow \begin{cases} 3a + 12b - 2c - 6d = 0 \\ -2a + b + c - d = 0 \\ 12a + 36b + c + 6d = 0 \\ -6a + 6b - c - \quad = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d = 0 \end{cases}$$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9) $p(x,y,z) = x^2+y^2+z^2+2xy-2zy-2xz$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$Q_A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x\ y\ z)\, A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = p(x,y,z)$$

$$p(x,y,z) = (x+y-z)^2 \geq 0 \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

$$p(x,y,z) = 0 \iff x+y-z=0$$

$\Rightarrow Q_A$ È SEMIDEFINITA POSITIVA

$$p(1,0,0) = 1 > 0$$

$$p(1,1,2) = 1+1+4+2-4-4 = 8-8 = 0$$

10) $A \in M_n(\mathbb{Q})$ TALE CHE $\det(A) = 2$

SE $\exists\, B \in M_n(\mathbb{Q}) : ABA = BAB$, ALLORA $\det(B) \in \{0, 2\}$

DIM.

$ABA = BAB \Rightarrow \det(ABA) = \det(BAB)$

$$\det(ABA) \underset{\substack{\uparrow \\ \text{BINET}}}{=} \det(A)\det(B)\det(A) = \det^2(A) \cdot \det(B) = 4\det(B)$$

$$\det(BAB) = \det^2(B)\det(A) = 2\det^2(B)$$

$$\Rightarrow 4\det(B) = 2\det^2(B) \Rightarrow \det^2(B) - 2\det(B) = 0 \Rightarrow \det(B) \in \{0, 2\}$$

11) $\rho_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$A_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \Rightarrow \rho_\alpha\begin{pmatrix} x \\ y \end{pmatrix} = A_\alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\alpha = \frac{\pi}{3} \Rightarrow A_{\frac{\pi}{3}} = \begin{pmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\rho_\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x + \frac{y}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x - \sqrt{3}y \\ \sqrt{3}x + y \end{pmatrix}$$

$$\det\left(A_{\frac{\pi}{3}} - \lambda I_2\right) = \det\begin{pmatrix} \frac{1}{2} - \lambda & -\sqrt{3}/2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} - \lambda \end{pmatrix} =$$

$$= \left(\frac{1}{2} - \lambda\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + \lambda^2 - \lambda + \frac{3}{4} =$$

$$= \lambda^2 - \lambda + 1 = 0 \qquad \Delta = 1 - 4 = -3 < 0$$

NON HA SOLUZIONI REALI $\Rightarrow$ NON HA AUTOVALORI REALI

$$\rho_\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 - \sqrt{3} \\ -\sqrt{3} + 1 \end{pmatrix} = \begin{pmatrix} \frac{-1-\sqrt{3}}{2} \\ \frac{-\sqrt{3}+1}{2} \end{pmatrix}$$

$$\left\|\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\left\|\rho_\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} \frac{-1-\sqrt{3}}{2} \\ \frac{\sqrt{3}+1}{2} \end{pmatrix}\right\| = \sqrt{\left(\frac{-1-\sqrt{3}}{2}\right)^2 + \left(\frac{-\sqrt{3}+1}{2}\right)^2} = \frac{\sqrt{1+3+\cancel{2\sqrt{3}}+3+1\cancel{-2\sqrt{3}}}}{2} =$$

$$= \frac{\sqrt{2 \cdot 4}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$